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COMPLEMENTED ORDERED SETS

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Dedicated to Professor M. Novotný on the occasion of his seventieth birthday

Abstract. We introduce the concept of complementary elements in ordered sets. If an ordered set S is a lattice, this concept coincides with that for lattices. The connections between distributivity and the uniqueness of complements are shown and it is also shown that modular complemented ordered sets represents "geometries" which are more general than projective planes.

It was shown in [2], [4] and [6] that some of lattice properties like distributivity or modularity can be generalized also for ordered sets. We can generalize also the concept of lattice complement. The aim of this paper is to show some basic properties of complements in ordered sets and to determine connections between complementarity and distributivity.

Let A be an ordered set, denote by $\leq$ its ordering. For $M \subseteq A$, put

$$U(M) = \{x \in A;\ m \leq x \ \text{ for each } m \in M\}$$
$$L(M) = \{y \in A;\ y \leq m \ \text{ for each } m \in M\}\ .$$

If M is a finite set, e.g. $M = \{a_1, \dots, a_n\}$, we write briefly $U(a_1, \dots, a_n)$ and $L(a_1, \dots, a_n)$ instead of $U(M)$ and $L(M)$, respectively, and for any $P, Q \subseteq A$ we write $L(P,Q) = L(P \cup Q)$ and $U(P,Q) = U(P \cup Q)$. If $P \subseteq M \subseteq A$, then clearly $U(P) \supseteq U(M)$ and $L(P) \supseteq L(M)$. Therefore, $U(\emptyset) = L(\emptyset) = A$. If A has the greatest element 1 (the least element 0), then $L(1) = A$, $U(A) = \{1\}$ $\ (U(0) = A,\ L(A) = \{0\}$, respectively). If A has not the greatest (the least) element, then $U(A) = \emptyset$ $\ (L(A) = \emptyset$ respectively).

Definition 1. Let S be an ordered set and $A \subseteq S,\ B \subseteq S$. We say that A, B are complementary if

$$L(U(A,B)) = S \quad \text{and} \quad U(L(A,B)) = S\ .$$

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Elements a, b of S are complementary (or a is a complement of b) if the sets $\{a\}, \{b\}$ are complementary. An ordered set S is complemented if each $a \in S$ has at least one complement in S.

The following assertions are evident:

(1) If S is an ordered set, then the subsets $\emptyset$ and S are complementary;
(2) If $A, B \subseteq S$ are complementary and $B \subseteq C \subseteq S$, then also A, C are complementary;
(3) If S has the greatest element 1 and the least element 0, then a, b of S are complementary if and only if $U(a, b) = \{1\}$ and $L(a, b) = \{0\}$, i.e. there exist $\sup(a, b) = 1$ and $\inf(a, b) = 0$.

Remark 1. If an ordered set S is a lattice with 0 and 1, then $a, b \in S$ are complementary by Definition 1 if and only if a, b are complementary in the lattice sence.

Remark 2. If S is at least two element antichain, then any two different elements of S are complementary.

By Remark 2, there exist complemented ordered sets which are not lattices.

Example 1. An ordered set S in Fig. 1 is not a lattice but it is complemented (it is immedietely clear that S is not a lattice and elements 0 and 1 are complementary; moreover, y is a complement of a and d; v is a complement of a; x is a complement of c).

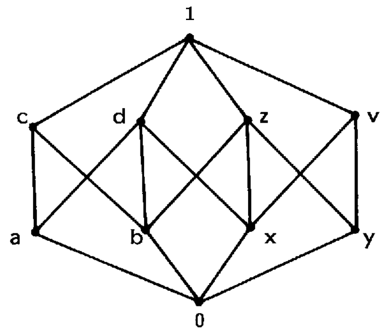

Fig. 1

Example 2. An ordered set S in Fig 2 is a complemented set without the greatest and the least element (therefore, it is not a lattice). The element a is a complement of c; b is a complement of d; x is a complement of r, q, y is a complement of r, q; z is a complement of p.

