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A WINTGEN TYPE INEQUALITY AND CHARACTERIZATION OF HYPERBOLIC POINTS ON SURFACES SITTING IN THE EUCLIDEAN FOUR-SPACE

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Abstract. The aim of this paper is to explore the relations between some geometric invariants associated with surfaces immersed in the Euclidean four-space, by investigating the extrinsic and intrinsic geometries of our surfaces from a global point of view, as well as considering the curvature ellipses of a given surface and their associated isoptic curves. We establish an elegant formula which shows how the isoptic curves of the curvature ellipses of a given surface are related to the asymptotic directions on the surface, and derive a Wintgen type inequality which provides both a simple relationship between the main intrinsic and extrinsic invariants, and a natural and geometric characterization of the hyperbolic points, on the given surface. To indicate an application of our Wintgen type inequality to the theory of Möbius invariant Euclidean submanifolds, we conclude the paper with a novel geometric result on a remarkable family of surfaces known as Wintgen ideal surfaces.

1. Introduction

The quest for discovering nice and novel relations between intrinsic and extrinsic invariants associated with a submanifold is one of the most fundamental problems in submanifold theory; this is actually due to the classical fact that the celebrated embedding theorem of John F. Nash Jr. allows us to view each Riemannian manifold as a submanifold of a Euclidean space (for more details and a through treatment, we refer the reader to [6] and [8]). Moreover, understanding the smoothly embedded surfaces in a 4-manifold is intimately related to understanding the smooth structures on a 4-manifold; for instance, this was exhibited in [16] where it has been shown that the minimum genus of such a surface often depends on the particular smooth structure on the 4-manifold. One more interesting fact concerning the topic of our paper and the smooth structure of the ambient space of our surfaces is the existence of an uncountable family of non-diffeomorphic smooth structures on $\mathbb{R}^4$, as was shown by Clifford Henry Taubes in [25]; however, throughout this paper, we equip $\mathbb{R}^4$ with its standard smooth structure.

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The problem of describing the *lines of curvature* near an umbilic point, as a remarkable problem in submanifold theory, attracted the attention of Cayley, Frost and Hamilton – to name only some of the prominent researchers among others, cf. [3], [4] and [11]; in addition, Darboux [7] fully discussed the same problem in its generic case. Gullstrand, whose interest in the concept of umbilicity arose out of his work on eyesight aberration [12], seems to be the first to put due emphasis on the discriminant of the cubic form giving the principal directions at an umbilic point.

On any given surface M immersed in the flat space-form $\mathbb{R}^4$ there exist three basic local invariants associated with M: the *first* and the *second fundamental forms*, and the *normal connection*. These three invariants are related by the Gauss, Codazzi and Ricci equations, and they determine the isometric immersion of M into $\mathbb{R}^4$ uniquely up to isometries of $\mathbb{R}^4$.

To a given surface M in $\mathbb{R}^4$ and a globally defined normal field ν on M, there corresponds a shape operator S_ν on M intrinsically related to the second fundamental form II_ν along ν on M. The ν-lines of curvature of M are determined by the eigenvectors of S_ν, and the corresponding eigenvalues give the ν-principal curvatures. If the two ν-principal curvatures coincide at some point $p \in M$, then p is called a *ν-umbilic* point. The ν-umbilic points can, in fact, be characterized as the critical points of the corresponding principal direction fields. If all points on a surface are umbilic with respect to some globally defined normal vector field ν, then the surface is referred to as a ν-umbilic surface. A surface M is said to be "totally umbilic" if it is ν-umbilic with respect to all normal vector fields ν on M.

Moreover, associated with any point p on an embedded surface M in $\mathbb{R}^4$ we shall consider an elegant configuration in the normal plane N_pM, which is actually another local invariant associated with M, called the *curvature ellipse* of M at p (this was referred to as the "curvature conic" by Y.-C. Wong in [27]). The point $p \in M$, being thought of as the origin of the normal vector space N_pM, is said to be a *hyperbolic point* if it lies outside the curvature ellipse of M at p. In addition, Wong [27] considered the possibility of defining the so-called *asymptotic directions* over the points of the surfaces M in $\mathbb{R}^4$.

In the spirit of [27, 28], we establish in this paper some novel relations between the above-mentioned local invariants associated with a surface $M \subset \mathbb{R}^4$; in fact, employing the analytic features of asymptotic directions on M, we develop a theory exhibiting harmony of these local invariants: a *Wintgen type inequality* is derived which provides both a simple relationship between the main intrinsic and extrinsic invariants, and a natural and geometric characterization of the points, of the given surface M, that lie on the *isoptic curves* of the corresponding curvature ellipses (see Theorem 3.14) – accordingly, we shall refer to these points as the *isoptic points* of M (by Definition 3.13, to each of these points on M there corresponds a unique angle $\alpha \in (0, \pi)$ and therefore such a given point is called an "α-isoptic point"). In fact, the primary purpose of this paper is to prove the following theorem:

Theorem 3.14. *A point $p \in M$ is an α-isoptic point, for some angle $\alpha \in (0, \pi)$, if and only if the normal curvature N and the mean curvature vector H read*

$$|N(p)| \leq (-\Delta(p))^{\frac{1}{2}}\, E_\alpha + \|H(p)\|^2,$$

with $(-\Delta(p))^{\frac{1}{2}}\, E_\alpha$ as a new geometric quantity on the curvature ellipse (to be considered in Section 3 with a clear geometric interpretation as in Corollary 3.19) *where the equality holds when and only when either p is an umbilic point or the curvature ellipse of M at p degenerates to a circle.*

However, as is shown in the proof of Proposition A.1, *the set of all isoptic points and that of the hyperbolic points coincide.* Hence, Theorem 3.14 provides a geometric characterization of the *hyperbolic points* on M as well.

Our local constructions lead to global results on the existence of singularities of M as well. It is worth noting that the *semi-umbilic* points (i.e., the points at which the curvature ellipses degenerate to line segments) can be characterized as singularities of corank 2 for some special type of mapping given by the composition of an embedding and the standard Euclidean distance function taken from some focal centers of the surface (see [23] for an introduction to the geometric interpretation of the singularities of those special functions on submanifolds, and [19] for the especial case of surfaces in $\mathbb{R}^4$) – recent developments on the geometry of surfaces in Euclidean and Minkowski spaces from a singularity theory viewpoint can be found in [13]. The main difference between classical works on geometric invariants on surfaces and the present work is that this paper enables us, in a novel way, to systematically compare the geometries of the surface M and its ambient space $\mathbb{R}^4$, and provides us with an infinitesimal description of the shape of M in $\mathbb{R}^4$; in fact, throughout this paper when dealing with connections, we would employ the two standpoints of considering a connection *globally* as a differential operator, and *locally* as connection 1-forms appropriately.

This paper is organized as follows. Section 2 is devoted to the study of the lines of curvature as an analytic tool for considering the principal directions on a surface $M \subset \mathbb{R}^4$, and presents some preliminaries on the extrinsic and intrinsic invariants of M. In Section 3, we will be concerned with the geometric aspects of the above-mentioned invariants and the analytic features of asymptotic directions of the surfaces by looking more closely at the normal connection and extrinsic geometry of M, and investigating the curvature ellipse of M and its *isoptic curves*; in fact, in this section, referring to the reconciliation between extrinsic and intrinsic geometries of $M \subset \mathbb{R}^4$, we obtain – as our main result – a Wintgen type inequality given by Theorem 3.14. To indicate an application of our Wintgen type inequality to the theory of Möbius invariant Euclidean submanifolds, we conclude the paper with a novel geometric result (in Corollary 3.19) on a remarkable family of surfaces known as Wintgen ideal surfaces:

Corollary 3.19. *Let $M \subset \mathbb{R}^4$ be a Wintgen ideal surface with no umbilic points. A point $p \in M$ is an α-isoptic point, for some angle $\alpha \in (0, \pi)$, if and only if the*

Gaussian curvature G of M at p reads $G(p) = -\left(-\Delta(p)\right)^{\frac{1}{2}} E_\alpha$.
Accordingly, the Gaussian curvature $G(p)$ of M at an α-isoptic point p is negative (respectively, positive, or else zero) when and only when $\alpha \in (0, \frac{\pi}{2})$ (respectively, $\alpha \in (\frac{\pi}{2}, \pi)$, or else $\alpha = \frac{\pi}{2}$).

We have also included in this paper, for the convenience of the reader, an appendix in which we derive the equation of isoptic curves of a given ellipse (see Appendix A).

Perspectives. It is worth pointing out here that the stereographic projection is a construction that connect the field of study of asymptotic lines on surfaces in $\mathbb{R}^4$ with that of curvature lines on those in $\mathbb{R}^3$. Hence, any new result regarding the former field yields a generalization of similar issue concerning the latter. Moreover, according to Lemma 3.6 and Remark 3.7, one can expect to see some other nice and novel geometric features in the study of the invariants associated with surfaces or submanifolds with codimensions bigger than two sitting in some (flat or curved) space forms of higher dimensions. In fact, applications of the main result of this paper to the analysis of some *constrained* Newtonian three-body motions (both planar and non-planar), and extension of some of these formalisms to the case of non-Euclidean ambient spaces and investigation of the *curved* three-body motions, e.g., in the realm of hyperbolic geometry (also known as Lobachevsky-Bolyai-Gauss geometry) will be the subject of our future publications (cf. [9, 14, 15]).

2. The integral curves of principal direction fields on a surface $M \subset \mathbb{R}^4$

Let M be an orientable smooth surface isometrically immersed in $\overline{M} := \mathbb{R}^4$. Considering the standard Riemannian metric on the ambient space $\mathbb{R}^4$, we equip the surface M with the Riemannian metric induced by the isometric immersion. For any point $p \in M$, we would consider the following decomposition

$$T_p\mathbb{R}^4 = T_pM \oplus N_pM,$$

where the normal plane N_pM of M at p is the orthogonal complement of the tangent plane T_pM in $\mathbb{R}^4$.

