ARCHIVUM MATHEMATICUM (BRNO)
Tomus 28 (1992), 51 – 55

REMARK CONCERNING OSCILLATORY PROPERTIES OF SOLUTIONS OF A CERTAIN NONLINEAR EQUATION OF THE THIRD ORDER

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Dedicated to Professor M. Novotný and Professor F. Šik on the occasion of their seventieth birthdays

Abstract. Sufficient conditions for oscillation of a certain nonlinear trinomial third order differential equation are proved.

1. In the present paper we use the results of linear differential equation of the third order [1] to derive sufficient conditions for oscillation or nonoscillation of solutions of the following equation of the third order:

$$u''' + q(t)u' + p(t)u^{\alpha} = 0 \ , \tag{1}$$

where $p(t)$, $q(t)$, $q'(t)$ are continuous functions $t \in (a, \infty)$, $-\infty < a < \infty$, $a > 1$ is a ratio of two odd relatively prime natural numbers.

The results suitably supplement known results of P. Šoltés [2], P. Waltman [3] and other authors. In this paper under a solution of differential equation (1) we will understand a nontrivial solution of equation (1) defined on the interval $\langle \bar{t}, \infty)$, $\bar{t} \geq a$. A nontrivial solution of (1) is said to be oscillatory if it has zeros for arbitrary large values of (the independent variable) t.

2. Here we will introduce some results on the linear differential equation of the third order

$$y''' + 2A(t)y' + [A'(t) + b(t)]y = 0 \ , \tag{a}$$

where $A'(t)$, $b(t)$ are continuous functions on (a, ∞) and $b(t) \geq 0$ for $t \in (a, \infty)$ with the condition that $b(t) \equiv 0$ does not hold on any subinterval.

1991 *Mathematics Subject Classification*: 34C30.
Key words and phrases: ordinary differential equation, oscillatory.
Received June 13, 1991.

The adjoint equation to (a) is

$$z''' + 2A(t)z' + [A'(t) - b(t)]z = 0 \ . \tag{b}$$

Let $w(t) > 0$ for $t \in \langle t_0, \infty)$, $a \leq t_0 < \infty$ be a solution of the equation (b). Then there exists two-parameter family of solutions y of the differential equation (a) that satisfy the equation

$$\left[\frac{1}{w(t)}y'\right]' + \frac{w''(t) + 2A(t)w(t)}{w^2(t)}y = 0 \ . \tag{c}$$

Lemma A. [1, Theorem 1.7 and Remark 1.5]. *Suppose b has the above mentioned property and let $A(t) \leq 0$ and $A'(t) + b(t) \geq 0$ for $t \in (a, \infty)$. Then the solution w of the differential equation (b) with the property $w(t_0) = w''(t_0) = 0$, $w'(t_0) > 0$, $t_0 > a$ has the property $w(t) > 0$, and $w''(t) + 2A(t)w(t) > 0$ for $t > t_0$.*

Remark 1. If we substitute the solution w from Lemma A into (c) then solutions y of (c) have the property $y'(t_0) = 0$. Differentiating equation (c) term by term we obtain equation (a), hence all solutions y of (c) are at the same time solutions of equation (a).

Theorem B. [1, Theorem 2.14]. *Let $A(t) \leq 0$, $A'(t) + b(t) \geq 0$ for $t \in (a, \infty)$ with $b(t) \not\equiv 0$ on any subinterval. Besides, let*

$$\int_{t_0}^{\infty} \left[A'(t) + b(t) - \frac{4}{3}\left(-\frac{2}{3}A^3(t)\right)^{1/2}\right] dt = \infty, \quad t_0 > a \ .$$

Then the differential equation (a) is oscillatory in (a, ∞) (i.e. each of its solutions with one null-point has infinitely many null-points in (a, ∞)).