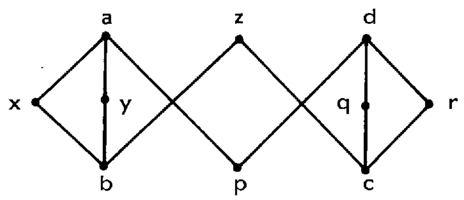

Fig. 2

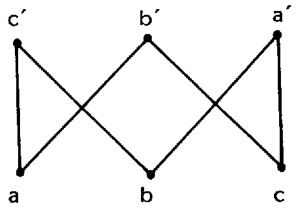

Fig. 3

Proposition 1. *Let S be a complemented ordered set. S has the greatest element 1 if and only if it has the least element 0.*

Proof. Suppose e.g. $0 \in S$. If x is a complement of 0, then $0 \leq x$ implies $S = L(U(0,x)) = L(U(x)) = L(x)$, hence x is the greatest element of S. Dually for the case $1 \in S$. $\square$

It follows directly from Remark 2 that in the two element antichain, every element has the unique complement. If S is an ordered set in which each $a \in S$ has the unique complement, S is called uniquely complemented; in this case, denote by a' the complement of $a \in S$ Evidently, $(a')' = a$ for each $a \in S$.

Example 3. Ordered sets in Fig. 3 and Fig. 4 are uniquely complemented (and are not lattices)

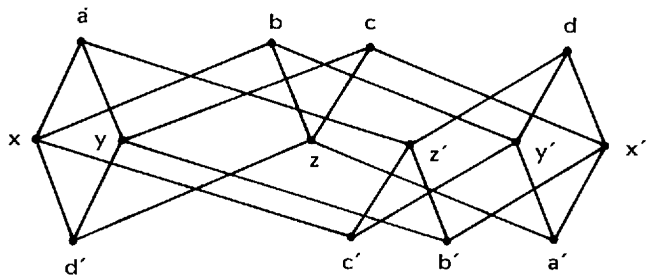

Fig. 4

It is well-known (see e.g. [1], [3], [7]) that there is a close connection between distributivity and unique complementness in lattices. We proceed to show that similar relationship is valid also for ordered sets. At first we recall the definition of distributive and modular ordered set from [2], [4]:

Definition 2. An ordered set S is distributive if

$$L(U(a,b),c) = L(U(L(a,c),L(b,c)))$$

holds for each elements a, b, c of S. An ordered set S is modular if for each a, b, c of S, $a \leq c$ implies

$$L(U(a,b),c) = L(U(a,L(b,c))) \quad .$$

Clearly a lattice L is distributive (modular) in the lattice sense if and only if it is distributive (modular) as an ordered set by Definition 2.

Sets in Fig. 3 and Fig. 4 are distributive (it can be verified by the methods of [2]). We can prove the following:

Theorem 1. *Let S be a distributive ordered set. Then each $a \in S$ has at most one complement in S.*

Proof. Let $a \in S$ and suppose b, c are complements of a. The distributivity of S implies:

$$
\begin{aligned}
L(b) =& S \cap L(b) = L(U(a,c)) \cap L(b) = L(U(a,c),b) = \\
=& L(U(L(a,b),L(c,b))) = L(U(L(a,b)) \cap U(L(c,b))) = \\
=& L(S \cap U(L(c,b))) = L(U(L(c,b))) = L(c,b)\ , \\
L(c) =& S \cap L(c) = L(U(a,b)) \cap L(c) = L(U(a,b),c) = \\
=& L(U(L(a,c),L(b,c))) = L(U(L(a,c)) \cap U(L(b,c))) = \\
=& L(S \cap U(L(c,b))) = L(U(L(c,b))) \doteq L(c,b)
\end{aligned}
$$

$$\text{whence} \qquad L(b) = L(c) \quad \text{which implies} \quad b = c\ .$$

Notation. Let S be a uniquely complemented ordered set and $A \subseteq S, x, y \in S$. Denote by $A' = \{a';\ a \in A\}$ and $(U(x,y))' = U(x,y)', (L(x,y))' = L(x,y)'$.