One of the basic tools for the study of manifold and/or submanifold geometry is the Levi-Civita connection. The Levi-Civita connection can be defined both globally as a differential operator (by Koszul's definition, see for instance [21, Ch. 3: Th. 11]) and locally as connection 1-forms (using Cartan's formulation and the Maurer-Cartan equation, cf. [22, Th. 1.2.2]). While the global definition is better for interpreting the geometry, the local definition is easier to compute with; in fact, throughout this paper, we would employ both definitions appropriately.

Suppose that $\overline{\nabla}$ is the Riemannian connection (i.e., the Levi-Civita connection) of $\mathbb{R}^4$. For any local vector fields X, Y on M, a local extension of these fields to $\mathbb{R}^4$ will be respectively denoted by $\overline{X}, \overline{Y}$. Then it can be shown that the Riemannian connection ∇ of M is the tangent component of the Riemannian connection of $\mathbb{R}^4$; that is, $\nabla_X Y = \tan\left(\overline{\nabla}_{\overline{X}}\overline{Y}\right)$, where "tan" stands for the tangent component of vectors (cf. [21, Ch. 4: Lemma 3]).

We denote the space of all smooth vector fields tangent to the surface M by $\mathfrak{X}(M)$, and that of all smooth vector fields normal to M by $\mathfrak{X}^{\perp}(M)$. In addition, we will use the symbol $\mathfrak{F}(M)$ to denote the set of all smooth real-valued functions on M. The *second fundamental map II* is defined by

$$II : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}^{\perp}(M), \quad II(X,Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y,$$

which is an $\mathfrak{F}(M)$-bilinear and symmetric mapping. In fact, II is also referred to as either the *shape tensor* or the *second fundamental tensor* of $M \subset \mathbb{R}^4$ (while the *first fundamental tensor* is just the metric tensor of M). Thus at each point $p \in M$, II determines an $\mathbb{R}$-bilinear function

$$T_pM \times T_pM \to N_pM$$

sending an ordered pair of tangent vectors (x,y) to the normal vector $II(x,y)$.

For any normal vector field $\nu \in \mathfrak{X}^{\perp}(M) \setminus \{0\}$, define

$$H_\nu : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathbb{R}, \quad H_\nu\left(X,Y\right) = \left\langle II(X,Y), \nu \right\rangle.$$

The quadratic form of H_ν is called the *second fundamental form II_ν of M along ν*,

$$II_\nu : \mathfrak{X}(M) \to \mathbb{R}, \quad II_\nu(X) = H_\nu(X,X).$$

Accordingly, the *shape operator of $M \subset \mathbb{R}^4$ derived from ν* is given by the tensor field

$$S_\nu : \mathfrak{X}(M) \to \mathfrak{X}(M), \quad S_\nu(X) = -\tan\left(\overline{\nabla}_{\overline{X}}\overline{\nu}\right),$$

and for any vector fields $X, Y \in \mathfrak{X}(M)$ reads

$$\langle S_\nu(X), Y\rangle = H_\nu(X,Y).$$

As usual, S_ν determines a linear operator $S_\nu : T_pM \to T_pM$ at each point $p \in M$; evidently, this linear operator is self-adjoint.

So the second fundamental form along ν reads

$$II_\nu(X) = \left\langle S_\nu(X), X\right\rangle.$$

Hence, for any $p \in M$, there exists an orthonormal basis consisting of the eigenvectors of S_ν on T_pM with respect to which the restriction $II_\nu|_{S^1}$, to the unit vectors, of the second fundamental form attains its maximum and minimum values. We will denote the corresponding eigenvalues, respectively, by k_1 and k_2 and refer to them as the *ν- principal curvatures* of M. With the above notation, we call a point $p \in M$ a *ν-umbilic* point if $k_1 = k_2$. The surface M is said to be ν-umbilic if every point on M is ν-umbilic.

Now let $\mathcal{U}_\nu$ be the set of all ν-umbilic points on M. Then for any point $p \in M \setminus \mathcal{U}_\nu$, there exist two principal directions with respect to ν, given by the eigenvectors of S_ν, whose corresponding vector fields are smooth and integrable. To study the integral curves of these vector fields, we make the following definition.

Definition 2.1. For any point $p \in M \setminus \mathcal{U}_\nu$, integral curves of the vector fields defined by the ν-principal directions of M form two orthogonal families of curves which are called the *ν-principal lines of curvature.*

To be more precise, a *ν-line of curvature* is, by definition, a differentiable curve $\gamma : (-1,1) \to M$ which passes through $\gamma(0) = p$ on M with the velocity vector $\gamma'(0) = X(\gamma(0))$, where $X \in \mathfrak{X}(M)$ is the vector field given by a ν-principal direction on M.

The differential equation of ν-lines of curvature is given by

$$S_\nu (X(p)) = \lambda(p) X(p). \tag{2.1}$$

Now for a suitable open subset $U \subset \mathbb{R}^2$, assume that $\varphi : U \to \mathbb{R}^4$ is an immersion of $\varphi(U) = M$, with the local coordinates (u, v), into $\mathbb{R}^4$. Then the coefficients E, F, and G of the first fundamental form, and those of the second fundamental form along $\nu \in \mathfrak{X}^\perp(M)$, denoted by e_ν, f_ν, and g_ν, are defined in this coordinate system as

$$E = \langle \varphi_u, \varphi_u \rangle, \; F = \langle \varphi_u, \varphi_v \rangle, \; G = \langle \varphi_v, \varphi_v \rangle,$$

$$\begin{aligned} e_\nu &= -II_\nu(\partial_u) = -\langle II(\partial_u, \partial_u), \nu \rangle, \\ f_\nu &= -\langle II(\partial_u, \partial_v), \nu \rangle = -\langle II(\partial_v, \partial_u), \nu \rangle, \\ g_\nu &= -II_\nu(\partial_v) = -\langle II(\partial_v, \partial_v), \nu \rangle, \end{aligned}$$

where for instance $\partial_u = \partial/\partial u$ and $\varphi_u = d\varphi(\partial_u)$.

In fact, in the above-mentioned coordinate system, the differential equation (2.1) of the ν-lines of curvature can be expressed (as can be found in [24]) in the following form

$$(f_\nu E - e_\nu F) du^2 + (g_\nu E - e_\nu G) du dv + (g_\nu F - f_\nu G) dv^2 = 0, \tag{2.2}$$

and assuming our coordinate system to be an *isothermic* one (i.e., having $E = G > 0$, $F = 0$) we derive

$$f_\nu du^2 + (g_\nu - e_\nu) du dv - f_\nu dv^2 = 0. \tag{2.3}$$

Given any $p \in M$, (2.3) just amounts to the fact that the linear shape operator $S_\nu : T_pM \to T_pM$ can be defined in the matrix form by

$$S_\nu = \frac{1}{E} \begin{pmatrix} e_\nu & f_\nu \\ f_\nu & g_\nu \end{pmatrix}. \tag{2.4}$$

Accordingly, the surface M is ν-umbilic if and only if $(e_\nu - g_\nu) = 0 = f_\nu$.

3. The extrinsic and intrinsic invariants of $M \subset \mathbb{R}^4$

In fact, at the very beginning and actually prior to Gauss, every property was thought of as being extrinsic. Shortly after Gauss drew our attention to the difference between intrinsic and extrinsic properties, Riemann followed the track and redefined geometric objects in an intrinsic way. The consequence was that geometric objects could not necessarily be globally realized in any ambient space. The embedding theorems of the 1950's (see [8] for more details) suggested a link back in time, to realize a Riemannian manifold as a submanifold of some Euclidean space, and open up for giving extrinsic definitions of intrinsic properties.

Suppose that $M \subset \mathbb{R}^4$ is an embedded surface, and $\psi : M \to \mathbb{R}^4$ is an embedding. If we take an orthonormal frame $\{X_i\}_{i=1}^4$ on M, then the coframe $\{\omega_i\}_{i=1}^4$ dual to our frame is given by the 1-forms $\omega_i = \langle d\psi, X_i \rangle$, and the $\overline{\nabla}$-connection 1-forms $\{\omega_{ij}\}_{i,j=1}^4$ on $\mathbb{R}^4$ are defined by $\omega_{ij} = \langle \overline{\nabla} X_i, X_j \rangle$.

In what follows, we assume that $\{X_j\}_{j=1}^2 \subset \mathfrak{X}(M)$ gives a tangent frame and $\{X_k\}_{k=3}^4 \subset \mathfrak{X}^\perp(M)$ does a normal frame on M. So, it is clear that $\omega_3 = 0 = \omega_4$ and hence, by the Maurer-Cartan structure equation,

$$0 = d\omega_k = \omega_{k1} \wedge \omega_1 + \omega_{k2} \wedge \omega_2, \quad k = 3, 4.$$

On the other hand, since the 1-forms ω_1, ω_2 are linearly independent, applying Cartan's lemma twice, yields

$$\begin{cases} \omega_{13} = a\omega_1 + b\omega_2 \\ \omega_{23} = b\omega_1 + c\omega_2 \end{cases}, \quad \begin{cases} \omega_{14} = \hat{a}\omega_1 + \hat{b}\omega_2 \\ \omega_{24} = \hat{b}\omega_1 + \hat{c}\omega_2 \end{cases},$$

for some real-valued smooth functions $a, b, c, \hat{a}, \hat{b}, \hat{c} \in \mathfrak{F}(M)$. In fact, these functions are

$$a = e_{X_3},\ b = f_{X_3},\ c = g_{X_3}, \quad \hat{a} = e_{X_4},\ \hat{b} = f_{X_4},\ \hat{c} = g_{X_4},$$

since, for instance, we have

$$\begin{aligned} a = \omega_{13}(X_1) = \langle \overline{\nabla}_{X_1} X_3, X_1 \rangle &= -\left\langle -\tan\left(\overline{\nabla}_{X_1} X_3\right), X_1 \right\rangle \\ &= -\langle S_{X_3}(X_1), X_1 \rangle = -\langle II(X_1, X_1), X_3 \rangle = e_{X_3}. \end{aligned}$$

Hence, the connection 1-forms ω_{jk} (for $j = 1, 2$ and $k = 3, 4$) can be expressed in terms of the 1-forms ω_1, ω_2 as

$$\begin{cases} \omega_{1k} = e_{X_k}\omega_1 + f_{X_k}\omega_2, \\ \omega_{2k} = f_{X_k}\omega_1 + g_{X_k}\omega_2. \end{cases} \qquad (k = 3, 4) \tag{3.1}$$

We shall now see that the second fundamental map II may be written as

$$II = \sum_{k=3}^{4} \left(e_{X_k}\omega_1^2 + 2f_{X_k}\omega_1\omega_2 + g_{X_k}\omega_2^2 \right) X_k, \tag{3.2}$$

such that, for any vector fields $X, Y \in \mathfrak{X}(M)$, the (normal) component $\langle II(X,Y), X_k \rangle$ of $II(X,Y)$ equals

$$e_{X_k}\omega_1(X)\omega_1(Y) + f_{X_k}\left(\omega_1(X)\omega_2(Y) + \omega_2(X)\omega_1(Y)\right) + g_{X_k}\omega_2(X)\omega_2(Y).$$

To see this, given an embedding $\psi : M \to \mathbb{R}^4$, it suffices to show that the components of $II = \langle II(d\psi, d\psi), X_3 \rangle X_3 + \langle II(d\psi, d\psi), X_4 \rangle X_4$ can be expressed in the desired form as above.

