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PRINCIPAL SOLUTIONS AND TRANSFORMATIONS OF LINEAR HAMILTONIAN SYSTEMS

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Dedicated to Professor M. Novotný on the occasion of his seventieth birthday

Abstract. Sufficient conditions are given which guarantee that the linear transformation converting a given linear Hamiltonian system into another system of the same form transforms principal (antiprincipal) solutions into principal (antiprincipal) solutions.

1. Introduction.

Consider a linear Hamiltonian system

$$Y' = A(t)Y + B(t)Z, \quad Z' = -C(t)Y - A^T(t)Z, \tag{1.1}$$

where A, B, C are $n \times n$ matrices of continuous, real valued functions, $t \in I = [a, \infty)$, B, C are symmetric, i. e., $B^T = B$, $C^T = C$, and Y, Z are $n \times n$ matrices. If the matrices B, C are nonnegative definite, it is known that (1.1) is nonoscillatory at ∞ (for terminology see Section 2) if and only if the so-called reciprocal system

$$U' = -A^T(t)U + C(t)V, \quad V' = -B(t)U + A(t)V \tag{1.2}$$

is nonoscillatory at ∞, see [2,5,8,9].

Recently the author established a more general duality in oscillation behaviour of various linear Hamiltonian systems which may be described in the following way. If we set

$$U = H(t)Y + M(t)Z, \quad V = K(t)Y + N(t)Z, \tag{1.3}$$

where H, K, M, N are $n \times n$ matrices of continuously differentiable real-valued functions such that $M(t)$ is nonsingular on I and the $2n \times 2n$ matrix

$$R(t) = \begin{pmatrix} H(t) & M(t) \\ K(t) & N(t) \end{pmatrix} \tag{1.4}$$

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is J-unitary, i.e.,

$$R^T(t)JR(t) = J, \tag{1.5}$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, I_n being the $n \times n$ identity matrix, then (U, V) is also a solution of a linear Hamiltonian system which is under certain additional assumptions (corresponding to nonnegativeness of B and C in the case of reciprocal systems) nonoscillatiory at ∞. Obviously, if $H = 0 = N$, $M = I_n$, $K = -I_n$, the duality in oscillation behaviour of mutually reciprocal systems (1.1) and (1.2) follows from this result.

Ahlbrandt derived in [1] conditions under which a principal (antiprincipal) solution (Y, Z) of (1.1) at ∞ is also coprincipal (anticoprincipal) at ∞, i. e., the solution $(U, V) = (Z, -Y)$ is a principal (antiprincipal) solution of (2.1) at ∞. Here we generalize this result giving conditions which quarantee that a principal (antiprincipal) solution of (1.1) is transformed by (1.3) into a principal (antiprincipal) solution of the new Hamiltonian system.

2. Definitions and preliminary results.

Simultaneously with the matrix system (1.1) consider its vector modification

$$y' = A(t)y + B(t)z, \quad z' = -C(t)y - A^T(t)z, \tag{2.1}$$

where y, z are n-dimensional vectors. Throughout the paper we shall suppose that all differential systems are *identically normal* on I (a linear Hamiltonian system of the form (2.1) is said to be identically normal on I whenever the trivial solution $(y, z) \equiv (0, 0)$ is the only solution for which $y(t) \equiv 0$ on a nondegenerate subinterval of I).

Oscillation and nonoscillation of (2.1) are defined by means of the concept of *conjugate points*. Two points t_1, t_2 are said to be *conjugate* relative to (2.1) if there exists a solution (y, z) of (2.1) such that $y(t_1) = 0 = y(t_2)$ and $y(t)$ is not identically zero between t_1 and t_2. System (2.1) is said to be *conjugate* on an interval I whenever there exist t_1, $t_2 \in I$ which are conjugate relative to (2.1), in the opposite case (2.1) is said to be *disconjugate* on I. If there exists $c \in I$ such that (2.1) is disconjugate on (c, ∞) then (2.1) is said to be *nonoscillatory* at ∞, in the opposite case (2.1) is said to *oscillatory* at ∞. In the sequel the concepts oscillatory and nonoscillatory mean always oscillatory or nonoscillatory at ∞.