Proposition 2. *Let S be a uniquely complemented set and $\emptyset \neq A \subseteq S$. Then A and A' are complementary.*

Proof. Let $a \in A$. Clearly $\{a'\} \subseteq A'$ thus

$$S = L(U(a,a')) \subseteq L(U(A,A')) \subseteq S \quad .$$

Dually we can prove $U(L(A,A')) = S$.

Definition 3. Let S be a uniquely complemented set. We say that S satisfies De Morgan Laws if $U(x,y)' = L(x',y')$ and $L(x,y)' = U(x',y')$ for each x, y of S.

Theorem 2. *Let S be a uniquely complemented ordered set. The following conditions are equivalent:*

(a) *$x \leq y$ implies $y' \leq x'$;*
(b) *S satisfies De Morgan laws.*

Proof. (a) $\Rightarrow$ (b): Let $q \in U(x,y)'$. Then $q = a'$ for some $a \in U(x,y)$, i.e. $a \geq x$, $a \geq y$. By (a), we have $a' \leq x', a' \leq y'$ which is equivalent to $q = a' \in L(x',y')$. Clearly $U(x,y)' \subseteq L(x',y')$. The converse inclusion can be proved analogously and the second De Morgan law can be proved dually.

(b) $\Rightarrow$ (a): Let $x, y \in S$ and $x \leq y$. Then $y \in U(x,y)$ and, by (b), $y \in L(x',y')'$, thus $y' = in\ L(x',y')$ with respect to the unique complementarity in S. Hence, $y' \leq x'$.

Definition 4. An ordered set S is called boolean if it is distributive and complemented.

Evidently, ordered sets in Fig. 3 and Fig. 4 are boolean sets wich are not lattices.

Theorem 3. *Let S be a boolean set. Then*

$$a \le b \quad \textit{implies} \quad b' \le a'$$

for each a, b, of S, i.e. S satisfies De Morgan laws.

Proof. Let $a, b \in S$. Clearly $U(a, a', b') \subseteq U(a, a')$, which implies

$$S \supseteq L(U(a, a', b')) \supseteq L(U(a, a')) = S\ .$$

Hence,

$$\text{(i)} \qquad L(U(a, U(a', b'))) = L(U(a, a', b')) = S\ .$$

Suppose now $a \le b$. Then $L(a, b') \subseteq L(b, b')$ whence

$$U(L(a, b')) = U(L(b, b')) = S\ .$$

By distributivity of S, we obtain

$$\begin{aligned} & U(L(a, U(a', b'))) = U(L(U(L(a, a'), L(a, b')))) = U(L(a, b')) = \\ \text{(ii)} \qquad & = U(L(S)) = S\ . \end{aligned}$$

Therefore, (i) and (ii) yield that $\{a\}$ and $U(a', b')$ are complementary. Since a has the unique complement a', it gives $\{a'\} \subseteq U(a', b')$, whence $b' \le a'$ is evident.

Corollary 1. *Let S be a boolean set and $a, b \in S$. Then*

$$\begin{aligned} L(a, b) = \emptyset \quad & \textit{if and only if}\ U(a', b') = \emptyset \quad \textit{and} \\ U(a, b) = \emptyset \quad & \textit{if and only if}\ L(a', b') = \emptyset\ . \end{aligned}$$

Proof. Suppose e.g. $L(a, b) \ne \emptyset$, thus $x \in L(a, b)$. Then $x \le a, x \le b$ and, by Theorem 3, $a' \le x', b' \le x'$ i.e. $x' \in U(a', b')$.