We know that $\omega_k = \langle d\psi, X_k \rangle = 0$, as $\{X_k\}_{k=3}^4$ is a normal frame on M. It follows that

$$0 = d\omega_k = \langle \overline{\nabla} d\psi, X_k \rangle + \langle d\psi, \overline{\nabla} X_k \rangle, \quad k = 3, 4.$$

Therefore

$$\langle II(d\psi, d\psi), X_k \rangle = \langle \overline{\nabla} d\psi, X_k \rangle = -\langle d\psi, \overline{\nabla} X_k \rangle, \quad k = 3, 4. \tag{3.3}$$

On the other hand, by (3.1) and the fact that $\omega_{kj} = -\omega_{jk}$ (for $j = 1, 2$ and $k = 3, 4$), we have

$$\begin{aligned} -\left\langle d\psi, \overline{\nabla} X_k \right\rangle &= -\left\langle \left(\langle d\psi, X_1 \rangle X_1 + \langle d\psi, X_2 \rangle X_2 \right), \overline{\nabla} X_k \right\rangle \\ &= -\left(\omega_1 \left\langle X_1, \overline{\nabla} X_k \right\rangle + \omega_2 \left\langle X_2, \overline{\nabla} X_k \right\rangle \right) \\ &= \omega_1 \omega_{1k} + \omega_2 \omega_{2k} \\ &= \omega_1 \left(e_{X_k} \omega_1 + f_{X_k} \omega_2 \right) + \omega_2 \left(f_{X_k} \omega_1 + g_{X_k} \omega_2 \right). \end{aligned}$$

Thus

$$-\left\langle d\psi, \overline{\nabla} X_k \right\rangle = e_{X_k} \omega_1^2 + 2 f_{X_k} \omega_1 \omega_2 + g_{X_k} \omega_2^2, \quad k = 3, 4. \tag{3.4}$$

Combining (3.3) and (3.4) establishes the formula (3.2).

3.1. **The curvature ellipse.** We shall now consider a local invariant associated with an embedded surface M in $\mathbb{R}^4$ which will be referred to as the *curvature ellipse.*

The local invariants associated with surfaces embedded in $\mathbb{R}^4$ have been rather thoroughly studied in the classical works [10, 20, 27, 28]. In fact, it was found (in [20]) that the invariants are those of a simple configuration; namely, a point and an ellipse in the normal plane. To describe the configuration, we need the following definitions.

By (3.1), the connection 1-forms $\{\omega_{jk};\ j = 1, 2,\ k = 3, 4\}$ on a surface M in $\mathbb{R}^4$ read $\omega_{jk} = A_{kj1}\omega_1 + A_{kj2}\omega_2$, where $A_{k11} = e_{X_k}$, $A_{k22} = g_{X_k}$, and $A_{k12} = A_{k21} = f_{X_k}$. Put this way, the *mean curvature vector field* $H \in \mathfrak{X}^{\perp}(M)$ is defined as

$$H = \frac{1}{2} \sum_{k=3}^{4} \left(A_{k11} + A_{k22} \right) X_k = \frac{1}{2} \sum_{k=3}^{4} \left(e_{X_k} + g_{X_k} \right) X_k. \tag{3.5}$$

Let M be a surface embedded by ψ in $\mathbb{R}^4$ as above. Given $p \in M$, consider the unit circle in the tangent vector space T_pM, centered at the origin 0_{T_pM}, and being parametrized by $\theta \mapsto \cos(\theta)X_1 + \sin(\theta)X_2$ for the angle $\theta \in [0, 2\pi]$. Then the *normal section of $\psi(M)$ in the direction θ* is defined as the curve γ_θ obtained by intersecting $M \subset \mathbb{R}^4$ with the hyperplane at the point $p \in M$ composed by the direct sum of the normal plane N_pM and the straight line in the tangent direction $\mathrm{span}_{\mathbb{R}} \{\cos(\theta)X_1 + \sin(\theta)X_2\} \subset T_pM$ represented by θ.

We begin by analyzing the curvature vector of γ_θ at p, which is a vector in the normal plane N_pM.

Proposition 3.1. *Given any point $p \in M$, if we denote by $\eta(\theta)$ the curvature vector of the curve γ_θ at p (as above) then*

$$\eta(\theta) = H + \sum_{k=3}^{4} \left(\frac{(e_{X_k} - g_{X_k})}{2} \cos(2\theta) + f_{X_k} \sin(2\theta) \right) X_k,$$

where H is the mean curvature vector field, as in (3.5). In addition, varying θ on the interval $[0, 2\pi]$, the locus of vectors $\eta(\theta)$ is an ellipse in the normal vector space N_pM which will be referred to as the curvature ellipse *of M at the point p.*

Proof of Proposition 3.1. For any given point $p \in M$, parametrize the curve γ_θ of the normal section of $\psi(M)$ at p in the direction θ by the arc length s so that $\gamma_\theta(0) = p$. Then the curvature vector of γ_θ at $p \in M$ is, by definition, equals

$$\eta(\theta) = \frac{d^2\gamma_\theta}{ds^2}|_{s=0}.$$

Setting $\dot{\gamma}_\theta(0) := d\gamma_\theta/ds(0)$, we simply get $\eta(\theta) = \ddot{\gamma}_\theta(0)$. Since any arc-length parametrized curve (like γ_θ) is of constant (unit) speed, it is a simple matter to see that $\langle \eta(\theta), \dot{\gamma}_\theta(0)\rangle = 0$. By the definition of a normal section, as above, this amounts to saying that $\eta(\theta)$ has no component in the tangent vector space T_pM, and hence $\eta(\theta) = \ddot{\gamma}_\theta(0) \in N_pM$.

On the other hand, by [21, Ch. 4: Corollary 9], γ_θ as a curve in $M \subset \bar{M} = \mathbb{R}^4$ reads

$$\ddot{\gamma}_\theta = \gamma_\theta'' + II(\dot{\gamma}_\theta, \dot{\gamma}_\theta),$$

where $\ddot{\gamma}_\theta$ is the acceleration of γ_θ in $\bar{M}$, and γ_θ'' its acceleration in M. Thus $\gamma_\theta''(0) = 0$, and

$$\eta(\theta) = II\left(\dot{\gamma}_\theta(0), \dot{\gamma}_\theta(0)\right). \tag{3.6}$$

We may write $\gamma_\theta(0) = \cos(\theta)X_1 + \sin(\theta)X_2$, since γ_θ is an arc-length parametrized curve; on substituting it into (3.6) and applying the formula (3.2), we deduce that

$$\eta(\theta) = \sum_{k=3}^{4} \left(e_{X_k}\cos^2(\theta) + 2f_{X_k}\sin(\theta)\cos(\theta) + g_{X_k}\sin^2(\theta)\right) X_k.$$

Using (3.5) and the trigonometric identities for double angles, we derive the desired formula for the curvature vector $\eta(\theta)$ of the curve γ_θ at p.

Moreover, it should now be clear that $(\eta - H)$ is in fact a linear transformation in θ from the unit circle in the tangent vector space T_pM into the normal vector space N_pM, whose 2×2 matrix with respect to $\{X_1, X_2\}$ and $\{X_3, X_4\}$, respectively, as the tangent and normal frames on M is

$$\begin{bmatrix} \frac{(e_{X_3} - g_{X_3})}{2} & f_{X_3} \\ \frac{(e_{X_4} - g_{X_4})}{2} & f_{X_4} \end{bmatrix},$$

and hence η can be considered as an affine transformation in θ. Thus, due to the fact that the image of a circle under an affine transformation is an ellipse, we see that the normal curvature vector $\eta(\theta)$ moves on an ellipse in the normal plane N_pM about the mean curvature vector H. This finishes the proof (see also [17, Section 2]). $\square$

On account of the Jordan-Brouwer theorem, since an ellipse (in the non-degenerate case, i.e., when its area is not zero) is a simple closed curve in the plane, the points on M can be classified as follows. A point $p \in M$, being thought of as the origin 0_{N_pM} of the normal vector space N_pM, is said to be a *hyperbolic*, a *parabolic*, or else an *elliptic* point if it lies, respectively, outside, on, or inside the curvature ellipse of M at p.

Furthermore, $p \in M$ is called a *semi-umbilic* point if the curvature ellipse of M at p degenerates to a line segment in the normal plane N_pM.

One of the fundamental concepts in this paper is that of an *asymptotic direction* on a surface M embedded by an embedding ψ in $\mathbb{R}^4$.

Definition 3.2. We call a θ-direction in $T_{\psi(p)}\psi(M)$ an *asymptotic direction* on M at p if the position vector of the point $\eta(\theta)$ and the tangent vector $\frac{d\eta}{d\theta}$ at the same point $\eta(\theta)$ of the curvature ellipse of M at p are collinear.

