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ON GLOBAL ATTRACTIVITY OF A HIGHER ORDER DIFFERENCE EQUATION WITH ASYMPTOTIC CONSTANT COEFFICIENTS

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Abstract. Consider the following higher order difference equation

$$x_{n+1} = a_n x_n + b_n f(x_n) + c_n f(x_{n-k}),\ n = 0, 1, \dots,$$

where $f\colon [0,\infty) \to [0,\infty)$ is a continuous function with $f(x) > 0$ for $x > 0$, $\{a_n\}$ is a sequence in $(0,1)$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1)$ with $a_n + b_n + c_n = 1$ and a_n, b_n and c_n are convergent, and k is a positive integer. Our aim in this paper is to study the global attractivity of positive solutions of this equation and its applications.

1. Introduction

Recently, the global attractivity of positive solutions of the following higher order difference equation

$$x_{n+1} = a x_n + b f(x_n) + c f(x_{n-k}),\ n = 0, 1, \dots, \tag{1.1}$$

has been studied in [1, 2, 27], where a, b and c are constants with $0 < a < 1, 0 \le b < 1, 0 \le c < 1$ and $a + b + c = 1$, $f \in C[[0,\infty), [0,\infty)]$ with $f(x) > 0$ for $x > 0$ and k is a positive integer. The case when the sum of the main coefficients of a higher order difference equation is equal to one is of a great interest and has been studied a lot, see for example, [1, 2, 16, 17, 20, 21, 22, 23, 24, 27] and the references cited therein. One of the reasons for this, is the fact that such difference equations frequently model some processes in nature or society. For instance, consider the following difference system

$$\begin{cases} x_{n+1} = (1-\epsilon) f(x_n) + \epsilon y_n, & \\ y_{n+1} = (1-\epsilon) y_n + \epsilon f(x_n), & n = 0, 1, \dots, \\ x_0 \ge 0,\ y_0 \ge 0,\ x_0 + y_0 > 0, & \end{cases} \tag{1.2}$$

where $0 < \epsilon < 1$ is a positive constant, and $f\colon [0,\infty) \to [0,\infty)$ is a continuous function with $f(x) > 0$ for $x > 0$. Sys. (1.2) is a population model proposed by Newman et al. [14] which assumes symmetric dispersal between active population

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x_n and refuge population y_n. The chaotic behavior of positive solutions of Sys. (1.2) is studied in [14] by numerical simulations, whereas in [5] various properties of solutions of (1.2) are studied and several results on the asymptotic behavior of solutions of Sys. (1.2) are obtained. However, note that by a simple calculation, Sys. (1.2) can be converted into the second order difference equation

$$x_{n+1} = (1-\epsilon)x_n + (1-\epsilon)f(x_n) + (2\epsilon - 1)f(x_{n-1}). \tag{1.3}$$

Eq. (1.3) is in the form of (1.1) with $a = b = 1-\epsilon, c = 2\epsilon - 1$ and $k = 1$. Hence, by studying the global attractivity of (1.3), several global attractivity results on the positive solutions of Sys. (1.2) are obtained in [1, 2, 27].

Motivated by theoretical interest and the fact that in applications there are often some unknown factors which might affect the coefficients of difference models (see, for example, the model in [24] and a related system in [25]), in the present paper we study the following more general higher order difference equation

$$x_{n+1} = a_n x_n + b_n f(x_n) + c_n f(x_{n-k}),\ n = 0, 1, \dots \tag{1.4}$$

where $f\colon [0,\infty) \to [0,\infty)$ is a continuous function with $f(x) > 0$ for $x > 0$, $\{a_n\}$ is a sequence in $(0,1)$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1)$ with $a_n + b_n + c_n = 1$ and a_n, b_n and c_n are convergent, and k is a positive integer.

When $b_n = 0$, then $c_n = 1 - a_n$ and so Eq. (1.4) reduces to the difference equation

$$x_{n+1} = a_n x_n + (1-a_n)f(x_{n-k}),\ n = 0, 1, \dots, , \tag{1.5}$$

The asymptotic behavior of positive solutions of Eq. (1.5) and applications has been studied by numerous authors, see for example, [6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 23, 24, 26] and the references cited therein. In particular, when $k = 0$, Eq. (1.5) becomes

$$x_{n+1} = a_n x_n + (1-a_n)f(x_n),\ n = 0, 1, \dots,$$

which is often said to be a segmenting Mann interation or to be of Krasnoselskii-type [3, 4].

In the following discussion, we assume that f has a unique positive fixed point $\bar{x}$, and that f satisfies the negative feedback condition

$$(x - \bar{x})(f(x) - x) < 0,\ x \neq \bar{x}. \tag{1.6}$$

Clearly, $\bar{x}$ is the only positive equilibrium of Eq. (1.4). In Section 2, we establish some sufficient conditions for $\bar{x}$ to be a global attractor of all positive solutions of Eq. (1.4). Then, in Section 3, we will apply our results obtained in Section 3 to some equations derived from mathematical biology.

In the following discussion, for the sake of convenience, we adopt the notation $\prod_{i=m}^{n} s_i = 1$ and $\sum_{i=m}^{n} s_i = 0$ whenever $\{s_n\}$ is a real sequence and $m > n$.

2. Main Results

In this section, we establish some sufficient conditions for the global attractivity of positive solutions of Eq. (1.4). By noting the assumption (1.6) and by using an argument similar to that for Lemma 2.1 in [1], we may have the following conclusion

Lemma 2.1. *Every positive solution $\{x_n\}$ of Eq. (1.4) is bounded and persistent.*

First, we deal with the case when $\{a_n\}$ converges to 1 as $n \to \infty$.

Theorem 2.2. *Assume that*

$$\lim_{n\to\infty} a_n = 1 \tag{2.1}$$

and

$$\sum_{n=0}^{\infty}(b_n + c_n) = \infty. \tag{2.2}$$

Then every positive solution $\{x_n\}$ *of Eq.* (1.4) *converges to* $\bar{x}$ *as* $n \to \infty$.