Theorem C. [1, Theorem 2.10]. *Let $A(t) \leq 0$ and $b(t) \geq 0$ for $t \in (a, \infty)$, and $b(t) \not\equiv 0$ on any subinterval. Then there exists at least one solution y of the differential equation (a), $y(t) \neq 0$ for $t \in (a, \infty)$, moreover y, y' are monotone functions in (a, ∞) and sgn $y(t) =$ sgn $y''(t) \neq$ sgn $y'(t)$ for $t \in (a, \infty)$.*

3. Now we return to the differential equation (1).

Theorem 1. *Let $q(t) \leq 0$, $q'(t) \leq 0$, and $p(t) > 0$ for $t \in (a, \infty)$ and let $-k^2 < q(t)$, $k \neq 0$ and $\lim_{t \to \infty} p(t) = +\infty$. Then each solution u of the differential equation (1) defined on $\langle t_0, \infty)$ with the property $u(t) \neq 0$ for $t \in \langle t_0, \infty)$ has also the property: There exists a $T \geq t_0$ such that for $t \geq T$ we have sgn $u(t) \neq$ sgn $u'(t)$.*

Proof. Let $u_1(t)$ be a solution of the differential equation (1), $u_1(t) \neq 0$ for $t > t_0$. Suppose that $u_1'(t) < 0$ does not hold for $t > T$. Then we have two cases:
a) $u_1'(t) \geq 0$; b) $u_1'(t)$ is an oscillatory function in (T, ∞).
a) Let $u_1'(t) \geq 0$. The solution $u_1(t)$ satisfies the (linear) equation

$$u''' + q(t)u' + p(t)u_1^{\alpha-1}(t)u = 0 \ . \tag{2}$$

The coefficients of equation (2) satisfy the assumptions of the Theorem C because $A(t) = \frac{1}{2}q(t) \le 0$, $b(t) = p(t)u_1^{\alpha-1}(t) - \frac{1}{2}q'(t) > 0$ for $t > a$. Theorem C ensures the existence of a least one solution $u_2(t)$ of equation (2) with the property $u_2(t) > 0$, sgn $u_2(t) \ne$ sgn $u_2'(t)$ and $u_2(t)$, $u_2'(t)$ are monotone functions.

Let $k > 0$ be a constant such that $v(t) = u_1(t) - ku_2(t)$ and $v(t_1) = 0$, $t_1 \ge T$. The first derivative $v'(t)$ of the function $v(t)$ is positive for $t > t_1$. We will show that this is not possible. Equation (2) satisfies the assumptions of Theorem B because $A = \frac{1}{2}q(t) \le 0$, $A' + b = p(t)u_1^{\alpha-1}(t) > 0$, $b(t) = p(t)u_1^{\alpha-1}(t) - \frac{1}{2}q'(t) > 0$ for $t > t_1$ and

$$\int_{t_0}^{\infty}\left[A'(t) + b(t) - \frac{4}{3}\left(\frac{2}{3}\left(-A^3(t)\right)\right)^{1/2}\right]dt =$$
$$= \int_{t_0}^{\infty}\left[p(t)u_1^{\alpha-1}(t) - \frac{4}{3}\left(\frac{2}{3}\left(-\frac{1}{8}q^3(t)\right)\right)^{1/2}\right]dt\ .$$

But

$$\int_{t_0}^{\infty}\left[p(t)u_1^{\alpha-1}(t) - \frac{4}{3}\left(\frac{2}{3}\left(-\frac{1}{8}q^3(t)\right)\right)^{1/2}\right]dt \ge$$
$$\ge \int_{t_0}^{\infty}\left[p(t)K^{\alpha-1} - \frac{4}{3}\left(\frac{2}{3}\left(\frac{1}{8}k^6\right)\right)^{1/2}\right]dt\ ,$$

where $u_1(t) > K$ for $t > t_1$. From the assumptions of Theorem 1 for the function $p(t)$ it follows that the integral on the right-hand side of the last inequality diverges to infinity. Theorem B shows that the solution $v(t)$ oscillates in $\langle t_1, \infty)$, hence $v'(t) > 0$ for $t > t_1$ cannot hold and therefore also $u_1' \ge 0$ for $t > T$.
b) Let $u_1'(t)$ be an oscillatory function in $\langle T_1, \infty)$ and let $u_1'(T_1) = 0$. Lemma A implies that the solution u_1 fulfils equation (c), where the function w satisfies $w(T_1) = w''(T_1) = 0$, $w'(T_1) > 0$ and $w''(t) + q(t)w(t) > 0$ for $t > T_1$. Let $T_2 > T_1$ be the next null-point of $u_1'(t)$.