Accordingly, at any hyperbolic or parabolic point on the surface M, the number of asymptotic directions on M, respectively, equals two or one. But there exists no asymptotic direction on M at any of its elliptic points.

3.2. **The normal connection and extrinsic geometry of the surface $M \subset \mathbb{R}^4$.** To systematically compare the geometries of M and its ambient space $\overline{M} = \mathbb{R}^4$, and to provide an infinitesimal description of the shape of M in $\mathbb{R}^4$, we would need the following definitions.

The *normal connection* of the surface $M \subset \overline{M} = \mathbb{R}^4$ is, by definition, the function $\nabla^\perp : \mathfrak{X}(M) \times \mathfrak{X}^\perp(M) \to \mathfrak{X}^\perp(M)$ given by

$$\nabla^\perp_V Z = \operatorname{nor}\left(\overline{\nabla}_V Z\right),$$

where the term "nor" stands for the normal component on M of vectors, and the mapping $\overline{\nabla} : \mathfrak{X}(M) \times \overline{\mathfrak{X}}(M) \to \overline{\mathfrak{X}}(M)$ is the induced connection on $M \subset \mathbb{R}^4$ such that

$$\overline{\mathfrak{X}}(M) = \left\{X : M \to T\overline{M}; \quad X(p) \in T_p\overline{M}, \text{ for every } p \in M\right\}$$

consists of what are known as the *$\overline{M}$-vector fields X* on M. It is worth noting that, since the induced connection on $M \subset \mathbb{R}^4$ is very closely related to the Levi-Civita connection of $\overline{M} = \mathbb{R}^4$ and shares most of the Levi-Civita properties, we have used the same notation $\overline{\nabla}$ for both. An $\overline{M}$-vector field X on M is smooth if $f \in \mathfrak{F}(\overline{M})$ implies $Xf \in \mathfrak{F}(M)$.

Furthermore, the function $R^\perp : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}^\perp(M) \to \mathfrak{X}^\perp(M)$ given by

$$R^\perp(X,Y)Z = \nabla^\perp_{[X,Y]}Z - \left[\nabla^\perp_X, \nabla^\perp_Y\right] Z$$

is the curvature tensor of the normal connection $\nabla^\perp$, and is referred to as the *normal curvature tensor* of $M \subset \mathbb{R}^4$.

In fact, for submanifolds of codimension greater than one the normal connection $\nabla^\perp$ becomes more important. Just as the shape tensor of $M \subset \mathbb{R}^4$ measures the difference between ∇ and $\overline{\nabla}$, it can also be used to measure the difference between $\nabla^\perp$ and $\overline{\nabla}$ but in a different formulation, as follows.

For any given vector fields $X \in \mathfrak{X}(M)$ and $Z \in \mathfrak{X}^\perp(M)$, we define the tensor $\widetilde{II} : \mathfrak{X}(M) \times \mathfrak{X}^\perp(M) \to \mathfrak{X}(M)$ as $\widetilde{II}(X,Z) = \tan\left(\overline{\nabla}_X Z\right)$. Thus

$$\overline{\nabla}_X Z = \underbrace{\widetilde{II}(X,Z)}_{\text{tangent to } M} + \underbrace{\nabla^\perp_X Z}_{\text{normal to } M}.$$

Since for any vector fields $X, Y \in \mathfrak{X}(M)$ and $Z \in \mathfrak{X}^\perp(M)$ by differentiating $\langle Z, Y\rangle = 0$ we get $\langle \overline{\nabla}_X Z, Y\rangle = -\langle Z, \overline{\nabla}_X Y\rangle$, the shape tensors $\widetilde{II}$ and II read

$$\left\langle \widetilde{II}(X,Z), Y\right\rangle = -\langle II(X,Y), Z\rangle\,;$$

in addition, as the identity $\langle II(X,Y), Z\rangle = \langle S_Z(X), Y\rangle$ always holds for the shape operator S_Z, we conclude that

$$\widetilde{II}(X,Z) = -S_Z(X), \tag{3.7}$$

the last equality (3.7) being due to the fact that the metric $\langle\ ,\ \rangle$ is non-degenerate.

Using the curvature form of the normal bundle of the surface M, the *normal curvature* N of M is given by $d\omega_{34} = -N\omega_1 \wedge \omega_2$. Now we would like to derive a formula to compute the normal curvature N of M.

Lemma 3.3. *Let $R^\perp$ be, as above, the normal curvature tensor of $M \subset \mathbb{R}^4$. Then the normal curvature N of the surface M reads*

$$N = \left\langle R^\perp_{X_1X_2}X_3, X_4\right\rangle = (e_{X_3} - g_{X_3})\, f_{X_4} - (e_{X_4} - g_{X_4})\, f_{X_3}.$$

Proof of Lemma 3.3. Since $\overline{M} = \mathbb{R}^4$ is a flat manifold, the Ricci equation of $M \subset \mathbb{R}^4$ takes the form

$$\left\langle R^\perp_{XY}V, W\right\rangle = \left\langle \widetilde{II}(X,V), \widetilde{II}(Y,W)\right\rangle - \left\langle \widetilde{II}(X,W), \widetilde{II}(Y,V)\right\rangle, \tag{3.8}$$

for any vector fields $X, Y \in \mathfrak{X}(M)$ and $V, W \in \mathfrak{X}^\perp(M)$.

From (3.7) and (3.8) we obtain

$$\begin{aligned}\left\langle R^\perp_{XY}V, W\right\rangle &= \langle S_V X, S_W Y\rangle - \langle S_W X, S_V Y\rangle\\ &= \langle II\,(Y, S_V X)\,, W\rangle - \langle II\,(X, S_V Y)\,, W\rangle\\ &= \langle II\,(Y, S_V X) - II\,(X, S_V Y)\,, W\rangle\,,\end{aligned}$$

which, by the non-degeneracy of the metric, yields

$$R^\perp_{XY}V = II\,(Y, S_V X) - II\,(X, S_V Y)\,. \tag{3.9}$$

Now, for any two objects a, b, define an endomorphism $a \diamond b$ on the same set of objects by

$$a \diamond b(c) := \langle a, c\rangle\, b - \langle b, c\rangle\, a.$$

Then, setting $II_{ij} := II(X_i, X_j)$ for $i, j = 1, 2$, and using (3.9), we see at once that for any vector fields $V, W \in \mathfrak{X}^\perp(M)$,

$$\langle (II_{11} - II_{22}) \diamond II_{12}(V), W\rangle = \left\langle R^\perp_{X_1X_2}V, W\right\rangle$$

which gives

$$R^\perp_{X_1X_2} = (II_{11} - II_{22}) \diamond II_{12}. \tag{3.10}$$

Applying (3.10) and the definition of the endomorphism $a \diamond b$ (as above), we deduce that if $R^\perp \neq 0$ then (for any point $p \in M$) the vectors $(II_{11} - II_{22})(p)$ and $II_{12}(p)$ are linearly independent vectors in the 2-dimensional normal vector space N_pM; moreover, we conclude that

$$\left\langle R^\perp_{X_1X_2}X_3, X_4\right\rangle = (e_{X_3} - g_{X_3})\, f_{X_4} - (e_{X_4} - g_{X_4})\, f_{X_3}. \tag{3.11}$$

On the other hand, using the connection 1-forms, the Ricci equation for $M \subset \mathbb{R}^4$ reads as

$$d\omega_{34} = \omega_{31} \wedge \omega_{14} + \omega_{32} \wedge \omega_{24}$$

(see for instance [22, p. 28]). By (3.1) and the fact that $\omega_{ki} = -\omega_{ik}$, it follows easily that

$$d\omega_{34} = -\left((e_{X_3} - g_{X_3})\, f_{X_4} - (e_{X_4} - g_{X_4})\, f_{X_3}\right)\; \omega_1 \wedge \omega_2,$$

which establishes an interesting formula for the normal curvature N of the surface M as

$$N = (e_{X_3} - g_{X_3})\, f_{X_4} - (e_{X_4} - g_{X_4})\, f_{X_3}. \tag{3.12}$$

Combining (3.11) and (3.12), we derive the assertion of Lemma 3.3. □

The following lemma establishes a geometric relation between the normal curvature and the curvature ellipse at any point on the surface M .

Lemma 3.4. *Given any $p \in M$. The area of the curvature ellipse of the surface $M \subset \mathbb{R}^4$ at the point p equals $\frac{1}{2}\pi|N(p)|$.*

Proof of Lemma 3.4. As in the proof of Lemma 3.3, equation (3.10) guarantees that in the non-trivial case of non-vanishing $R^\perp \neq 0$ the normal vectors $\widetilde{u} := 1/2(II_{11} - II_{22})(p)$ and $\widetilde{v} := II_{12}(p)$ are always linearly independent in N_pM (for any $p \in M$). So, given any normal frame $\{X_3, X_4\}$ on the surface M, we may choose the tangent frame $\{X_1, X_2\}$ on M in such a way that makes $\widetilde{u}, \widetilde{v}$ a pair of orthogonal vectors. By (3.2), we have

$$\widetilde{u} = \frac{1}{2}\sum_{k=3}^{4} (e_{X_k} - g_{X_k})\, X_k, \quad \widetilde{v} = \sum_{k=3}^{4} f_{X_k} X_k, \tag{3.13}$$

and Proposition 3.1 then shows that $\widetilde{u}, \widetilde{v}$ are coincided with the semimajor and semiminor axes of the curvature ellipse of M at p.

Without loss of generality, we may set $X_3 := \widetilde{u}/\|\widetilde{u}\|$ and $X_4 := \widetilde{v}/\|\widetilde{v}\|$. Then (3.10) shows, by Lemma 3.3, that $R^\perp_{X_1X_2} = 2\widetilde{u} \diamond \widetilde{v}$ and

$$\begin{aligned}
\frac{1}{2}\pi|N| = \frac{1}{2}\pi|\left\langle R^\perp_{X_1X_2}X_3, X_4\right\rangle| &= \pi|\left\langle \widetilde{u}, X_3\right\rangle \left\langle \widetilde{v}, X_4\right\rangle - \left\langle \widetilde{v}, X_3\right\rangle \left\langle \widetilde{u}, X_4\right\rangle| \\
&= \pi|\|\widetilde{u}\|\|\widetilde{v}\| - \frac{\langle \widetilde{v}, \widetilde{u}\rangle}{\|\widetilde{u}\|}\frac{\langle \widetilde{u}, \widetilde{v}\rangle}{\|\widetilde{v}\|}| = \pi\|\widetilde{u}\|\|\widetilde{v}\|,
\end{aligned}$$

which is the desired conclusion (see also [17, Section 2]). □

As a corollary of Lemmas 3.3 and 3.4, we can state the following result.