Proof. Let $\{x_n\}$ be a positive solution of Eq. (1.4). From Lemma 2.1 we know that $\{x_n\}$ is bounded and persistent. Hence, there are positive constants R and r such that $\limsup_{n\to\infty} x_n = R$ and $\liminf_{n\to\infty} x_n = r$. We claim that $R = r$. Otherwise, $R > r$. Let q be any number with $R > q > r$. Suppose that $f(q) \neq q$. Then there are two cases: $f(q) > q$ or $f(q) < q$. Suppose that $f(q) > q$. So, we can find a constant β such that $0 < \beta < 1$, $R - r > \beta$ and $f(x) > x$ for $q - \beta \le x \le q + \beta$. Let

$$\lambda = \min_{q-\beta\le x\le q+\beta} \big(f(x) - x\big).$$

By continuity of f, we see that $\lambda > 0$. Noting $a_n + b_n + c_n = 1$, Eq. (1.4) yields

$$x_{n+1} - x_n = b_n(f(x_n) - x_n) + c_n(f(x_{n-k}) - x_n).$$

Since (2.1) holds, $b_n \to 0$ and $c_n \to 0$ as $n \to \infty$. Hence, it follows that

$$\lim_{n\to\infty}(x_{n+1} - x_n) = \lim_{n\to\infty} b_n(f(x_n) - x_n) + \lim_{n\to\infty} c_n(f(x_{n-k}) - x_n) = 0.$$

Let N be large enough so that

$$|x_{n+1} - x_n| < \frac{\beta\lambda}{(k+1)(\lambda+1)} \quad \text{for } n > N - k. \tag{2.3}$$

We see that if $x_n < q$ for every $n > N$, then $R \le q$ which is a contradiction. So, there is an $\bar{n} > N$ such that $x_{\bar{n}} > q$. We claim that $x_n > q$ for $n \ge \bar{n}$. In fact, if $x_n > q + \frac{\beta\lambda}{(k+1)(\lambda+1)}$, then by noting (2.3) we see that

$$x_n - \frac{\beta\lambda}{(k+1)(\lambda+1)} < x_{n+1} < x_n + \frac{\beta\lambda}{(k+1)(\lambda+1)},$$

and it follows that $q < x_{n+1}$. If $q + \frac{\beta\lambda}{(k+1)(\lambda+1)} \ge x_n > q$, then

$$\begin{aligned} |x_{n-k} - q| &\le \sum_{m=0}^{k-1} |x_{n-k+m} - x_{n-k+1+m}| + |x_n - q| \\ &\le k\frac{\beta\lambda}{(k+1)(\lambda+1)} + \frac{\beta\lambda}{(k+1)(\lambda+1)} = (k+1)\left(\frac{\beta\lambda}{(k+1)(\lambda+1)}\right) < \beta. \end{aligned}$$

Since $f(x) > x$ for $|x-q| \le \beta$ and $\lambda = \min_{|x-q|\le\beta}(f(x)-x)$, we see that $f(x_{n-k}) - x_{n-k} \ge \lambda$. Then it follows from Eq. (1.4) and (2.3) that

$$\begin{aligned}
x_{n+1} &= x_n + b_n(f(x_n)-x_n) + c_n(f(x_{n-k})-x_n)\\
&\ge x_n + b_n\lambda + c_n\left(f(x_{n-k}) - x_{n-k} - \sum_{m=1}^{k}(|x_{n-m}-x_{n-m+1}|)\right)\\
&\ge x_n + b_n\lambda + c_n\left(\lambda - k\frac{\beta\lambda}{(k+1)(\lambda+1)}\right)\\
&\ge x_n + b_n\lambda + c_n\lambda\left(1 - \frac{k}{(k+1)(\lambda+1)}\right) \ge x_n > q.
\end{aligned}$$

By induction, we see that $x_n > q$ for all $n > N$ and thus $r \ge q$. This is a contradiction. Hence, $f(q) > q$ is not true. By a similar argument, we may show that $f(q) < q$ can not be true either. So, we have $f(q) = q$. However, this contradicts the fact that f has a unique fixed point since q is any number between r and R. Hence, we must have $r = R$ and it follows that $\{x_n\}$ converges.

Since x_n converges, there is a constant p such that $x_n \to p$ as $n \to \infty$. We claim that $f(p) = p$. Otherwise, $f(p) \neq p$. Then,

$$\lim_{n\to\infty}(f(x_n)-x_n) = \lim_{n\to\infty}(f(x_{n-k})-x_n) = f(p)-p \neq 0,$$

and it follows that

$$\sum_{n=0}^{\infty}(b_n[f(x_n)-x_n] + c_n[f(x_{n-k})-x_n]) = +\infty \ (or -\infty) \tag{2.4}$$

since (2.2) holds. However, from Eq. (1.4), we know that

$$x_{n+1} - x_n = b_n(f(x_n)-x_n) + c_n(f(x_{n-k})-x_n),\ n = 0, 1, \ldots,$$

and so

$$x_{n+1} - x_0 = \sum_{i=0}^{n}(b_i[f(x_i)-x_i] + c_i[f(x_{i-k})-x_i]).$$

This together with (2.4) implies that $\lim_{n\to\infty} x_n = +\infty$ or $-\infty$ which contradicts the fact that $\{x_n\}$ is bounded. So, $f(p) = p$, that is, p is a fixed point of f. Since f has a unique fixed point $\bar{x}$, we see that $p = \bar{x}$. Hence $\{x_n\}$ converges to $\bar{x}$ as $n \to \infty$. The proof is complete. □

Our next theorem deals with the case when $\{a_n\}, \{b_n\}$ and $\{c_n\}$ have limits, but the limit of $\{a_n\}$ is not 1.

Theorem 2.3. *Assume that $\{a_n\}, \{b_n\}$ and $\{c_n\}$ have limits a, b, c, respectively, with $a \neq 1$. Suppose also that $a_n x + b_n f(x)$ is increasing in x and that f is L-Lipschitz with*

$$\frac{c(1-a^{k+1})}{c+a^k b}L < 1. \tag{2.5}$$

Then every positive solution $\{x_n\}$ of Eq. (1.4) converges to $\bar{x}$ as $n \to \infty$.