Theorem 4. *Let S be a boolean set and $a, b, \in S$. If $U(a, b) \ne \emptyset$, then $U(a, b)$ and $L(a', b')$ are complementary. If $L(a, b) \ne \emptyset$, then $L(a, b)$ and $U(a', b')$ are complementary.*

Proof. Similarly as in the proof of Theorem 3, we obtain $L(U(a, a', b')) = S$ and $L(U(b, a', b')) = S$. Suppose e.g. $L(a, b) \ne \emptyset$. By Corollary 1, also $U(a', b') \ne \emptyset$ and the distributivity yields:

$$\begin{aligned} & L(U(L(a, b), U(a', b'))) = L(U(L(U(a, U(a', b')), U(b, U(a', b'))))) = \\ & \quad L(U(L(U(a, a', b'), U(b, a', b')))) = L(U(S)) = S\ . \end{aligned}$$

Dually we can prove $U(L(U(a',b'),L(a,b))) = S$ thus $L(a,b), U(a',b')$ are complementary. Similarly it can be shown for the sets $U(a,b), L(a',b')$.

Remark 3. The condition $U(a,b) \neq \emptyset$ (or $L(a,b) \neq \emptyset$) in Theorem 4 is essential. If e.g. $U(a,b) = \emptyset$, by Corollary 1 also $L(a',b') = \emptyset$, however, these subsets are not complementary.

Remark 4. Boolean sets in Fig. 3 and Fig. 4 are clearly partial lattices (if we add 0 and 1, we obtain boolean lattices). The following examples show that there exist also boolean sets which are not partial lattices (in this sense):

Example 4.

(a) An ordered set S in Fig. 5 is a boolean set without the least and the greatest element which is not a partial lattice (e.g. for $a, d \in S$, we have $L(a,d) \neq \emptyset$ but there does not exists $\inf(a,d)$);

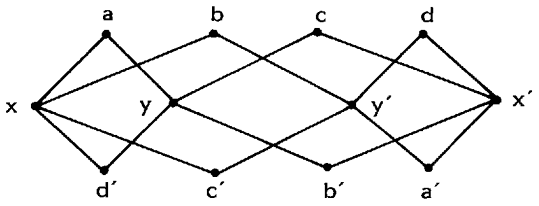

Fig. 5

(b) An ordered set S in Fig. 6, the so called pearl, is a boolean set with the least and the greatest element which is not a (partial) lattice (the same reason as in (a)).

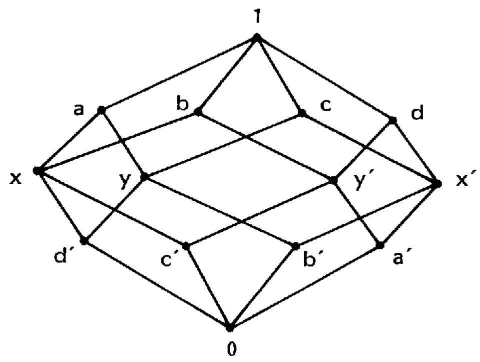

Fig. 6

Other boolean sets which are not partial lattices are visualized in Fig. 7 and Fig. 8.

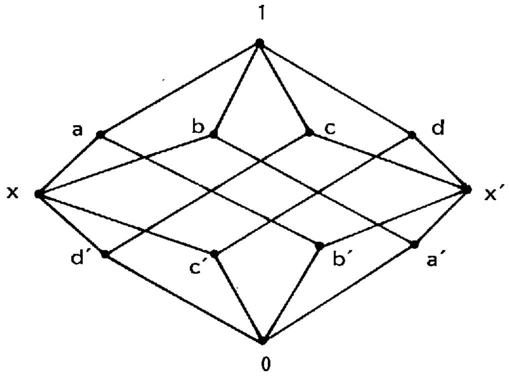

Fig. 7

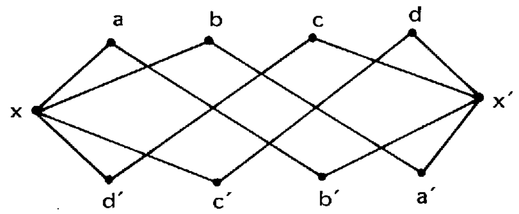

Fig. 8

Definition 5. Let A be an ordered set. An element p of A is called an atom of A whenever:

(a) If A has the least element 0 and $0 < q \le p$ for some $q \in A$, then $p = q$;

(b) If A has not the least element, then p is a minimal element of A.