A solution (Y, Z) of (1.1) is said to be *self-conjugate* (another terminology is *prepared* [7], *self-conjoined* [9], *isotropic* [4]) if $Y^T(t)Z(t) \equiv Z^T(t)Y(t)$. Two solutions (Y_1, Z_1), (Y_2, Z_2) are said to be *linearly independent* if any solution (Y, Z) of (2.1) can be expressed in the form $(Y, Z) = (Y_1C_1 + Y_2C_2, Z_1C_1 + Z_2C_2)$, where C_1, C_2 are constant $n \times n$ matrices. If $(Y_1, Z_1), (Y_2, Z_2)$ are self-conjugate then they are linearly independent if and only if the (constant) matrix $Y_1^T(t)Z_2(t) - Z_1^T(t)Y_2(t)$ is nonsingular. A self-conjugate solution (Y_0, Z_0) is said to be *principal* at ∞ if $Y_0(t)$ is nonsingular for large t and for any solution (Y, Z), linearly idependent of

(Y_0, Z_0), with Y nonsingular for large t, we have $\lim_{t\to\infty} Y^{-1}(t)Y_0(t) = 0$. Any solution linearly idependent of (Y_0, Z_0) is said to be *antiprincipal* at b (another terminology is *nonprincial*, see [7]). Equivalently, the solutions (Y_0, Z_0), (Y_1, Z_1) are principal resp. antiprincipal at ∞ if Y_0, Y_1 are nonsingular for large t and

$$\lim_{t\to\infty}\left(\int^t Y_0^{-1}(s)B(s)Y_0^{T-1}(s)ds\right)^{-1} = 0$$

resp.

$$\lim_{t\to\infty}\left(\int^t Y_1^{-1}(s)B(s)Y_1^{T-1}(s)ds\right)^{-1} = L,$$

L being a nonsingular $n \times n$ matrix.

Recall that a principal resp. nonprincipal solution of (1.1) at ∞ exist whenever (2.1) is nonoscillatory at ∞ and the principal solution is determined uniquely up to a right multiple by a constant nonsingular $n \times n$ matrix.

Lemma 1. *Let (Y, Z) be a self-conjugate solution of (1.1) such that $Y(t)$ is nonsingular on $I_0 \subseteq I$. Then*

$$\tilde{Y}(t) = Y(t)\int_c^t Y^{-1}(s)B(s)Y^{T-1}(s)ds, \quad c \in I_0$$
$$\tilde{Z}(t) = Z(t)\int_c^t Y^{-1}(s)B(s)Y^{T-1}(s)ds + Y^{T-1}(t)$$

is also a self-conjugate solution of (1.1) which is linearly independent of (Y, Z). If (Y, Z) is antiprincipal at ∞ then

$$Y_0(t) = Y(t)\int_t^\infty Y^{-1}(s)B(s)Y^{T-1}(s)ds$$
$$Z_0(t) = Z(t)\int_t^\infty Y^{-1}(s)B(s)Y^{T-1}(s)ds - Y^{T-1}(t)$$

is the principal solution at ∞.

Proof. [4, Chap. II]

Let (Y, Z) be a solution of (1.1) such that Y is nonsingular on I then $W = ZY^{-1}$ is a solution of the Riccati equation

$$W' + WB(t)W + WA(t) + A^T(t)W + C(t) = 0. \tag{2.2}$$

If (Y, Z) is principal at ∞ then W is said to be distinquished solution of (2.2) at ∞, this solution is determined uniquely. If $\tilde{W}$ is another solution of (2.2) which exists on the whole interval $[c, \infty)$, $c \geq a$, then $\tilde{W}(t) \geq W(t)$ on $[c, \infty)$ (this inequality means that the matrix $\tilde{W}(t) - W(t)$ is nonnegative definite).