Corollary 3.5. *The surface $M \subset \mathbb{R}^4$ is totally made of semi-umbilic points if and only if its normal curvature N is identically zero, or equivalently, the normal bundle of M is globally flat.*

Proof of Corollary 3.5. As was shown in the proof of Lemma 3.4, with respect to a suitable frame on M, we may consider $\widetilde{u}$ and $\widetilde{v}$ as the vectors being coincided with the semimajor and semiminor axes of the curvature ellipse. Therefore, since the area of the curvature ellipse equals $\pi\|\widetilde{u}\|\|\widetilde{v}\| = \frac{1}{2}\pi|N|$, the surface M is totally semi-umbilic if and only if $N \equiv 0$. On the other hand, analysis similar to that in the proof of Lemma 3.3 shows that M is totally semi-umbilic if and only if the vectors $\widetilde{u}$ and $\widetilde{v}$ are always linearly *dependent*; by (3.10), the latter statement just amounts to saying that $R^{\perp} \equiv 0$, or equivalently, the normal bundle of M is globally flat. □

The following lemma gives a necessary and sufficient condition under which the curvature ellipse at some point $p \in M$ degenerates into a line segment.

Lemma 3.6. *A point $p \in M \subset \mathbb{R}^4$ is ν-umbilic with respect to some normal vector field $\nu \in \mathfrak{X}^{\perp}(M) \setminus \{0\}$ if and only if it is a semi-umbilic point.*

Proof of Lemma 3.6. From (2.4) it follows that a point $p \in M$ is ν-umbilic with respect to some normal vector field $\nu \in \mathfrak{X}^{\perp}(M) \setminus \{0\}$ if and only if $e_\nu - g_\nu = 0 = f_\nu$, or equivalently, the linear shape operator on the tangent plane T_pM reads $S_\nu = \frac{e_\nu}{E} \mathrm{id}_{T_pM}$.

In addition, we may consider $\widetilde{u}$ and $\widetilde{v}$ as the vectors being coincided with the semimajor and semiminor axes of the curvature ellipse of M at p, by just taking a suitable orthonormal frame $\{X_i\}_{i=1}^4$ on the surface $M \subset \mathbb{R}^4$, as discussed in the proof of Lemma 3.4. Then

$$\begin{aligned}\langle \nu, \widetilde{u} \rangle &= \frac{1}{2} \left(\langle II(X_1, X_1), \nu \rangle - \langle II(X_2, X_2), \nu \rangle \right) \\ &= \frac{1}{2} \left(\langle S_\nu X_1, X_1 \rangle - \langle S_\nu X_2, X_2 \rangle \right) = \frac{e_\nu}{2E}(1-1) = 0,\end{aligned}$$

and also

$$\langle \nu, \widetilde{v} \rangle = \langle II(X_1, X_2), \nu \rangle = \langle S_\nu X_1, X_2 \rangle = \frac{e_\nu}{E} \langle X_1, X_2 \rangle = 0.$$

Therefore $\nu(p) \in N_pM$ is a non-zero vector perpendicular to a pair of orthogonal normal vectors $\widetilde{u}(p)$ and $\widetilde{v}(p)$ in the same normal plane N_pM. Since N_pM is in fact a 2-dimensional vector space, this implies that either of the normal vector fields $\widetilde{u}$ or $\widetilde{v}$ (or both) vanishes at $p \in M$; or equivalently, the curvature ellipse of M at p degenerates into a line segment (considering a single point as a segment of length zero), which amounts to saying that $p \in M$ is a semi-umbilic point, and the lemma follows. □

Remark 3.7. The result of Lemma 3.6 can be extended, by a similar argument, to surfaces or submanifolds M with codimensions bigger than two, sitting in some Euclidean space of higher dimension, as follows: *$p \in M$ is a semi-umbilic point if and only if p is umbilic with respect to $n = \mathrm{codim}(M) - 1$ linearly independent normal vector fields on the surface M.*

Recalling the parametrization of the unit circle in the tangent vector space T_pM by the angle $\theta \in [0, 2\pi]$, from the above and [18, Lemma 4], we have the following result on asymptotic directions.

Proposition 3.8. *Given $p \in M \subset \mathbb{R}^4$, the tangent direction θ is an asymptotic direction at p if and only if*

$$\exists \xi \in N_pM \qquad \mathrm{span}_{\mathbb{R}} \{\cos(\theta) X_1 + \sin(\theta) X_2\} \subseteq \ker\left(S_\xi(p)\right).$$

Given any real-valued smooth functions $u, v \in \mathfrak{F}(M)$, it is a simple matter to check that the wedge product of the 1-forms $\langle \overline{\nabla} e, X_3 \rangle$ and $\langle \overline{\nabla} e, X_4 \rangle$ for the tangent vector field $e := uX_1 + vX_2 \in \mathfrak{X}(M)$ is equal to

$$\langle \overline{\nabla} e, X_3 \rangle \wedge \langle \overline{\nabla} e, X_4 \rangle = \delta(u,v)\ \omega_1 \wedge \omega_2,$$

where $\delta(u, v)$ equals

$$(e_{X_3} f_{X_4} - f_{X_3} e_{X_4})\, u^2 + (e_{X_3} g_{x_4} - g_{X_3} e_{X_4})\, uv + (f_{X_3} g_{X_4} - g_{X_3} f_{X_4})\, v^2. \tag{3.14}$$

In fact, from $\overline{\nabla} e = duX_1 + u\overline{\nabla} X_1 + dvX_2 + v\overline{\nabla} X_2$ it follows easily that the 1-form $\langle \overline{\nabla} e, X_k \rangle = u\omega_{1k} + v\omega_{2k}$, for $k = 3, 4$. Then, by (3.1), we derive the desired relation as above.

We are now in a position to define a geometric quantity which expresses how the curvature ellipse is oriented with respect to the direction given by the mean curvature vector H. To do this, using (3.14), we consider a 2×2 matrix δ that reads $\delta(u,v) = \begin{bmatrix} u & v \end{bmatrix} \delta \begin{bmatrix} u \\ v \end{bmatrix}$. Then,

$$\delta = \begin{pmatrix} e_{X_3} f_{X_4} - f_{X_3} e_{X_4} & \frac{1}{2}\,(e_{X_3} g_{X_4} - g_{X_3} e_{X_4}) \\ \frac{1}{2}\,(e_{X_3} g_{x_4} - g_{X_3} e_{X_4}) & f_{X_3} g_{X_4} - g_{X_3} f_{X_4} \end{pmatrix}. \tag{3.15}$$

Now we define a real-valued smooth function $\Delta : M \to \mathbb{R}$ by $\Delta = \det \delta$. So,

$$\Delta = (e_{X_3} f_{X_4} - f_{X_3} e_{X_4})\,(f_{X_3} g_{X_4} - g_{X_3} f_{X_4}) - \frac{1}{4}\,(e_{X_3} g_{X_4} - g_{X_3} e_{X_4})^2. \tag{3.16}$$

The following lemma shows that $\Delta \in \mathfrak{F}(M)$ is our desired geometric quantity as claimed above.

Lemma 3.9. *A point $p \in M \subset \mathbb{R}^4$ is of elliptic, parabolic, or hyperbolic type if and only if we have $\Delta(p) > 0$, $\Delta(p) = 0$, or $\Delta(p) < 0$ respectively.*

Proof of Lemma 3.9. By Definition 3.2, we show equivalently that the number of asymptotic direction(s) on the surface M at the point p equals zero, one, or two if and only if $\Delta(p) > 0$, $\Delta(p) = 0$, or $\Delta(p) < 0$ respectively. As before, we may suppose that $\{X_1, X_2\}$ and $\{X_3, X_4\}$ are respectively a tangent and a normal frame on M, and choose the frame's vector fields so that $\{X_i\}_{i=1}^4$ gives an isothermic coordinate system on M. Since the shape operator is bilinear, it follows from Proposition 3.8 that the tangent direction given by θ is an asymptotic direction at $p \in M$ if and only if

$$\exists \xi \in N_pM \qquad (\cos(\theta), \sin(\theta)) \in \ker(S_\xi).$$

It is evident that for the normal vector $\xi = aX_3 + bX_4$, for some $a, b \in \mathbb{R}$, we get $S_\xi = aS_{X_3} + bS_{X_4}$. Since the linear shape operator S_ξ on T_pM, in terms of the above-mentioned isothermic coordinate system, can be given by $S_\xi = \frac{1}{E}\begin{bmatrix} e_\xi & f_\xi \\ f_\xi & g_\xi \end{bmatrix}$, the tangent direction θ is an asymptotic direction at $p \in M$ if and only if

$$\begin{aligned} &\frac{a}{E}\left[(e_{X_3}\cos(\theta) + f_{X_3}\sin(\theta))\, X_1 + (f_{X_3}\cos(\theta) + g_{X_3}\sin(\theta))\, X_2\right] \\ &\quad + \frac{b}{E}\left[(e_{X_4}\cos(\theta) + f_{X_4}\sin(\theta))\, X_1 + (f_{X_4}\cos(\theta) + g_{X_4}\sin(\theta))\, X_2\right] = 0, \end{aligned}$$

or equivalently,

$$\begin{vmatrix} e_{X_3}\cos(\theta)+f_{X_3}\sin(\theta) & f_{X_3}\cos(\theta)+g_{X_3}\sin(\theta) \\ e_{X_4}\cos(\theta)+f_{X_4}\sin(\theta) & f_{X_4}\cos(\theta)+g_{X_4}\sin(\theta) \end{vmatrix} = 0,$$

i.e.,

$$\begin{aligned}(e_{X_3}f_{X_4} - f_{X_3}e_{X_4})\cos^2(\theta) + (e_{X_3}g_{X_4} - g_{X_3}e_{X_4})\sin(\theta)\cos(\theta) \\ + (f_{X_3}g_{X_4} - g_{X_3}f_{X_4})\sin^2(\theta) = 0.\end{aligned}$$