Proof. First, assume that $\{x_n\}$ does not oscillate about $\bar{x}$. We show that $\{x_n\}$ converges to $\bar{x}$ when $x_n - \bar{x}$ is eventually positive. The proof for the case that $x_n - \bar{x}$ is eventually negative is similar and will be omitted. By Lemma 2.1, we know that $\{x_n\}$ is bounded and persistent. Let $\limsup_{n\to\infty} x_n = R$, then $\bar{x} \le R < \infty$. We need to show that $R = \bar{x}$. First assume that $\{x_n\}$ is decreasing eventually. Then $\lim_{n\to\infty} x_n = R$. If $\bar{x} < R$, it follows from Eq. (1.4) that

$$x_{n+1} - x_n = b_n(f(x_n) - x_n) + c_n(f(x_{n-k}) - x_n),$$

and so, by noting $\sum_{n=0}^{\infty}(b_n + c_n) = \infty$,

$$x_{n+1} - x_0 = \sum_{m=0}^{n} (b_m(f(x_m) - x_m)) + \sum_{m=0}^{n} (c_m(f(x_{m-k}) - x_m)) \to -\infty \text{ as } n \to \infty$$

which is a contradiction. Hence, $R = \bar{x}$. Next, assume that $\{x_n\}$ is not eventually decreasing. Then, there is a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that

$$\lim_{m\to\infty} x_{n_m} = R \text{ and } x_{n_m} > x_{n_m-1} \tag{2.6}$$

and so it follows from Eq. (1.4) that

$$b_{n_m-1}(f(x_{n_m-1}) - x_{n_m}) + c_{n_m-1}(f(x_{n_m-1-k}) - x_{n_m}) = a_{n_m-1}(x_{n_m} - x_{n_m-1}) > 0.$$

Hence there is a subsequence $\{x_{n_{m_j}}\}$ of $\{x_{n_m}\}$ such that either

$$f(x_{n_{m_j}-1}) > x_{n_{m_j}}, \tag{2.7}$$

or

$$f(x_{n_{m_j}-1-k}) > x_{n_{m_j}}. \tag{2.8}$$

Suppose (2.7) holds for certain j. Since $x_{n_{m_j}-1} > \bar{x}$, $f(x_{n_{m_j}-1}) < x_{n_{m_j}-1}$. Then, from (2.7) we see that $x_{n_{m_j}-1} > x_{n_{m_j}}$ which contradicts (2.6). Hence, (2.7) can not hold for any j and so (2.8) must hold for $j = 1, 2, \ldots$. Since $x_{n_{m_j}-1-k} > \bar{x}$, $f(x_{n_{m_j}-1-k}) < x_{n_{m_j}-1-k}$ which yields $x_{n_{m_j}-1-k} > x_{n_{m_j}}$. Hence, $\lim\limits_{j\to\infty} x_{n_{m_j}-1-k} = R$ and $\lim\limits_{j\to\infty} f(x_{n_{m_j}-1-k}) = f(R)$. Then by taking limit on (2.8), we see that $f(R) \ge R$ which yields $R \le \bar{x}$. Hence, $R = \bar{x}$ and so it follows that $x_n \to \bar{x}$ as $n \to \infty$.

Next, assume that $\{x_n\}$ oscillates about $\bar{x}$. We show that $\{x_n\}$ converges to $\bar{x}$ also. To this end, let $y_n = x_n - \bar{x}$. Then $\{y_n\}$ satisfies the equation

$$y_{n+1} = a_n y_n + b_n(f(y_n + \bar{x}) - \bar{x}) + c_n(f(y_{n-k} + \bar{x}) - \bar{x}). \tag{2.9}$$

Since (2.5) holds, we can find a sufficiently small constant δ and a positive integer N such that

$$\frac{(c+\delta)(1-(a+\delta)^{k+1})}{(c-2\delta) + (a+\delta)^k(b+\delta)} L < 1, \tag{2.10}$$

and

$$\begin{cases} 0 < a - \delta < a_n < a + \delta < 1, \\ 0 < b - \delta < b_n < b + \delta < 1, \qquad n > N \\ 0 < c - \delta < c_n < c + \delta < 1, \end{cases} \tag{2.11}$$

Let $y_u < y_v$ with $v > u > N$ be two consecutive members of the solution $\{y_n\}$ such that

$$y_u \leq 0, \;\; y_{v+1} \leq 0 \text{ and } y_n > 0 \text{ for } u + 1 \leq n \leq v. \tag{2.12}$$

Let

$$y_r = \max\{y_{u+1}, y_{u+2}, \ldots, y_v\}. \tag{2.13}$$

We claim that

$$r - (u + 1) \leq k. \tag{2.14}$$

Suppose that $r - (u + 1) > k$. Then,

$$y_r \geq y_{r-1} > 0 \text{ and } y_r \geq y_{r-1-k} > 0.$$

By noting $y_{r-1} + \bar{x} > \bar{x}$, $y_{r-1-k} + \bar{x} > \bar{x}$ and $f(x) < x$ for $x > \bar{x}$. We see that

$$\begin{aligned} &b_{r-1}(f(y_{r-1} + \bar{x}) - y_r - \bar{x}) + c_{r-1}(f(y_{r-1-k} + \bar{x}) - y_r - \bar{x}) \\ &\qquad < \; b_{r-1}(y_{r-1} - y_r) + c_{r-1}(y_{r-1-k} - y_r) \leq \; 0. \end{aligned} \tag{2.15}$$

On the other hand, (2.9) yields

$$b_{r-1}(f(y_{r-1} + \bar{x}) - y_r - \bar{x}) + c_{r-1}(f(y_{r-1-k} + \bar{x}) - y_r - \bar{x}) = a_{r-1}(y_r - y_{r-1}) > 0,$$

which contradicts (2.15). Hence, (2.14) must hold. Now, observe that (2.9) yields

$$\frac{y_{n+1}}{\prod_{i=0}^{n} a_i} - \frac{y_n}{\prod_{i=0}^{n-1} a_i} = \frac{b_n}{\prod_{i=0}^{n} a_i}[f(y_n + \bar{x}) - \bar{x}] + \frac{c_n}{\prod_{i=0}^{n} a_i}[f(y_{n-k} + \bar{x}) - \bar{x}]. \tag{2.16}$$

Summing up (2.16) from u to $r - 1$, we see that

$$\begin{aligned} \frac{y_r}{\prod_{i=0}^{r-1} a_i} - \frac{y_u}{\prod_{i=0}^{u-1} a_i} = &\sum_{n=u}^{r-1} \left(\frac{b_n}{\prod_{i=0}^{n} a_i}[f(y_n + \bar{x}) - \bar{x}] \right) \\ &+ \sum_{n=u}^{r-1} \left(\frac{c_n}{\prod_{i=0}^{n} a_i}[f(y_{n-k} + \bar{x}) - \bar{x}] \right). \end{aligned}$$

Then it follows that

$$\begin{aligned} y_r = \prod_{i=0}^{r-1} a_i \Bigg(\frac{y_u}{\prod_{i=0}^{u-1} a_i} + &\sum_{n=u}^{r-1} \left(\frac{b_n}{\prod_{i=0}^{n} a_i}[f(y_n + \bar{x}) - \bar{x}] \right) \\ &+ \sum_{n=u}^{r-1} \left(\frac{c_n}{\prod_{i=0}^{n} a_i}[f(y_{n-k} + \bar{x}) - \bar{x}] \right) \Bigg) \end{aligned}$$

which can be written as

$$y_r = \prod_{i=0}^{r-1} a_i \left(\frac{1}{\prod_{i=0}^{u} a_i}(a_u y_u + b_u(f(y_u+\bar{x}) - \bar{x})) + \sum_{n=u+1}^{r-1} \left(\frac{b_n}{\prod_{i=0}^{n} a_i}[f(y_n+\bar{x}) - \bar{x}] \right) \right) + \prod_{i=0}^{r-1} a_i \left(\sum_{n=u}^{r-1} \left(\frac{c_n}{\prod_{i=0}^{n} a_i}[f(y_{n-k}+\bar{x}) - \bar{x}] \right) \right). \tag{2.17}$$