Lemma 2. *Let W_0 and $\tilde{W}$ be distinquished solutions at ∞ of (2.2) and*

$$W' + WB(t)W + WA(t) + A^T(t)W + \tilde{C}(t) = 0, \tag{2.3}$$

respectively. If $\tilde{C}(t) \geq C(t)$ on I then $\tilde{W}(t) \geq W_0(t)$ on I.

Proof. [4, Chap. II]

Now recall some results concerning transformations of linear Hamiltonian systems. Let $R(t)$ be a $2n \times 2n$ J-unitary matrix of the form (1.4), then substituting into (1.5) and the equivalent relation $RJR^T = J$ we get

$$\begin{aligned} &H^TK = K^TH, \quad M^TN = N^TM, \quad H^TN - K^TM = I_n, \\ &HM^T = MH^T, \quad KN^T = N^KT, \quad HN^T - MK^T = I_n. \end{aligned} \tag{2.4}$$

The transformation (1.3) transforms (1.1) into the system

$$U' = \bar{A}(t)U + \bar{B}(t)V, \quad V' = -\bar{C}(t)U - \bar{A}^T(t)V \tag{2.5}$$

and the matrices $\bar{A}, \bar{B}, \bar{C}$ are related to A, B, C by the equalities

$$\begin{aligned} A &= N^T(-H' + \bar{A}H + \bar{B}K) + M^T(K' + \bar{C}H + \bar{A}^TK), \\ B &= N^T(-M' + \bar{A}M + \bar{B}N) + M^T(N' + \bar{C}M + \bar{A}^TN), \\ C &= H^T(K' + \bar{C}H + \bar{A}^TK) + K^T(-H' + \bar{A}H + \bar{B}K), \end{aligned} \tag{2.6}$$

see, e.g., [3].

The main results of [6] are summarized in the next theorem.

Theorem A. *Suppose that the matrix $R(t)$ given by (1.4) is J-unitary, the matrix M is nonsingular on I and the matrices $B(t)$, $\bar{B}(t)$ are nonnegative definite on I. Then system (1.1) is nonoscillatory if and only (2.5) is nonoscillatory.*

Finally, for the later comparison, recall the results of [1] which were the main motivation for our investigation.

Theorem B. *Let $D(t)$ be the fundamental matrix of the equation $D' = A(t)D$. Suppose that the matrices $B(t)$, $C(t)$ are nonnegative definite in I, both systems (1.1) and (1.2) are identically normal on this interval and*

$$\lim_{t\to\infty} \left[\int^t D^{-1}(s)B(s)D^{T-1}(s)ds\right]^{-1} = 0.$$

If (Y_0, Z_0) is the principal solution of (1.1) at ∞ then $(U_0, V_0) = (Z_0, -Y_0)$ is the principal solution of (1.2) at ∞. Moreover, a solution (Y_1, Z_1) of (1.1) is antiprincipal at ∞ if and only if $(U_1, V_1) = (Z_1, -Y_1)$ is an antiprincipal solution of (1.2) at ∞.

3. Main results.

Our main results are based on the following lemma which generalizes a similar result of [1].

Lemma 3. *Let (Y, Z) and (U, V) be self-conjugate solutions of (1.1) and (1.2), respectively, related by (1.3), such that Y and U are nonsingular. If $M(t)$ is nonsingular and (2.4) holds (i.e., $R(t)$ given by (1.4) is J-unitary in I), then*