Therefore θ is an asymptotic direction if and only if it satisfies the equation

$$\begin{aligned}(f_{X_3}g_{X_4} - g_{X_3}f_{X_4})\tan^2(\theta) + (e_{X_3}g_{X_4} - g_{X_3}e_{X_4})\tan(\theta) \\ + (e_{X_3}f_{X_4} - f_{X_3}e_{X_4}) = 0.\end{aligned} \tag{3.17}$$

But, by (3.16), the discriminant of the polynomial (3.17) in the variable $\tan(\theta)$ is -4Δ. Hence, the desired conclusion is obtained due to the fact that π is a period for the trigonometric function $\tan(\theta)$; see also [17]. □

Lemma 3.10. *Let θ_1, θ_2 be two asymptotic directions on M. Then*

$$\tan^2(\theta_1 - \theta_2) = -\frac{4\Delta}{N^2}.$$

Proof of Lemma 3.10. By assumption, $\tan(\theta_1)$ and $\tan(\theta_2)$ are two different real roots of the quadratic polynomial (3.17) in the variable $\tan(\theta)$. Thus, the discriminant of the polynomial (3.17) must be strictly positive: $-4\Delta > 0$, the function Δ being as in (3.16).

Using the formula (3.12) and some straightforward calculations, we get

$$\tan^2(\theta_1 - \theta_2) = \left(\frac{\sqrt{-4\Delta}/\mathcal{A}}{N/\mathcal{A}}\right)^2, \quad \text{for } \mathcal{A} = (f_{X_3}g_{X_4} - g_{X_3}f_{X_4}),$$

which precisely gives the assertion of the lemma. □

From what has just been proved, we can derive the following result.

Corollary 3.11. *Let M be an isometrically immersed surface in $\mathbb{R}^4$. Then the normal curvature N of the surface M is identically zero, or equivalently, the normal bundle of M is globally flat if and only if there exist two orthogonal global vector fields of asymptotic lines on M.*

Proof of Corollary 3.11. By a similar argument as in the proof of Lemma 3.9, it is seen that the existence of two orthogonal globally defined fields of asymptotic directions θ_1, θ_2 on M amounts to having

$$-4\Delta > 0, \quad \text{and} \quad \theta_1 - \theta_2 = \pi/2,$$

which, due to Lemma 3.10, hold if and only if N is identically zero. On the other hand, the existence of two globally defined vector fields of asymptotic lines on M is guaranteed when $N \equiv 0$. In fact, by (3.12) and (3.15), we have $N = \operatorname{trace}\delta$. So, having $N \equiv 0$, we derive

$$\Delta = \det\delta = -\Big((e_{X_3}f_{X_4} - f_{X_3}e_{X_4})^2 + (e_{X_3}g_{X_4} - g_{X_3}e_{X_4})^2/4\Big) < 0,$$

and hence $-4\Delta > 0$ (cf. Lemma 3.9). □

3.3. **A Wintgen type inequality, and the isoptic curves of the curvature ellipse.** One of the remarkable families of Euclidean surfaces that are invariant under Möbius transformations consists of the so-called *Wintgen ideal surfaces*. They arise in connection with the following pointwise inequality relating intrinsic and extrinsic invariants of a given surface M isometrically immersed in $\mathbb{R}^4$; in fact, at any point $p \in M$, letting $G(p)$ denote the Gaussian curvature of M at p, we have

$$G(p) + |N(p)| \leq \|H(p)\|^2. \tag{3.18}$$

This inequality was proved by Wintgen [26], and is referred to as the *Wintgen inequality*; in fact, the isometrically immersed surfaces $M \subset \mathbb{R}^4$ for which the equality in this inequality is attained at any point are known as *Wintgen ideal surfaces*. It was also shown in [26] that "Wintgen ideal surfaces $M \subset \mathbb{R}^4$ are precisely those surfaces sitting in $\mathbb{R}^4$ whose curvature ellipses are circular at all points $p \in M$." Thus, we can make the following definition of a Wintgen ideal surface.

Definition 3.12. We call an isometrically immersed surface $M \subset \mathbb{R}^4$ a *Wintgen ideal surface* if one of the following equivalent conditions is satisfied:

(1) The equality in the Wintgen inequality (3.18) is attained everywhere on M; i.e., at any point $p \in M$ we have $G(p) + |N(p)| = \|H(p)\|^2$; or,
(2) The curvature ellipse of M at any point $p \in M$ is circular.

For instance, the Whitney 2-sphere is a Wintgen ideal surface. It can also be seen that the rotation surface of Vrănceanu, which is defined by the immersion $f : \mathbb{R} \times (0, 2\pi) \to \mathbb{R}^4$ with

$$f(u, v) = r(u) (\cos u \cos v, \cos u \sin v, \sin u \cos v, \sin u \sin v)$$

for some smooth real-valued function r, is a Wintgen ideal surface if and only if $r(u)$ reads either of the following expressions:

$$r(u) = \frac{A}{\sqrt{|\cos(2u + a)|}}, \text{ or } r(u) = B\sqrt{|\cos(2u + b)|},$$

where $A > 0, B > 0$ and $a, b \in \mathbb{R}$. With the former expression for r, the rotation surface of Vrănceanu is the tensor product of an equilateral hyperbola and a unit circle; and with the latter one, the surface is the tensor product of a lemniscate of Bernoulli and a unit circle. For a through treatment on Wintgen ideal surfaces, we refer the reader to [5].

In order to formulate the main result of this paper, we need to make the following definition as well.

Definition 3.13. Given an angle $\alpha \in (0, \pi)$, the *α-isoptic curve* of a given ellipse is, by definition, the geometric locus of intersection of tangents to the ellipse making the angle α (i.e., the locus of points on the plane of our ellipse from which the ellipse is seen under the angle α).
Accordingly, we call a point $p \in M$ an *α-isoptic point* of M if p lies on the α-isoptic curve of the curvature ellipse of the surface M at p.

It is worth noting here that the word *isoptic* has been coined from two Greek words: *Isos* meaning "equal", and *optikos* doing "relative to sight".

We have exhibited the α-isoptic curves of a given ellipse for some values of the angle α in Fig. 1 below.

In fact, we have derived the general form of the equation of an α-isoptic curve of a given ellipse in Proposition A.1 (see Appendix A for the proof). In particular, Proposition A.1 shows that the $\frac{\pi}{2}$-isoptic curve of a given ellipse is actually a circle centered at the center of the ellipse, with radius $\sqrt{a^2+b^2}$, where a and b are the lengths of the semimajor and semiminor axes of the ellipse. This circle is called the *director circle* (also known as the *orthoptic circle* or the *Fermat-Apollonius circle*) of the given ellipse. For another independent proof in the case of $\alpha = \frac{\pi}{2}$ we refer the reader to [1, Th. 1.5].

We can now formulate our main result, which yields a Wintgen type inequality (3.19); cf. the Wintgen's original inequality (3.18).

Theorem 3.14. *A point $p \in M$ is an α-isoptic point (i.e., it lies on the α-isoptic curve of the curvature ellipse of the surface M at p), for a given angle $\alpha \in (0, \pi)$, if and only if*

$$|N(p)| \leq (-\Delta(p))^{\frac{1}{2}} E_\alpha + \|H(p)\|^2, \tag{3.19}$$

with $E_\alpha = \left(\alpha_\circ^2 - 1\right)^{\frac{1}{2}} \left(\cot^2(\alpha/2) - 1\right)$ for a unique real number $\alpha_\circ > 1$ corresponding to α, where the equality holds when and only when either p is an umbilic point or the curvature ellipse of M at p degenerates to a circle of radius $\frac{\|H(p)\|}{\sqrt{2C_\alpha}}$, in which $C_\alpha = 1 - (\alpha_\circ^2 - 1)^{1/2} \cot \alpha$.

Before starting our proof of the above theorem, we would like to make some important remarks on and indicate some applications of our main result.

Remark 3.15. In fact, the constant parameter $\alpha_\circ$ in Theorem 3.14 reads

$$\alpha_\circ^2 = \left(c_1 \cos\zeta + c_2 \sin\zeta\right)^2 / \|\widetilde{u}\|^2 + \left(c_1 \sin\zeta - c_2 \cos\zeta\right)^2 / \|\widetilde{v}\|^2,$$

(please refer to the proof of our theorem) where $c_i = \left(e_{X_{i+2}} + g_{X_{i+2}}\right)/2$ for $i = 1, 2$ are the components of the mean curvature vector H, as in (3.5), with respect to the orthogonal frame $\{X_3, X_4\}$ on the normal plane N_pM, the normal vectors $\widetilde{u}$ and $\widetilde{v}$ being coincided with the semimajor and semiminor axes of the curvature ellipse (cf. Proposition 3.1) are as in (3.13), and the angle ζ measures the smallest amount of rotation one has to make at the origin 0_{N_pM} of the normal plane so that the orthogonal directions given by the pair of vectors of the frames $\{\widetilde{u}, \widetilde{v}\}$ and $\{X_3, X_4\}$ are coincided.

It follows immediately from Theorem 3.14 that for any point $p \in M$ lying on the director circle (i.e., the $\frac{\pi}{2}$-isoptic curve) of the curvature ellipse of the surface M at p we have $|N(p)| \leq \|H(p)\|^2$; the equality being occurred exactly when either p is an umbilic point, or else the curvature ellipse of M at p degenerates to a circle of radius $\frac{\|H(p)\|}{\sqrt{2}}$ concentric with the director circle, because $E_{\frac{\pi}{2}} = 0$ and $C_{\frac{\pi}{2}} = 1$.