An ordered set A is atomic if for each $a \in A$ there exists an atom $p \in A$ such that $p \le a$.

Proposition 3. *Let S be a uniquely complemented set and p, q be different atoms of S. Then $p \le q'$.*

Proof. Let $p \nleq q'$. Then $L(p, q') \subseteq L(p)$ and $L(p, q') \neq L(p)$, thus either $L(p, q') = \{0\}$ if S has the least element 0 or $L(p, q') = \emptyset$ in the opposite case, i.e. $U(L(p, q')) = S$. Suppose $x \in U(p, q')$.

(a) Suppose $q \nleq x$. Then again either $L(x, q) = 0$ or $L(x, q) = \emptyset$ if $0 \in S$ or $0 \notin S$, respectively, thus $U(L(x, q)) = S$ in both of these cases. Further, $U(x, q) \subseteq U(q', q)$ (because $x \in U(p, q')$ implies $q' \le x$), thus

$$S \supseteq L(U(x, q)) \supseteq L(U(q', q)) = S\ .$$

Thus we proved that x is a complement of q, i.e. $x = q'$. By the assumption, $x \in U(p, q')$ which gives $p \leq q'$, which is a contradiction.

(b) Suppose $q \leq x$. Since $x \in U(p, q')$, also $q' \leq x$. Thus $U(x) \subseteq U(q, q')$, whence

$$L(x) = L(U(x)) \supseteq L(U(q, q')) = S \ ,$$

i.e. $x = 1 \in S$. It gives $U(p, q') = \{1\}$, thus p is a complement of q'. With respect to the uniqueness of complementation in $S, p = q$ which is a contradiction again.

Definition 6. We say that an ordered set$S$ satisfies the condition (P) if for each elements a and x of S the following assertion is true:

(P) if A is a set of all atoms which are contained in a and $p \leq x$ for each $p \in A$, then $a \leq x$

Remark 5. All sets in Fig. 3, 4, 5 , 6, 7, 8 are uniquely complemented atomic ordered sets (which are not lattices)satisfying the condition (P). An example of as uniquely complemented atomic ordered set which does not satisfy (P) is visualized in Fig. 9.

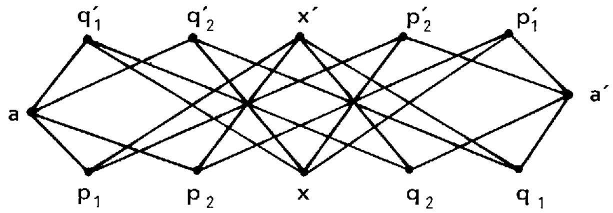

Fig. 9

Theorem 5. *Let S be a uniquely complemented atomic ordered set satisfying the condition (P). Then there exists a mapping h of S into a boolean lattice B such that:*

(a) *h is injective and isotone;*
(b) *h preserves complementation;*
(c) *if there exists $a \vee b$ for some a, b of S, then*

$$h(a \vee b) = h(a) \vee h(b) \ ,$$

if there exists $a \wedge b$ for some $a, b,$ of S, then

$$h(a \wedge b) = h(a) \wedge h(b) \ ;$$

(d) $h(c) \geq h(a) \vee h(b)$ *for each* $c \in U(a, b)$ *and*
$h(d) \leq h(a) \wedge h(b)$ *for each* $d \in L(a, b)$.