$$[(Y^T M^{-1} U)^{-1}]' = -Y^{-1} B Y^{T-1} + U^{-1} \bar{B} U^{T-1}$$

Proof. We have

$$\begin{aligned}
&[(Y^T M^{-1} U)^{-1}]' = \\
&-(Y^T M^{-1} U)^{-1}[Y^{T\prime} M^{-1} U - Y^T M^{-1} M' M^{-1} U + Y^T M^{-1} U'](Y^T M^{-1} U)^{-1} = \\
&-(Y^T M^{-1} U)^{-1}[Y^T A^T + Z^T B) M^{-1}(HY + MZ) - Y^T M^{-1} M' M^{-1}(HY + \\
&BZ) + Y^T M^{-1}(\bar{A} U + \bar{B} V)](Y^T M^{-1} U)^{-1} = -(Y^T M^{-1} U)^{-1}[Y^T A^T M^{-1} HY + \\
&Y^T A^T Z + Z^T B M^{-1} HY + Z^T BZ - Y^T M^{-1} M' M^{-1} HY - Y^T M^{-1} M' M^{-1} BZ + \\
&Y^T M^{-1} \bar{A}(HY + MZ) + Y^T M^{-1} \bar{B}(KY + NZ) - U^T M^{T-1} B M^{-1} U + \\
&U^T M^{T-1} B M^{-1} U](Y^T M^{-1} U)^{-1} = -(Y^T M^{-1} U)^{-1}[Y^T(A^T M^{-1} H - \\
&M^{-1} M' M^{-1} H + M^{-1} \bar{A} H + M^{-1} \bar{B} N H^T M^{T-1} - H^T M^{T-1} B M^{-1} H) Y + \\
&Y^T(A^T - M^{-1} M' + M^{-1} \bar{A} M + M^{-1} \bar{B} N - H^T M^{T-1} B) Z + Z^T(-B M^{-1} H + \\
&B M^{-1} H) Y + Z^T(B - B) Z](Y^T M^{-1} U)^{-1} - (Y^T M^{-1} U)^{-1}[U^T M^{T-1} B M^{-1} U - \\
&Y^T M^{-1} B M^{T-1} Y](Y^T M^{-1} U)^{-1} = -U^{-1}(M A^T - M' + \bar{A} M + \bar{B} N - \\
&HB) H^T U^{T-1} - (M A^T - M' + \bar{A} M + \bar{B} N - HB) Z Y^{-1} M^T U^{T-1} - \\
&Y^{-1} B Y^{T-1} + U^{-1} \bar{B} U^{T-1},
\end{aligned}$$

where the relations (2.4) and the symmetry of the matrix $Y^T M^{-1} U = Y^T(M^{-1} H + ZY^{-1}) Y$ has been used. Computing the expression $M A^T - M' + \bar{A} M + \bar{B} N - HB$, using (2.5) and (2.6), we get

$$\begin{aligned}
&M A^T - M' + \bar{A} M + \bar{B} N - HB = M(-H^{T\prime} + H^T \bar{A}^T + K^T \bar{B}) N + M(K^{T\prime} + \\
&H^T \bar{C} + K^T \bar{A}) M + \bar{A} M + \bar{B} N - M' - H N^T(-M' + \bar{A} M + \bar{B} N) - H M^T(N' + \\
&\bar{C} M + \bar{A}^T N) = M(-H^{T\prime} + H^T \bar{A}^T + K^T \bar{B}) N + M(K^{T\prime} + H^T \bar{C} + \\
&K^T \bar{A}) M + \bar{A} M + \bar{B} N - M' + M' - M K^T M' + M H^T N' - M K^T \bar{A} M - \\
&M K^T \bar{B} N - H M^T \bar{C} M - H M^T \bar{A}^T N - \bar{A} M - \bar{B} N = M(-H^{T\prime} + H^T \bar{A}^T + \\
&K^T \bar{B}) N + M(K^{T\prime} + H^T \bar{C} + K^T \bar{A}) M + M(-K^{T\prime} N + H^{T\prime} N) - M K^T \bar{A} M - \\
&M K^T \bar{B} N - H M^T \bar{C} M - H M^T \bar{A}^T N = M(-H^{T\prime} + H^T \bar{A}^T + K^T \bar{B}) N + \\
&M(K^{T\prime} + H^T \bar{C} + K^T \bar{A}) M - M(-H^{T\prime} + H^T \bar{A}^T + K^T \bar{B}) N + \\
&M(K^{T\prime} + H^T \bar{C} + K^T \bar{A}) M = 0
\end{aligned}$$

which completes the proof.