It is evident that, for any $\alpha \in \left(\frac{\pi}{2}, \pi\right)$, the term $\left(\cot^2(\alpha/2) - 1\right)$ is *strictly negative* (cf. Fig. 2 below) and so is the expression E_α at any *non-umbilic* point

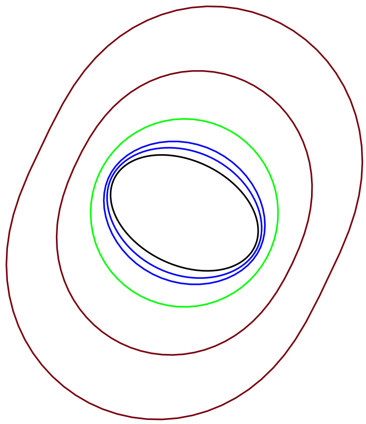

Fig. 1: The α-isoptic curves of a given ellipse for some values of the angle $\alpha \in (0, \pi)$. The innermost curve (in black) is an ellipse, and the circle (in green) is its director circle (i.e. the $\frac{\pi}{2}$-isoptic curve). The other curves are the α-isoptic curves for $\alpha < \frac{\pi}{2}$ (in red) and $\alpha > \frac{\pi}{2}$ (in blue); taking a smaller value for α, yields the α-isoptic curve with a bigger diameter

$p \in M$, as defined in the statement of Theorem 3.14. Hence, as a consequence of Theorem 3.14, an important necessary condition geometrically characterizing the non-umbilic points $p \in M$ which are α-isoptic with some wide isoptic angle $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ can be given as follows.

Corollary 3.16. *If a non-umbilic point $p \in M$ lies on the α-isoptic curve of the curvature ellipse of the surface M at p for some obtuse angle $\alpha \in \left(\frac{\pi}{2}, \pi\right)$, then the normal and mean curvatures of M at p read*

$$|N(p)| < \|H(p)\|^2.$$

On the other hand, as is shown in the proof of Proposition A.1, any point outside of a given ellipse lies on some α-isoptic curve of the ellipse, hence

Remark 3.17. The set of isoptic points on M and that of the hyperbolic points coincide.

Therefore, Theorem 3.14 provides a geometric characterization of the hyperbolic points on M as well.

Moreover, due to the fact that the term $\left(\cot^2(\alpha/2) - 1\right) \geq 0$ is non-negative, for any $\alpha \in \left(0, \frac{\pi}{2}\right]$ (cf. Fig. 2), and so is the expression E_α in the statement of Theorem 3.14, we can make the following nice and interesting remark which is in fact the contra-positive of Corollary 3.16.

Remark 3.18. If $|N(p)| \geq \|H(p)\|^2$ for some point $p \in M$, then either p is an umbilic point or it is α-isoptic for some angle $\alpha \in \left(0, \frac{\pi}{2}\right]$.

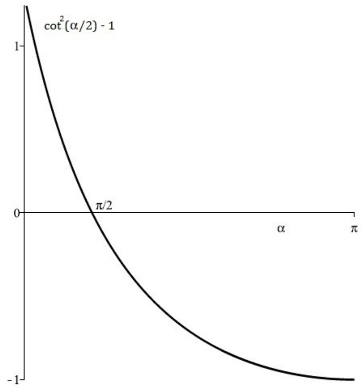

FIG. 2: The graph of the mapping $\alpha \mapsto \left(\cot^2(\alpha/2) - 1\right)$ illustrates the argument supporting Remark 3.18 and Corollaries 3.16 and 3.19

As another consequence of our Wintgen type inequality (3.19), given by Theorem 3.14, we can now state the following corollary for a family of Wintgen ideal surfaces that are free of umbilic points; in fact, the following result shows how the Gaussian curvatures of our surfaces at isoptic points can be related to the isoptic angles, and hence yields a clearer geometric interpretation of the term $(-\Delta(p))^{\frac{1}{2}} E_\alpha$ in the inequality (3.19).

Corollary 3.19. *Let $M \subset \mathbb{R}^4$ be a Wintgen ideal surface with no umbilic points. A point $p \in M$ is an α-isoptic point, for some angle $\alpha \in (0, \pi)$, if and only if the Gaussian curvature G of M at p reads*

$$G(p) = -\left(-\Delta(p)\right)^{\frac{1}{2}} E_\alpha.$$

Accordingly, the Gaussian curvature $G(p)$ of M at an α-isoptic point p is negative (respectively, positive, or else zero) when and only when $\alpha \in (0, \frac{\pi}{2})$ (respectively, $\alpha \in (\frac{\pi}{2}, \pi)$, or else $\alpha = \frac{\pi}{2}$).

Proof of Corollary 3.19. Let $M \subset \mathbb{R}^4$ be a Wintgen ideal surface. By Definition 3.12, at any point $p \in M$, we then have

$$G(p) + |N(p)| = \|H(p)\|^2, \tag{3.20}$$

and that the curvature ellipses are all circular.

The proof falls naturally into two parts, respectively, concerning the values (Part A) and the signs (Part B) of the Gaussian curvatures of a given surface at its isoptic points.

Part A. Let $p \in M$ be an α-isoptic point for some angle $\alpha \in (0, \pi)$. Employing Theorem 3.14, we deduce that

$$|N(p)| = \left(-\Delta(p)\right)^{\frac{1}{2}} E_\alpha + \|H(p)\|^2 \tag{3.21}$$

since, by the assumptions, M is free of umbilic points and its curvature ellipse at p is circular. Comparison of (3.20) and (3.21) shows that the Gaussian curvature G of M at p must then read

$$G(p) = -\left(-\Delta(p)\right)^{\frac{1}{2}} E_\alpha.$$

Now, suppose that for a point $p \in M$ and some angle $\alpha \in (0, \pi)$ we have the Gaussian curvature $G(p) = -\left(-\Delta(p)\right)^{\frac{1}{2}} E_\alpha$. Hence, as M is a Wintgen ideal surface, by Definition 3.12, we know that

$$G(p) + |N(p)| = \|H(p)\|^2 = |N(p)| - \left(-\Delta(p)\right)^{\frac{1}{2}} E_\alpha;$$

but the latter equality is just the equality case in our Wintgen type inequality (3.19). Thus, by Theorem 3.14, we conclude that p is an α-isoptic point.

Part B. Taking into account the fact that the expression $-E_\alpha$ is negative (respectively, positive, or else zero) when and only when $\alpha \in (0, \frac{\pi}{2})$ (respectively, $\alpha \in (\frac{\pi}{2}, \pi)$, or else $\alpha = \frac{\pi}{2}$), cf. Fig. 2, proves our assertion concerning the sign of the Gaussian curvature $G(p)$ of M at an α-isoptic point p. □

The remainder of this section will be devoted to the proof of our main result, Theorem 3.14.

We begin with establishing an elegant formula which shows how α-isoptic curves of the curvature ellipses of our surface M are related to the asymptotic directions on M (cf. Definitions 3.2 and 3.13).

Lemma 3.20. *If $p \in M$ lies on the α-isoptic curve of the curvature ellipse of M at p, for a given angle $\alpha \in (0, \pi)$, then there exist asymptotic directions θ_1 and θ_2 on M at p that read $|\theta_2 - \theta_1| = \alpha/2$.*

Proof of Lemma 3.20. By Definitions 3.2 and 3.13, the fact that $p \in M$ lies on the α-isoptic curve of the curvature ellipse of the surface M at p amounts to saying that there exist two asymptotic directions θ_1 and θ_2 on M at p so that the vectors $\eta(\theta_1)$ and $\eta(\theta_2)$ on the normal plane N_pM make an angle α. Since $\left\{\eta(\theta_i), \frac{\partial \eta}{\partial \theta}(\theta_i)\right\}$ is a set of two collinear vectors (for $i = 1, 2$), the angle between the tangent vectors $\frac{\partial \eta}{\partial \theta}(\theta_1)$ and $\frac{\partial \eta}{\partial \theta}(\theta_2)$ to the curvature ellipse is also equal to α; in what follows, we adopt the convention that the angle $\alpha \in (0, \pi)$ measures the smallest amount of rotation one has to make at the origin 0_{N_pM} of the normal plane by which the direction of $\frac{\partial \eta}{\partial \theta}(\theta_1)$ is coincided with that of $\frac{\partial \eta}{\partial \theta}(\theta_2)$.

Hence,

$$R_\alpha \frac{\partial \eta}{\partial \theta}(\theta_1) = \frac{\partial \eta}{\partial \theta}(\theta_2), \tag{3.22}$$

where R_α is the rotation matrix by the angle α in the normal plane N_pM, and by Proposition 3.1,

$$\frac{\partial \eta}{\partial \theta}(\theta_i) = \sum_{k=3}^{4} \left(-\left(e_{X_k} - g_{X_k}\right)\sin(2\theta_i) + 2f_{X_k}\cos(2\theta_i)\right) X_k, \quad \text{for } i = 1, 2.$$

Setting $\beta := \pm(\theta_2 - \theta_1)$ in accordance with our above-mentioned convention for determining the angle α so that $\beta \in \left(0, \frac{\pi}{2}\right)$, and then substituting θ_2 (in terms of θ_1 and β) into (3.22), we obtain

$$\begin{aligned}
&[\cos(2\beta) - \cos\alpha]\left((e_{X_3} - g_{X_3})\sin(2\theta_1) - 2f_{X_3}\cos(2\theta_1)\right)\\
&\quad + \sin\alpha\left((e_{X_4} - g_{X_4})\sin(2\theta_1) - 2f_{X_4}\cos(2\theta_1)\right)\\
&\quad + \sin(2\beta)\left((e_{X_3} - g_{X_3})\cos(2\theta_1) + 2f_{X_3}\sin(2\theta_1)\right) = 0,
\end{aligned}$$

and

$$\begin{aligned}
&[\cos(2\beta) - \cos\alpha]\left((e_{X_4} - g_{X_4})\sin(2\theta_1) - 2f_{X_4}\cos(2\theta_1)\right)\\
&\quad + \sin\alpha\left(-(e_{X_3} - g_{X_3})\sin(2\theta_1) + 2f_{X_3}\cos(2\theta_1)\right)\\
&\quad + \sin(2\beta)\left((e_{X_4} - g_{X_4})\cos(2\theta_1) + 2f_{X_4}\sin(2\theta_1)\right) = 0.
\end{aligned}$$

Summing the above two equations, yields

$$[\cos(2\beta) - \cos\alpha]\,\mathcal{A} + \mathcal{B}_1 + \mathcal{B}_2 = 0, \tag{3.23}$$

where

$$\begin{aligned}
\mathcal{A} &= \left([(e_{X_3} - g_{X_3}) + (e_{X_4} - g_{X_4})]\sin(2\theta_1) - 2\left(f_{X_3} + f_{X_4}\right)\cos(2\theta_1)\right),\\
\mathcal{B}_1 &= \sum_{k=3}^{4}(e_{X_k} - g_{X_k})\left(\cos(2\theta_1)\sin(2\beta) + (-1)^k\sin(2\theta_1)\sin(\alpha)\right), \text{ and}\\
\mathcal{B}_2 &= 2\sum_{k=3}^{4} f_{X_k}\left(\sin(2\theta_1)\sin(2\beta) + (-1)^{k-1}\cos(2\theta_1)\sin(\alpha)\right).
\end{aligned}$$

In fact, (3.23) shows how the angle α can be related to the asymptotic directions on M at p if $p \in M$ lies on the α-isoptic curve of our curvature ellipse.