Proof. Denote by P the set of all atoms in S. Let $B = 2^P$, i.e. B is a boolean lattice with respect to set inclusion (i.e. $\cap$ is the meet, $\cup$ is the join and $P - A$ is the complement of A in B). Let h be a mapping of S into B given by

$$h(a) = \{p;\ p \in P,\ p \leq a\} \ .$$

It is immediately evident that h is isotone. Prove (b): Let $a \in S$ and $A = h(a)$. If $h(a') \cap A \neq \emptyset$, then $y \in h(a') \cap h(a)$ for some $y \in P$, i.e. $y \leq a'$, $y \leq a$ and $y \neq 0$ (if $0 \in S$) which is a contradiction with $U(L(a', a)) = S$ Thus $h(a') \cap A = \emptyset$, i.e. $h(a') \subseteq P - A$. Suppose $h(a') \neq P - A$. Then there exists an element $x \in (P - A) - h(a')$, i.e. x is an atom. Then $x \neq 0$ and, by Theorem 3, $x' \neq 1$. By Proposition 3, $x' \geq p$ for each $p \in A$, thus by (P), also $x' \geq a$. Since $x \notin h(a')$, we have $x' \geq q$ (by Proposition 3) for each $q \in h(a')$, thus also $x' \geq a'$ which is a contradiction with $L(U(a', a)) = S$. Thus we have

$$h(a') = P - A$$

proving (b).

Now, we are ready to prove that h is an injection. Suppose $a, b \in S$ and $h(a) = h(b) = A$. With respect to (b), we have $h(a') = P - A$, $h(b') = P - A$. Thus $L(a, b')$ does not contain any atom of S, i.e. $U(L(a, b')) = S$. With respect to Proposition 3, $U(a)$ contains complements of all atoms of $P - A$, $U(b')$ contains complements of all atoms of A. By Theorem 3, the complement of an atom is a dual atom (it is defined dually as the atom), thus $U(a, b') = U(a) \cap U(b')$ contains no dual atom. hence $L(U(a, b') = S)$, i.e. b' is a complement of a. With respect to the unique complementation in S, we have $b = a$.

Prove (c): Let $a \vee b$ exists in S. Then clearly $h(a \vee b) = h(a) \cup h(b)$, since $a \vee b$ must contain all atoms of $h(a)$ and all atoms of $h(b)$. Dually we obtain the second assertion provided $a \wedge b$ exists in S.

The assertion (d) is evident directly from (a).

Corollary 2. *Let S be a finite uniquely complemented ordered set satisfying (P). Then there exists an embedding h of S into a finite boolean lattice such that h is monotone, h preserver complements and suprema or infima provided they exist.*

It is well-known (see e.g. [1]) or [3]) that every modular complemented lattice represents a projective geometry; if it is of the lenght 3, it represents a projective plane. It is a natural question what "geometry" or what "plane" is represented by a complemented ordered set. Clearly, it can satisfy usual axioms of projective plane but two different lines can meet in more than one point and two different points can lie on more than one line. For example, the set S in Fig. 10 is a complemented ordered set and it represent a "projective quasiplane" with 3 points x, y, z and with 4 lines a, b, c, d (visualized in Fig. 11). If instead of a plane the sphere is considered and lines are substituted by circles in this sphere, such "geometry" is easy to imagine.

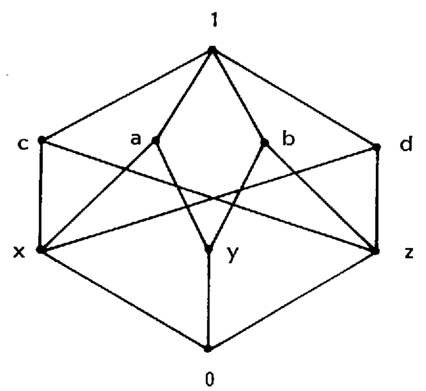

Fig. 10

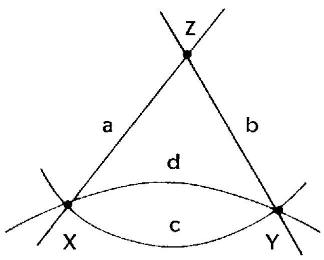

Fig. 11

Hence the theory of complemented sets is meaningfull and enables us to investigate geometrical systems (the so called incidence structures) which are more general than the projective geometries.

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