If we integrate equation (c) from T_2 to t, $t > T_2$, we get

$$u_1'(t) = -\frac{1}{w(t)}\int_{T_2}^{t}\frac{w''(\tau) + q(\tau)w(\tau)}{w^2(\tau)}u_1(\tau)d\tau < 0\ ,$$

what is in contradiction with the assumption that $u_1'(t)$ oscillates. Therefore $u_1'(t) < 0$ for $t > T_1$. □

Theorem 2. *Suppose the assumptions of Theorem 1 for $p(t)$ and $q(t)$ hold. Then each solution u of the differential equation (1) defined on $\langle t_0, \infty)$ with one null-point oscillates in $\langle t_0, \infty)$.*

Proof. Suppose u_1 is a solution of the differential equation (1) defined on $\langle t_0, \infty)$ and let $u_1(t_1) = 0$, $t_1 \ge t_0$. Suppose, e.g., $u_1(t) > 0$ for $t > t_1$. From Theorem

1, beginning at a certain T, it follows that u_1 must have the property $u_1(t) > 0$, $u_1'(t) < 0$. Therefore there exists $T_1 > t_1$ such that $u_1'(T_1) = 0$ and $u_1''(T_1) < 0$ and for $t > T_1$ we have $u_1(t) > 0$, $u_1'(t) < 0$. From equation (1) then for $t > T_1$ it follows that $u_1'''(t) = -q(t)u_1'(t) - p(t)u_1^\alpha(t) < 0$, $u_1''(t) < 0$ and $u_1'(t) < u_1'(T_2) < 0$, where $T_2 > T_1$, which, in turn, implies that $u_1(t)$ has another null-point. □

Corollary 1. *Let $q(t) \le 0$, $q'(t)$ and $p(t) > 0$, $t \in (a, \infty)$ be continuous functions. The sufficient condition for oscillation of each solution of (1) with one null-point is that each solution u of (1) without null-points in some neighbourhood of $+\infty$ has in some neighbourhood of the point $+\infty$ the property that $u(t)$, $u'(t)$ are monotone functions and sgn $u(t) \ne$ sgn $u'(t)$.*

The proof is completely analogous to the proof of Theorem 2 and therefore we do not include it.

P. Waltman in his paper [3] has derived that the sufficient condition for oscillation of solutions with one null-point of the equation

$$u''' + q(t)u' + p(t)f(u) = 0 \tag{3}$$

are the following: $q(t)$ and $p(t)$ are continuous, $q(t) \ge 0$, $p(t) \ge 0$ for $t \in (a, \infty)$ $f(y)/y > \kappa > 0$ for certain $\kappa > 0$ and $\kappa q(t) - p'(t) > 0$, and $\int_{t_0}^{\infty} t[\kappa q(t) - p'(t)]dt = \infty$. This result can be extended to equation (3) with the condition that $q(t) \le 0$ in the following way:

Corollary 2. *Let the coefficients $q(t)$ and $p(t)$ satisfy the assumptions of Theorem 1 and, moreover, let $f(u)$ be continuous for $u \in (-\infty, \infty)$, and $f(u)/u > \kappa > 0$ for $u \ne 0$. Then each solution u of the differential equation (3) defined on $\langle t_0, \infty)$ with $u(t) \ne 0$ for $t \in \langle t_0, \infty)$ has also the property: There exists $T \ge t_0$ such that for $t \ge T$ we have sgn $u(t) \ne$ sgn $u'(t)$.*

The proof is very similar to the proof of Theorem 1, only instead of equation (2) one must consider the equation

$$u''' + q(t)u' + p(t)\frac{f(u_1)}{u_1}u = 0 \ .$$

Corollary 3. *Suppose the assumptions on $p(t)$, $q(t)$ and $f(u)$ are the same as in Corollary 2. Then each solution of the differential equation (3) defined on $\langle t_0, \infty)$, $t_0 \ge a$, with one null-point oscillates in $\langle t_0, \infty)$.*

The proof is again similar to the proof of Theorem 2 and therefore is not included.

References

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