Since it is immaterial which indexes we choose for θ_i's from the beginning, replacing θ_1 by θ_2, and accordingly changing α to $(-\alpha)$ and β to $(-\beta)$ in (3.23), must give us exactly the same equation in terms of the parameters α, β, and θ_2 (instead of θ_1). In fact, under the simultaneous changes of $\theta_1 \rightsquigarrow \theta_2$, $\alpha \rightsquigarrow (-\alpha)$ and $\beta \rightsquigarrow (-\beta)$ in (3.23), either all or none of the terms on the left-hand side of (3.23) must change sign; but after making these changes, evidently, only the first term remains unchanged, contrary to the other terms in (3.23). Thus, the first term must be identically zero, and hence $\cos(2\beta) = \cos(\alpha)$ or equivalently $\beta = \alpha/2$. □

We are now in a position to proceed to the proof of Theorem 3.14.

Proof of Theorem 3.14. Lemmas 3.10 and 3.20 now show that

$$|N(p)| = 2\cot(\alpha/2)\left(-\Delta(p)\right)^{\frac{1}{2}}. \tag{3.24}$$

In addition, as was shown in the proof of Lemma 3.4, the lengths $\|\widetilde{u}\|$ and $\|\widetilde{v}\|$ of the semimajor and semiminor axes of the curvature ellipse of the surface M at p read

$$\|\widetilde{u}\|^2\|\widetilde{v}\|^2 = |N(p)|^2/4. \tag{3.25}$$

On the other hand, by the classical geometry of conics in the plane, it is obvious that to any given ellipse and any point *outside* of the ellipse, there corresponds a *unique* real number (strictly *bigger* than 1) as the scale factor of a uniform scaling transformation in the plane under which the given ellipse transforms into a bigger ellipse passing through the given point and being concentric with our given ellipse. Therefore, by Proposition A.1, $p \in M$ (as the origin of the normal vector space N_pM) lies on the α-isoptic curve of the curvature ellipse of M at p when and only when there exists a real number $\alpha_\circ > 1$ that reads

$$\tan\alpha = \frac{2\|\widetilde{u}\|\|\widetilde{v}\|(\alpha_\circ^2 - 1)^{\frac{1}{2}}}{(\|\widetilde{u}\|^2 + \|\widetilde{v}\|^2) - \|H\|^2}. \tag{3.26}$$

In fact, the unique real number $\alpha_\circ$ is the scale factor of a uniform scaling transformation that transforms the curvature ellipse to another concentric ellipse with $\alpha_\circ\|\widetilde{u}\|$ and $\alpha_\circ\|\widetilde{v}\|$ as the lengths of its semimajor and semiminor axes. Hence, it is clear that the real factor $\alpha_\circ$ can be uniquely determined by the explicit formula as in Remark 3.15 above. It is worth noting that the norm $\|H\|$ of the mean curvature vector appears in the denominator of the fraction in (3.26) just because H is the vector which connects the point p (as the origin of the normal plane N_pM) to the center of the curvature ellipse of M at p.

From (3.24), (3.25) and (3.26), it follows that

$$\|\widetilde{u}\|^2 + \|\widetilde{v}\|^2 = (\alpha_\circ^2 - 1)^{\frac{1}{2}}(\cot^2(\alpha/2) - 1)\,(-\Delta(p))^{\frac{1}{2}} + \|H\|^2, \tag{3.27}$$

using the trigonometric identity $\cot\alpha\cot(\alpha/2) = \ (\cot^2(\alpha/2) - \ 1)/2$.

By (3.25) and (3.27), which serve completely to determine the shape of the curvature ellipse, one may in fact consider the real numbers $\|\widetilde{u}\|^2$ and $\|\widetilde{v}\|^2$ as the roots of a quadratic polynomial in an auxiliary variable $\mathcal{X}$ as

$$\mathcal{X}^2 - \left((\alpha_\circ^2 - 1)^{\frac{1}{2}}(\cot^2(\alpha/2) - 1)\,(-\Delta(p))^{\frac{1}{2}} + \|H\|^2\right)\mathcal{X} + N^2/4 = 0.$$

Since the above polynomial has at least one iterated real root, its discriminant must be non-negative; or equivalently

$$|N(p)| \leq (\alpha_\circ^2 - 1)^{\frac{1}{2}}(\cot^2(\alpha/2) - 1)\,(-\Delta(p))^{\frac{1}{2}} + \|H\|^2,$$

with equality when and only when the curvature ellipse degenerates to a circle of radius $\|\widetilde{u}\| = \|\widetilde{v}\|$. Thus, for the case of equality, it follows from (3.26) that the radius of the circle is

$$\|\widetilde{u}\| = \|H\|/\left(\sqrt{2}\left(1 - (\alpha_\circ^2 - 1)^{\frac{1}{2}}\cot\alpha\right)^{\frac{1}{2}}\right).$$

This finishes the proof. □

Annexe A. The equation of isoptic curves of an ellipse

Here, we shall derive the equation of the α-isoptic curve of a given ellipse for any given α.

Given an angle $\alpha \in (0, \pi)$ and an ellipse, we shall derive in this section the equation of the α-isoptic curve of the ellipse. For the convenience of the reader, we first remind some basic facts from the geometry of conics.

In fact, the standard ellipse in the (x, y)-plane centered at the origin whose axes (of lengths a and b) lie along the coordinate axes is, by definition, represented by the canonical equation $x^2/a^2 + y^2/b^2 = 1$. Employing a suitable affine transformation on the coordinates (x, y), one can easily see that the equation of an ellipse centered at $c : (c_1, c_2)$ whose axes (of lengths a and b) make some angle ζ with the coordinate axes is

$$\frac{\left((x - c_1)\cos\zeta + (y - c_2)\sin\zeta\right)^2}{a^2} + \frac{\left((x - c_1)\sin\zeta - (y - c_2)\cos\zeta\right)^2}{b^2} = 1. \tag{A.1}$$

Proposition A.1. *Under the above assumptions, the α-isoptic curve of the given ellipse, as in (A.1), is represented by*

$$\tan\alpha = \frac{2ab\left(P^2/a^2 + Q^2/b^2 - 1\right)^{1/2}}{(a^2 + b^2) - ((X - c_1)^2 + (Y - c_2)^2)},$$

where (X, Y) gives the coordinates of an arbitrary point on our α-isoptic curve, and

$$\begin{aligned} P = P(X, Y) &:= (X - c_1)\cos\zeta + (Y - c_2)\sin\zeta, \\ Q = Q(X, Y) &:= (X - c_1)\sin\zeta - (Y - c_2)\cos\zeta. \end{aligned} \tag{A.2}$$

Proof of Proposition A.1. Take an arbitrary point (X, Y) *outside* of the ellipse (A.1), and consider the line $y = m(x - X) + Y$ passing through (X, Y) with some slope m. Substituting the equation $y = m(x - X) + Y$ of our line into (A.1), gives a univariate quadratic polynomial in terms of $(x - c_1)$ whose discriminant must be zero in order to have the line $y = m(x - X) + Y$ as a tangent line to our ellipse (this is, in fact, due to Bezout's theorem which states simply that a necessary and sufficient condition for a line to be tangent to an ellipse is to intersect the ellipse exactly at one point – please see [2, Ch. 16.4] for more details); after some lengthy but straightforward calculations, we can express this discriminant in the form of a univariate quadratic polynomial in terms of the slope m as

$$\left(a^2 - P^2\right)m^2 + 2mPQ + \left(b^2 - Q^2\right), \tag{A.3}$$

the polynomials P and Q being as in (A.2).

It is evident, by Definition 3.13, that our arbitrary chosen point (X, Y) can lie on *some* isoptic curve of our ellipse (A.1) when and only when the polynomial (A.3) has two *distinct* real roots $m_{1,2}$, which must be given by

$$m_{1,2} = \frac{1}{a^2 - P^2}\left(-PQ \pm \left(b^2P^2 + a^2Q^2 - a^2b^2\right)^{1/2}\right). \tag{A.4}$$

It is worth pointing out here that the polynomial (A.3) has indeed two distinct real roots $m_{1,2}$ as in (A.4), because we have assumed our arbitrary point (X, Y) to be outside of our ellipse (A.1) and therefore $\left(b^2P^2 + a^2Q^2 - a^2b^2\right) \gneq 0$. Consequently, *any point outside of a given ellipse lies on some α-isoptic curve of the ellipse.*

But in order to have the point (X, Y) precisely on the α-isoptic curve of our ellipse (A.1) for a given angle α, the slopes $m_{1,2}$ of our tangent lines to our ellipse must also read

$$\tan\alpha = \frac{m_1 - m_2}{1 + m_1m_2}, \tag{A.5}$$

(cf. Definition 3.13).

On substituting (A.4) into (A.5) we establish our desired formula. □

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