Theorem 1. *Let D be the fundamental matrix of the equation*

$$D' = (-B(t)M^{-1}(t)H(t) + A(t))D. \tag{3.2}$$

Suppose that the matrices $B(t)$, $\bar{B}(t)$ are nonnegative definite,

$$\lim_{t\to\infty}\left[\int^t D^{-1}(s)B(s)D^{T-1}(s)\,ds\right]^{-1} = 0 \tag{3.3}$$

and both systems (1.1) and (1.2) are identically normal on I. If (Y,Z) is a principal solution of (1.1) at ∞, then (U,V) given by (1.3) is a principal solution of (2.5) at ∞.

Proof. By Lemma 3

$$\begin{aligned}\int_a^t Y^{-1}(s)B(s)Y^{T-1}(s)\,ds + (Y^T(s)M^{-1}(s)U(s))^{-1}|_a^t = \\ \int_a^t U^{-1}\bar{B}(s)U^{T-1}(s)\,ds\end{aligned} \tag{3.4}$$

If (Y,Z) is a principal solution, by definition

$$\lim_{t\to\infty}\left(\int_a^t Y^{-1}BY^{T-1}\,ds\right)^{-1} = 0,$$

hence all eigenvalues of the matrix $\int_a^t Y^{-1}BY^{T-1}ds$ tend to ∞ as $t\to\infty$. Consequently, to prove the theorem it suffices to show that the (symmetric) matrix

$$Y^{T-1}(t)M^{-1}(t)U(t)$$

is nonnegative definite for large t, i. e., the matrix $M^{-1}H + ZY^{-1} = M^{-1}H + W_0$, W_0 being the distinquished solution of (2.2) at ∞, has this property. Since the matrix $\bar{B}$ is nonnegative definite, by (2.6) $MCM^T \geq -HM^{T\prime} + HAM^T - HBN^T + MH^{T\prime} + MA^TN^T$, hence $C \geq -M^{-1}HM^{T\prime}M^{T-1} + M^{-1}HA - M^{-1}HBN^TM^{T-1} + H^{T\prime}M^{T-1} + A^TN^TM^{T-1} =: \tilde{C}$. Using the symmetry of the matrix $M^{-1}H$, one can directly verify that $\tilde{W} = -M^{-1}H$ is a solution of (2.3). Let D be a solution of (3.2) and $F = \tilde{W}D$. Then (F,D) is a solution of (1.1) with $\tilde{C}$ instead of C and (3.3) implies that this solution is principal at ∞. Consequently, $\tilde{W} = FD^{-1} = -M^{-1}H$ is the distinquished solution of (2.3) at ∞ and by Lemma 2 $W_0 \geq -M^{-1}H$, i. e., the matrix $Y^TM^{-1}U$ is nonnegative definite for large t and the proof is complete.

Lemma 4. *Let the assumptions of Theorem 1 hold. If (Y, Z) is an antiprincipal solution of (1.1) and U is given by (1.3), then*

$$\lim_{t\to\infty} U^{-1}(t)M(t)Y^{T-1}(t) = 0 \tag{3.5}$$

Proof. Let

$$Y_2(t) = Y(t)\int_t^\infty Y^{-1}(s)B(s)Y^{T-1}(s)\,ds,$$
$$Z_2(t) = Z(t)\int_t^\infty Y^{-1}(s)B(s)Y^{T-1}(s)\,ds - Y^{T-1}(t).$$