Since $a_n x + b_n f(x)$ is increasing in x, $y_u \le 0$ and $f(\bar{x}) = \bar{x}$, we see that

$$a_u(y_u+\bar{x}) + b_u f(y_u+\bar{x}) \le a_u\bar{x} + b_u f(\bar{x})$$

which yields $a_u y_u + b_u f(y_u+\bar{x}) - b_u\bar{x} \le 0$. In addition, noting (2.12) and $0 < y_n \le y_r$ for $u+1 \le n \le r-1$, we see that $f(y_n+\bar{x}) \le y_n + \bar{x} \le y_r + \bar{x}$ for $u+1 \le n \le r-1$. Hence, it follows from (2.17) that

$$y_r \le \prod_{i=0}^{r-1} a_i \left(y_r \sum_{n=u+1}^{r-1} \left(\frac{b_n}{\prod_{i=0}^{n} a_i} \right) + \sum_{n=u}^{r-1} \left(\frac{c_n}{\prod_{i=0}^{n} a_i}[f(y_{n-k}+\bar{x}) - \bar{x}] \right) \right)$$

and so

$$\left(1 - \prod_{i=0}^{r-1} a_i \sum_{n=u+1}^{r-1} \frac{b_n}{\prod_{i=0}^{n} a_i} \right) y_r \le \prod_{i=0}^{r-1} a_i \left(\sum_{n=u}^{r-1} \left(\frac{c_n}{\prod_{i=0}^{n} a_i}[f(y_{n-k}+\bar{x}) - \bar{x}] \right) \right). \tag{2.18}$$

Noting $f(x)$ is L-Lipschitz, we see that

$$|f(y_{n-k}+\bar{x}) - \bar{x}| = |f(y_{n-k}+\bar{x}) - f(\bar{x})| \le L|y_{n-k}|, \quad n \ge k. \tag{2.19}$$

Since $\{x_n\}$ is bounded, there is a positive constant T such that $|y_n| = |x_n - \bar{x}| \le T$, $n \ge 0$. Then, from (2.19), we see that

$$|f(y_{n-k}+\bar{x}) - \bar{x}| \le LT, \quad n \ge k.$$

Hence, it follows from (2.18) that

$$y_r \le \frac{\sum_{n=u}^{r-1} \left(\prod_{i=n+1}^{r-1} a_i \right) c_n}{1 - \sum_{n=u+1}^{r-1} \left(\prod_{i=n+1}^{r-1} a_i \right) b_n} LT. \tag{2.20}$$

By noting (2.11), it follows from (2.20) that

$$y_r \le \frac{\sum_{n=u}^{r-1} \left(\prod_{i=n+1}^{r-1} (a+\delta) \right) (c+\delta)}{1 - \sum_{n=u+1}^{r-1} \left(\prod_{i=n+1}^{r-1} (a+\delta) \right) (b+\delta)} LT. \tag{2.21}$$

and so

$$y_r \le \frac{(c+\delta)(1-(a+\delta)^{k+1})}{(c-2\delta) + (a+\delta)^k (b+\delta)} LT. \tag{2.22}$$

Since y_u and y_v are two arbitrary members of the solution with property (2.12), we see that there is a positive integer N_1' such that

$$y_n \leq \frac{(c+\delta)(1-(a+\delta)^{k+1})}{(c-2\delta)+(a+\delta)^k(b+\delta)}LT \qquad n \geq N_1'.$$

Then, by a similar argument, it can be shown that there is a positive integer N_1'' such that

$$y_n \geq -\frac{(c+\delta)(1-(a+\delta)^{k+1})}{(c-2\delta)+(a+\delta)^k(b+\delta)}LT, \qquad n \geq N_1''.$$

Hence, there is a positive integer N_1 such that

$$|y_n| \leq \frac{(c+\delta)(1-(a+\delta)^{k+1})}{(c-2\delta)+(a+\delta)^k(b+\delta)}LT, \qquad n \geq N_1. \tag{2.23}$$

Now, by noting (2.23) and the Lipschitz property of f, we see that

$$|f(y_n+\bar{x})-\bar{x}| = |f(y_n+\bar{x})-f(\bar{x})| \leq L|y_n| \leq \frac{(c+\delta)(1-(a+\delta)^{k+1})}{(c-2\delta)+(a+\delta)^k(b+\delta)}L^2T, \quad n \geq N_1+k,$$

and

$$|f(y_{n-k}+\bar{x})-\bar{x}| = |f(y_{n-k}+\bar{x})-f(\bar{x})| \leq L|y_{n-k}| \leq \frac{(c+\delta)(1-(a+\delta)^{k+1})}{(c-2\delta)+(a+\delta)^k(b+\delta)}L^2T, \quad n \geq N_1+k.$$

Let y_u and y_v be two consecutive members of the solution $\{y_n\}$ with $N_1+k \leq u < v$ such that (2.12) holds. Let y_r be defined by (2.13). By a similar argument, we may show that (2.14) holds and

$$y_r \leq \left[\frac{(c+\delta)(1-(a+\delta)^{k+1})}{(c-2\delta)+(a+\delta)^k(b+\delta)}L\right]^2 T.$$

Then it follows that

$$y_r \leq \left[\frac{(c+\delta)(1-(a+\delta)^{k+1})}{(c-2\delta)+(a+\delta)^k(b+\delta)}L\right]^2 T, \qquad u \leq n \leq v,$$

and so again by noting y_u and y_v are two arbitrary members of the solution with property (2.12), there is a positive integer $N_2' \geq N_1+k$ such tat

$$y_n \leq \left[\frac{(c+\delta)(1-(a+\delta)^{k+1})}{(c-2\delta)+(a+\delta)^k(b+\delta)}L\right]^2 T, \qquad n \geq N_2'.$$

Similarly, it can be shown that there is a positive integer $N_2'' \geq N_1+k$ such that

$$y_n \geq -\left[\frac{(c+\delta)(1-(a+\delta)^{k+1})}{(c-2\delta)+(a+\delta)^k(b+\delta)}L\right]^2 T, \qquad n \geq N_2''.$$

Hence, there is a positive integer $N_2 \geq N_1+k$ such that

$$|y_n| \leq \left[\frac{(c+\delta)(1-(a+\delta)^{k+1})}{(c-2\delta)+(a+\delta)^k(b+\delta)}L\right]^2 T, \qquad n \geq N_2.$$

Finally, by induction, we find that for any positive integer m, there is a positive integer N_m with $N_m \to \infty$ as $m \to \infty$ such that

$$|y_n| \leq \left[\frac{(c+\delta)(1-(a+\delta)^{k+1})}{(c-2\delta)+(a+\delta)^k(b+\delta)}L\right]^m T, \qquad n \geq N_m.$$

Then, by noting (2.10), we see that $y_n \to 0$ as $n \to \infty$, and so it follows that $x_n \to \bar{x}$ as $n \to \infty$. The proof is complete. □

3. Applications

In this section, we apply our results obtained in the last section to a difference system derived from mathematical biology.

Consider the following difference system

$$\begin{cases} x_{n+1} = (1-\tau_n)f(x_n) + \tau_n y_n, \\ y_{n+1} = (1-\rho_n)y_n + \rho_n f(x_n), \quad n = 0, 1, \dots, \\ x_0 \geq 0, \ y_0 \geq 0, \ x_0 + y_0 > 0, \end{cases} \tag{3.1}$$

where $\{\tau_n\}, \{\rho_n\}$ are arbitrary sequences in $(0,1)$, and $\tau_n + \rho_n \geq 1$. When $\tau_n = \epsilon$ and $\rho_n = \epsilon$ are same constant, Sys. (3.1) reduces to Sys. (1.2) which is the population model mentioned in Section 1.

By a simple calculation, Sys. (3.1) can be converted into the second order difference equation

$$x_{n+1} = \frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}}x_n + (1-\tau_n)f(x_n) + \frac{\tau_n(\rho_{n-1}+\tau_{n-1}-1)}{\tau_{n-1}}f(x_{n-1}). \tag{3.2}$$

Clearly, Eq. (3.2) is in the form of Eq. (1.4) with $a_n = \frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}}$, $b_n = 1-\tau_n$, $c_n = \frac{\tau_n(\rho_{n-1}+\tau_{n-1}-1)}{\tau_{n-1}}$ and $k = 1$. Let us assume that f has a unique positive fixed point $\bar{x}$, and that f satisfies the negative feedback condition (1.6). It is easy to see that $(\bar{x}, \bar{x})$ is the unique positive equilibrium of Sys. (3.1). By Theorem 2.2, we have the following result

Theorem 3.1. *Assume that*

$$\lim_{n\to\infty} \frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} = 1 \tag{3.3}$$

and

$$\sum_{n=0}^{\infty}\left(1 + \frac{(\rho_{n-1}-1)\tau_n}{\tau_{n-1}}\right) = \infty. \tag{3.4}$$

Then every positive solution (x_n, y_n) *of Sys.* (3.1) *tends to its positive equilibrium* $(\bar{x}, \bar{x})$ *as* $n \to \infty$.

Proof. Since $a_n = \frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} \to 1$ as $n \to \infty$, and

$$\sum_{n=0}^{\infty}(b_n + c_n) = \sum_{n=0}^{\infty}\left(1 + \frac{(\rho_{n-1}-1)\tau_n}{\tau_{n-1}}\right) = \infty.$$

By Theorem 2.1, every positive solution $\{x_n\}$ of Eq. (3.2) converges to $\bar{x}$. Now, noting $\{\tau_n\}, \{\rho_n\}$ are arbitrary sequences in $(0,1)$, and $\tau_n + \rho_n \geq 1$, we see that

$$0 \leq \frac{1-\rho_{n-1}}{\tau_{n-1}} \leq 1 \text{ for all } n. \tag{3.5}$$

Let $r = \liminf_{n\to\infty} \frac{1-\rho_{n-1}}{\tau_{n-1}}$. We claim that $r = 1$. Otherwise $r < 1$. Then there is a constant $\eta > 0$ such that when n is sufficiently large,

$$\frac{1-\rho_{n-1}}{\tau_{n-1}} \leq 1-\eta$$

which yields

$$\frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} \leq \tau_n(1-\eta) \leq 1-\eta.$$

Hence, it follows that $\frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} \nrightarrow 1$ as $n \to \infty$ which contradicts (3.4). By noting (3.4), (3.5) and $r = 1$, we see that $\frac{1-\rho_{n-1}}{\tau_{n-1}} \to 1$ and $\tau_n \to 1$ as $n \to \infty$. Then it follows from Sys. (3.1) that

$$y_n = \frac{1}{\tau_n}(x_{n+1} - (1-\tau_n)f(x_n)) \to \bar{x} \text{ as } n \to \infty$$

Hence, every positive solution (x_n, y_n) of Sys. (3.1) converges to $(\bar{x}, \bar{x})$ as $n \to \infty$. The proof is complete. □

Next, by applying Theorem 2.3, we have the following result on the global attractivity of Sys. (3.1).

Theorem 3.2. *Assume that the limits*

$$\lim_{n\to\infty} \rho_n = \rho \text{ and } \lim_{n\to\infty} \tau_n = \tau \tag{3.6}$$

exist with $\rho > 0$ and $\tau > 0$. Suppose also that $\frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}}x + (1-\tau_n)f(x)$ is increasing in x and that f is L-Lipschitz with

$$\frac{(\rho+\tau-1)(2-\rho)}{\tau}L < 1. \tag{3.7}$$

Then every positive solution (x_n, y_n) of Sys. (3.1) tends to $(\bar{x}, \bar{x})$ as $n \to \infty$.