According to Lemma 1 (Y_2, Z_2) is the principal solution of (1.1) at ∞ and by Theorem 1 $(U_2, V_2) = (HY_2 + MZ_2, KY_2 + NZ_2)$ is the principal solution of (2.5). It follows $\lim_{t\to\infty} U^{-1}(t)U_2(t) = 0$. Substituting for U_2, we have

$$U^{-1}U_2 =$$
$$U^{-1}HY\int_t^\infty Y^{-1}BY^{T-1}\,ds - U^{-1}MY^{T-1} + U^{-1}MZ\int_t^\infty Y^{-1}BY^{T-1}\,ds,$$

hence

$$U^{-1}MY^{T-1} = -U^{-1}U_2 + U^{-1}(HY + MZ)\int_t^\infty Y^{-1}BY^{T-1}\,ds =$$
$$-\,U^{-1}U_2 + \int_t^\infty Y^{-1}BY^{T-1}\,ds,$$

i. e., (3.5) holds.

Theorem 2. *Suppose that the assumptions of Theorem 1 hold. A solution (Y, Z) of (1.1) is antiprincipal at ∞ if and only if the solution (U, V) of (2.5) given by (1.3) is antiprincipal at ∞.*

Proof. The statement follows immediately from (3.4), the previous lemma and the definition of the antiprincipal solution.

4. Remarks. i) If $H(t) \equiv 0$ in Theorems 1,2, then the statements of these theorems comply with the results of Ahlbrandt [1] given in Theorem B.

ii) Consider the second order system

$$(R(t)Y')' + P(t)Y = 0, \tag{4.1}$$

where R, P are symmetric $n \times n$ matrices, R is positive definite and let Γ be a constant symmetric $n \times n$ matrix such that $P(t) + \Gamma R(t)\Gamma =: Q(t)$ is positive definite. Then the combination $U = RY' + \Gamma Y$ is a solution of the system

$$\begin{aligned}
&[Q^{-1}U' + Q^{-1}PR^{-1}\Gamma Q^{-1}U]' - Q^{-1}\Gamma R^{-1}PQ^{-1}U' +\\
&Q^{-1}[PR^{-1}P - \Gamma(R^{-1})'P - P(R^{-1})'\Gamma + \Gamma R^{-1}PR^{-1}\Gamma + PR^{-1}\Gamma(P' +\\
&\Gamma(R^{-1})'\Gamma Q^{-1} + Q^{-1}(P' + \Gamma(R^{-1})'\Gamma)PR^{-1}\Gamma -\\
&(\Gamma R^{-1}P - PR^{-1}\Gamma)Q(PR^{-1}\Gamma - \Gamma R^{-1}P)]Q^{-1}U = 0
\end{aligned} \tag{4.2}$$

which is nonoscillatory if and only if (4.1) is nonoscillatory, see [6]. If Y is the principal solution of (4.1) and Γ is such that $W_0 = -\Gamma$ is the distinquished solution of the Riccati equation $W' + WR^{-1}W - \Gamma R^{-1}\Gamma = 0$ then $U = RY' + \Gamma Y$ is the principal solution of (4.2).

iii) Let (Y, Z) be the principal solution of (1.1) at ∞, i. e., all eigenvalues of the matrix $\int^t Y^{-1}(s)B(s)Y^{T-1}(s)\,ds$ tend to ∞ as $t \to \infty$. In order to show that the matrix $\int^t U^{-1}(s)\bar{B}(s)U^{T-1}(s)\,ds$ also has this property (i. e., that (U, V) given by (1.3) is the principal solution of (2.5)), we used the idea suggested in [1], we proved that under the assumptions of Theorem 1 the matrix $U^{-1}(t)M(t)Y^{T-1}(t)$ is nonnegative definite for large t. To the same end it suffices to prove that the last matrix is "bounded below", i. e., $c^T U^{-1}(t)M(t)Y^{T-1}(t)c$ is bounded from bellow for every $c \in \mathbb{R}^n$. We hope to follow this more general (but also more difficult) idea elsewhere.

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