Proof. Notice that Sys. (3.1) can be converted to Eq. (3.2) which is in the form of Eq. (1.4) with $k = 1$ and

$$a_n = \frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} \to a = 1-\rho < 1, \; b_n = 1-\tau_n \to b = 1-\tau < 1,$$

and

$$c_n = \frac{\tau_n(\rho_{n-1}+\tau_{n-1}-1)}{\tau_{n-1}} \to c = \rho+\tau-1 \text{ as } n \to \infty.$$

We see that

$$\frac{c(1-a^{k+1})}{c+a^k b}L = \frac{(\rho+\tau-1)(2-\rho)}{\tau}L < 1. \tag{3.8}$$

In addition, noting

$$a_n x + b_n f(x) = \frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} x + (1-\tau_n) f(x),$$

so $a_n x + b_n f(x)$ is increasing. Hence, by Theorem 2.3, every positive solution of Eq. (3.2) converges to $\bar{x}$. Then it follows from Sys. (3.1) that

$$y_n = \frac{1}{\tau_n}(x_{n+1} - (1-\tau_n) f(x_n)) \to \frac{1}{\tau}(\bar{x} - (1-\tau) f(\bar{x})) = \bar{x}.$$

Hence every positive solution (x_n, y_n) of Sys. (3.1) converges to $(\bar{x}, \bar{x})$ as $n \to \infty$. The proof is complete. ☐

When $\tau_n = \rho_n = \xi_n$, Sys. (3.1) reduces to the following difference system

$$\begin{cases} x_{n+1} = (1-\xi_n) f(x_n) + \xi_n y_n, \\ y_{n+1} = (1-\xi_n) y_n + \xi_n f(x_n), \quad n = 0, 1, \dots, \\ x_0 \geq 0, \ y_0 \geq 0, \ x_0 + y_0 > 0, \end{cases} \tag{3.9}$$

where $\{\xi_n\}$ is arbitrary sequence in $\left[\frac{1}{2}, 1\right)$. In this case Sys. (3.9) is converted into the following second order difference equation

$$x_{n+1} = \xi_n \frac{1-\xi_{n-1}}{\xi_{n-1}} x_n + (1-\xi_n) f(x_n) + \xi_n \frac{2\xi_{n-1}-1}{\xi_{n-1}} f(x_{n-1}). \tag{3.10}$$

Eq. (3.10) is in the form of Eq. (1.4) with $a_n = \xi_n \frac{1-\xi_{n-1}}{\xi_{n-1}}$, $b_n = 1-\xi_n$, $c_n = \xi_n \frac{2\xi_{n-1}-1}{\xi_{n-1}}$ and $k = 1$.

The following result is a direct consequence of Theorem 3.1.

Corollary 3.3. *Assume that*

$$\lim_{n\to\infty} \xi_n \frac{1-\xi_{n-1}}{\xi_{n-1}} = 1 \tag{3.11}$$

and

$$\sum_{n=0}^{\infty} \left(1 + \frac{\xi_n(\xi_{n-1}-1)}{\xi_{n-1}}\right) = \infty. \tag{3.12}$$

Then every positive solution (x_n, y_n) of Sys. (3.9) tends to $(\bar{x}, \bar{x})$ as $n \to \infty$

By Theorem 3.2, we have the following result immediately.

Corollary 3.4. *Assume that the limit*

$$\lim_{n\to\infty} \xi_n = \xi \tag{3.13}$$

exist. Suppose also that $\xi_n \frac{1-\xi_{n-1}}{\xi_{n-1}} x + (1-\xi_n) f(x)$ is increasing in x and that f is L-Lipschitz with

$$(2 - 1/\xi)(2\xi - 1) L < 1. \tag{3.14}$$

Then every positive solution (x_n, y_n) of Sys. (3.9) tends to $(\bar{x}, \bar{x})$ as $n \to \infty$.

In the following, we apply our results obtained above to Sys. (3.1) with the following two functions

$$f(x) = \sigma x e^{-\gamma x} \text{ and } f(x) = \frac{w}{q + x^m}$$

where $\sigma > 1$, $\gamma > 1$, $w > 0$, $q > 0$ and $m \geq 1$ are constants. The chaotic behavior and global stability of positive solutions of Sys. (1.2) with the unimodal function $f(x) = \sigma x e^{-\gamma x}$ have been studied in [14] and in [1,2] respectively, whereas the global attractivity of positive solutions of Sys. (1.2) with the decreasing function $f(x) = \frac{w}{q+x^m}$ has been studied in [1,25].

First, consider the function $f(x) = \sigma x e^{-\gamma x}$. Then Sys. (3.1) becomes

$$\begin{cases} x_{n+1} = (1-\tau_n)\sigma x_n e^{-\gamma x_n} + \tau_n y_n, \\ y_{n+1} = (1-\rho_n) y_n + \rho_n \sigma x_n e^{-\gamma x_n}, \quad n = 0, 1, \ldots, \\ x_0 \geq 0,\ y_0 \geq 0,\ x_0 + y_0 > 0, \end{cases} \tag{3.15}$$

and Eq. (3.2) becomes

$$x_{n+1} = \frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} x_n + (1-\tau_n)\sigma x_n e^{-\gamma x_n} + \frac{\tau_n(\rho_{n-1} + \tau_{n-1} - 1)}{\tau_{n-1}} \sigma x_{n-1} e^{-\gamma x_{n-1}}. \tag{3.16}$$

In addition, Sys. (3.9) becomes

$$\begin{cases} x_{n+1} = (1-\xi_n)\sigma x_n e^{-\gamma x_n} + \xi_n y_n, \\ y_{n+1} = (1-\xi_n) y_n + \xi_n \sigma x_n e^{-\gamma x_n}, \quad n = 0, 1, \ldots, \\ x_0 \geq 0,\ y_0 \geq 0,\ x_0 + y_0 > 0, \end{cases} \tag{3.17}$$

and Eq. (3.10) becomes

$$x_{n+1} = \xi_n \frac{1-\xi_{n-1}}{\xi_{n-1}} x(n) + (1-\xi_n)\sigma x_n e^{-\gamma x_n} + \xi_n \frac{2\xi_{n-1} - 1}{\xi_{n-1}} \sigma x_{n-1} e^{-\gamma x_{n-1}}. \tag{3.18}$$

It is clear that f has a unique positive fixed point $\bar{x} = \frac{\ln \sigma}{\gamma}$, and that f satisfies the negative feedback condition (1.6). Noting that

$$\left(\frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} x + (1-\tau_n)\sigma x e^{-\gamma x} \right)' = \frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} + (1-\tau_n)(1-\gamma x)\sigma e^{-\gamma x}$$

and

$$\left(\frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} x + (1-\tau_n)\sigma x e^{-\gamma x} \right)'' = (1-\tau_n)(\gamma^2 x - 2\gamma)\sigma e^{-\gamma x},$$

we see that when

$$\frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}(1-\tau_n)} e^2 \geq \sigma, \tag{3.19}$$

$$\begin{aligned} \left(\frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} x + (1-\tau_n)\sigma x e^{-\gamma x} \right)' &\geq \left(\frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} x + (1-\tau_n)\sigma x e^{-\gamma x} \right)' \Big|_{x=\frac{2}{\gamma}} \\ &= \frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} - (1-\tau_n)\sigma e^{-2} \geq 0, \end{aligned}$$

and so

$$\frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}}x+(1-\tau_n)\sigma x e^{-\gamma x}$$

is increasing in x. Now, observing that

$$f'(x)=(1-\gamma x)\sigma e^{-\gamma x}$$

and

$$f''(x)=(\gamma^2 x-2\gamma)\sigma e^{-\gamma x}.$$

we see that $\sup_{x\geq 0}\{|f'(x)|\}=f'(0)=\sigma$. Hence, f is L-Lipschitz with $L=\sigma$. Then by Theorem 3.1 and Theorem 3.2, every positive solution (x_n,y_n) of Sys. (3.15) tends to its positive equilibrium $\left(\frac{\ln\sigma}{\gamma},\frac{\ln\sigma}{\gamma}\right)$ as $n\to\infty$ if one of the following is satisfied
(i) (3.4) and (3.4) hold.
(ii) (3.6) and (3.19) hold with

$$\frac{(\rho+\tau-1)(2-\rho)}{\tau}\sigma<1. \tag{3.20}$$

In addition, by Corollary 3.3 and Corollary 3.4, every positive solution (x_n,y_n) of Sys. (3.17) tends to its positive equilibrium $\left(\frac{\ln\sigma}{\gamma},\frac{\ln\sigma}{\gamma}\right)$ as $n\to\infty$ if one of the following is satisfied
(i) (3.11) and (3.12) hold.
(ii) (3.13) holds with

$$\frac{\xi_n(1-\xi_{n-1})}{\xi_{n-1}(1-\xi_n)}e^2\geq\sigma$$

and

$$(2-1/\xi)(2\xi-1)\sigma<1.$$

Next consider the function $f(x)=\frac{w}{q+x^m}$. Then Sys. (3.1) becomes

$$\begin{cases} x_{n+1}=(1-\tau_n)\frac{w}{q+x_n^m}+\tau_n y_n, \\ y_{n+1}=(1-\rho_n)y_n+\rho_n\frac{w}{q+x_n^m}, \quad n=0,1,\ldots, \\ x_0\geq 0,\ y_0\geq 0,\ x_0+y_0>0, \end{cases} \tag{3.21}$$

and Eq. (3.2) becomes

$$x_{n+1}=\frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}}x_n+(1-\tau_n)\frac{w}{q+x_n^m}+\frac{\tau_n(\rho_{n-1}+\tau_{n-1}-1)}{\tau_{n-1}}\frac{w}{q+x_{n-1}^m}. \tag{3.22}$$

In addition, Sys. (3.9) becomes

$$\begin{cases} x_{n+1}=(1-\xi_n)\frac{w}{q+x_n^m}+\xi_n y_n, \\ y_{n+1}=(1-\xi_n)y_n+\xi_n\frac{w}{q+x_n^m}, \quad n=0,1,\ldots, \\ x_0\geq 0,\ y_0\geq 0,\ x_0+y_0>0, \end{cases} \tag{3.23}$$

and Eq. (3.10) becomes

$$x_{n+1}=\xi_n\frac{1-\xi_{n-1}}{\xi_{n-1}}x(n)+(1-\xi_n)\frac{w}{q+x_n^m}+\left(\xi_n\frac{2\xi_{n-1}-1}{\xi_{n-1}}\right)\frac{w}{q+x_{n-1}^m}. \tag{3.24}$$

Clearly, f is decreasing and has a unique positive fixed point $\bar{x}$, and f satisfies the negative feedback condition (1.6). In addition, we see that

$$f'(x) = \frac{-wmx^{m-1}}{(q+x^m)^2},$$

and

$$f''(x) = \frac{-wmx^{m-2}((m-1)-(m+1)x^m)}{(q+x^m)^3}.$$

Hence, it is easy to see that $f'(x)$ has minimum at $x^* = \left(\frac{q(m-1)}{m+1}\right)^{(1/m)}$, and

$$f'(x^*) = \frac{-w}{4m} q^{-(1+(1/m))}(m-1)^{(1-(1/m))}(1+m)^{(1+(1/m))}.$$

Then, f is L-Lipschitz with

$$L = \frac{w}{4m} q^{-(1+(1/m))}(m-1)^{(1-(1/m))}(1+m)^{(1+(1/m))}.$$

Noting that

$$\begin{aligned}(a_n x + b_n f(x))' &= \left(\frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} x + (1-\tau_n)\frac{w}{q+x^m}\right)' \\ &\geq \frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} + (1-\tau_n)\frac{-wmx^{m-1}}{(q+x^m)^2}|_{x=x^*},\end{aligned}$$

we see that when

$$\frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}(1-\tau_n)} \geq \frac{w}{4m} q^{-(1+(1/m))}(m-1)^{(1-(1/m))}(1+m)^{(1+(1/m))}, \tag{3.25}$$

$$\frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}} - (1-\tau_n)\frac{w}{4m} q^{-(1+(1/m))}(m-1)^{(1-(1/m))}(1+m)^{(1+(1/m))} > 0,$$

and so $a_n x + b_n f(x)$ is increasing in x. Hence, by Theorem 3.1 and Theorem 3.2 , every positive solution (x_n, y_n) of Sys. (3.21) tends to its positive equilibrium $(\bar{x}, \bar{x})$ as $n \to \infty$ if one of the following is satisfied
(i) (3.4) and (3.4) hold.
(ii) (3.6) and (3.25) hold with

$$\frac{w}{4m} q^{-(1+(1/m))}(m-1)^{(1-(1/m))}(1+m)^{(1+(1/m))}\frac{(\rho+\tau-1)(2-\rho)}{\tau} < 1 \tag{3.26}$$

In addition, by Corollary 3.3 and Corollary 3.4, every positive solution (x_n, y_n) of Sys. (3.23) tends to its positive equilibrium $(\bar{x}, \bar{x})$ as $n \to \infty$ if one of the following is satisfied
(i) (3.11) and (3.12) hold.
(ii) (3.13) holds with

$$\frac{\xi_n(1-\xi_{n-1})}{\xi_{n-1}(1-\xi_n)} \geq \frac{w}{4m} q^{-(1+(1/m))}(m-1)^{(1-(1/m))}(1+m)^{(1+(1/m))}$$

and

$$\frac{w}{4m} q^{-(1+(1/m))}(m-1)^{(1-(1/m))}(1+m)^{(1+(1/m))}(2-1/\xi)(2\xi-1) < 1.$$

Example 1. Consider the following difference system

$$\begin{cases} x_{n+1} = e^{\frac{-1}{n+2}} f(x_n) + \left(1 - e^{\frac{-1}{n+2}}\right) y_n, \\ y_{n+1} = 2^{-(n+2)} y_n + \left(1 - 2^{-(n+2)}\right) f(x_n), \quad n = 0, 1, \ldots, \\ x_0 \geq 0,\ y_0 \geq 0,\ x_0 + y_0 > 0, \end{cases} \tag{3.27}$$

Note that Sys. (3.27) is in the form of Sys. (3.1) with $\tau_n = 1 - e^{\frac{-1}{n+2}}$, and $\rho_n = 1 - 2^{-(n+2)}$. Thus, in this case, (3.2) reduces to

$$\begin{aligned} x_{n+1} = {} & \frac{(1 - 2^{-2-n}) e^{-1/(1+n)}}{1 - 2^{-1-n}} x_n + e^{\frac{-1}{n+2}} f(x_n) \\ & + \frac{(1 - 2^{-2-n})(1 - 2^{-1-n} - e^{\frac{-1}{1+n}})}{1 - 2^{-1-n}} f(x_{n-1}). \end{aligned} \tag{3.28}$$

It is clear that

$$\frac{(1 - 2^{-2-n}) e^{-1/(1+n)}}{1 - 2^{-1-n}} \to 1 \text{ as } n \to \infty,$$

and

$$\sum_{n=0}^{\infty} \left(e^{\frac{-1}{n+2}} + \frac{(1 - 2^{-2-n})(1 - 2^{-1-n} - e^{\frac{-1}{1+n}})}{1 - 2^{-1-n}} \right) = \infty.$$

From the above discussion, we see that when $f(x) = 6xe^{-2x}$, every positive solution (x_n, y_n) of Sys. (3.27) tends to $\left(\frac{\ln(6)}{2}, \frac{\ln(6)}{2}\right)$ as $n \to \infty$; when $f(x) = \frac{2}{1+x^3}$, every positive solution (x_n, y_n) of Sys. (3.27) tends to $(1, 1)$ as $n \to \infty$.

Example 2. Consider the following difference system

$$\begin{cases} x_{n+1} = \left(1 - (\frac{3}{4} + e^{-(n+2)})\right) f(x_n) + \left(\frac{3}{4} + e^{-(n+2)}\right) y_n, \\ y_{n+1} = \left(1 - \frac{2^{n+1}+2}{3(2^{n+1}+1)}\right) y_n + \frac{2^{n+1}+2}{3(2^{n+1}+1)} f(x_n), \qquad n = 0, 1, \ldots, \\ x_0 \geq 0,\ y_0 \geq 0,\ x_0 + y_0 > 0, \end{cases} \tag{3.29}$$

Note that Sys. (3.29) is in the form of Sys. (3.1) with $\tau_n = \frac{3}{4} + e^{-(n+2)}$, $\rho_n = \frac{2^{n+1}+2}{3(2^{n+1}+1)}$, and

$$\tau_n + \rho_n = \frac{3}{4} + e^{-(n+2)} + \frac{2^{n+1} + 2}{3(2^{n+1} + 1)} > \frac{3}{4} + \frac{2^{n+1} + 1}{3(2^{n+1} + 1)} + \frac{1}{3(2^{n+1} + 1)} > \frac{3}{4} + \frac{1}{3} > 1,$$

Thus, in this case, (3.2) becomes

$$\begin{aligned} x_{n+1} = {} & \frac{(2^{n+1})(3e^{n+2} + 4)}{3e(2^n + 1)(3e^{n+1} + 4)} x_n + \frac{1}{4}(1 - 4e^{-(n+2)}) f(x_n) \\ & + \frac{e^{-(n+2)}((3)2^{n+2} + 5e^{n+1} + 2^2 e^{n+1} + 12)((3)2^{n+2} + 4)}{12(2^n + 1)(3e^{n+1} + 4)} f(x_{n-1}). \end{aligned} \tag{3.30}$$

Clearly,

$$\tau_n = \frac{3}{4} + e^{-(n+2)} \to \tau = \frac{3}{4} \text{ as } n \to \infty.$$

and

$$\rho_n = \frac{2^{n+1}+2}{3(2^{n+1}+1)} \to \rho = \frac{1}{3}$$

Let $f(x) = \sigma x e^{-x}$ where $\sigma = 5$. Observe that

$$\frac{(\rho+\tau-1)(2-\rho)}{\tau}\sigma = \left(\frac{5}{27}\right)(5) = \frac{25}{27} < 1.$$

yields (3.20). In addition, by noting that

$$\frac{\left(\frac{3}{4}+e^{-(n+2)}\right)\left(1-\frac{2^n+2}{3(2^n+1)}\right)}{\left(\frac{3}{4}+e^{-(n+1)}\right)\left(1-\left(\frac{3}{4}+e^{-(n+2)}\right)\right)} > 1$$

we see that

$$\frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}(1-\tau_n)}e^2 = \frac{\left(\frac{3}{4}+e^{-(n+2)}\right)\left(1-\frac{2^n+2}{3(2^n+1)}\right)}{\left(\frac{3}{4}+e^{-(n+1)}\right)\left(1-\left(\frac{3}{4}+e^{-(n+2)}\right)\right)}e^2 \geq e^2 > 5.$$

Hence (3.19) is satisfied. From the above discussion, we see that every positive solution (x_n, y_n) of Sys. (3.29) with $f(x) = 5xe^{-x}$ tends to $(\ln(5), \ln(5))$ as $n \to \infty$.

Next, let $f(x) = \frac{w}{q+x^m}$ where $w = q = 1$ and $m = 7$. Then, by noting

$$\begin{aligned}\frac{\tau_n(1-\rho_{n-1})}{\tau_{n-1}(1-\tau_n)} &\geq 1 > \left(\frac{4}{7}\right)\left(\frac{2}{3}\right)^{2/7} \\ &\geq \frac{w}{4m}q^{-(1+(1/m))}(m-1)^{(1-(1/m))}(1+m)^{(1+(1/m))},\end{aligned}$$

and

$$\begin{aligned}\frac{w}{4m}q^{-(1+(1/m))}(m-1)^{(1-(1/m))}(1+m)^{(1+(1/m))}&\frac{(\rho+\tau-1)(2-\rho)}{\tau} \\ &= \left(\frac{4}{7}\right)\left(\frac{2}{3}\right)^{2/7}\left(\frac{5}{27}\right) < 1.\end{aligned}$$

we see that (3.25) and (3.26) are satisfied. Hence it follows from the above discussion that every positive solution (x_n, y_n) of Sys. (3.29) with $f(x) = \frac{1}{3+x^7}$ tends to $(\bar{x}, \bar{x}) \approx (0.33, 0.33)$ as $n \to \infty$.

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