ARCHIVUM MATHEMATICUM (BRNO)
Tomus 61 (2025), 101–132

A GENERALISED RICCI-HESSIAN EQUATION ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we prove new rigidity results related to some generalised Ricci-Hessian equation on Riemannian manifolds.

1. Introduction

In this article, which follows [8], we continue investigating those Riemannian manifolds (M^n, g) supporting a non-identically-vanishing function f satisfying what we call the *generalised Ricci-Hessian equation* [8, Eq. (1)]

$$\nabla^2 f = -f \cdot \mathrm{Ric} \tag{1}$$

on M, where $\nabla^2 f := \nabla\nabla f$ denotes the Hessian of f and Ric the Ricci-tensor of (M^n, g), both seen as $(1,1)$-tensor fields. Recall that this equation was first considered when studying the so-called *skew-Killing-spinor-equation* [9], where f is a particular function built out of a so-called skew-Killing spinor, see [8] for more details. Recall also that, although equation (1) looks like those considered by other authors in the search of warped product Einstein metrics [14, 12], Lorentzian Einstein metrics [6], quasi-Einstein metrics [4, 10] or gradient Ricci solitons [7], it is not connected to any of those frameworks and therefore needs very different kinds of techniques, hence leading to very different results. Again, we refer to the introduction of [8] for more references and details.

In [8], we proved that, provided sufficiently many symmetries preserving a solution f are available on the underlying manifold (M^n, g), only one of the following can occur: unless f is constant and then (M^n, g) is Ricci-flat, either (M^n, g) is isometric to the Riemannian product of a real interval with a Ricci-flat manifold and f is an affine-linear function on the interval; or (M^n, g) is isometric to the Riemannian product of a Ricci-flat manifold with either the 2-sphere or the hyperbolic plane and f is the trivial extension of a solution to the Obata resp. Tashiro equation on the second factor.

In this article, we show that, in many further situations, some of which are more general than those from [8], mostly only those two possibilities can occur: namely when M has harmonic curvature tensor (Theorem 2.3), is a warped product

2020 *Mathematics Subject Classification*: primary 53C20; secondary 58J60.

Key words and phrases: Obata equation, Tashiro equation, virtual Einstein equation.
Received June 17, 2025. Editor J. Slovák.
DOI: 10.5817/AM2025-3-101

(Theorem 3.4), when the space of solutions is of dimension at least 2 (Theorem 4.1), when M is homogeneous (Theorem 5.1) and when M is Kähler (Theorem 6.1).

The article is structured as follows. After preliminary remarks in Section 2, we describe and partially classify those warped products carrying solutions to (1). In Section 4, we turn to the case where the space of solutions to (1) is at least 2-dimensional. Section 5 is dedicated to the homogeneous case, which remains partially open. We dedicate Section 6 to the case where the manifold is Kähler. We conclude by an outlook (Section 7) about further work related to the above equation.

In order for the article to remain as self-contained as possible, we included parts of [8] in Section 2.

We underline that no full classification is available yet. This is the object of future work.

Acknowledgement. Part of this work was done while the second-named author received the support of the Humboldt Foundation, the French embassy in Beirut via the SAFAR programme and the Alfried Krupp Wissenschaftskolleg, which he would like to thank. We thank the Centre International de Rencontres Mathématiques (CIRM) where the article was finished. We also thank Ines Kath for her support and interest in the first part of that work and Gilles Carron for very interesting feedback. Last but not the least, we thank the referee as well as the editors of Archivum Mathematicum for the positive remarks and efficient work.

2. Preliminary remarks

We start with preliminary results, some of which are already contained in [8] but, for the sake of self-containedness, we give and reprove them here. From now on, we shall denote by S the scalar curvature of M and, for any function h on M, by ∇h the gradient vector field of h w.r.t. g on M. First observe that the equation $\nabla^2 f = -f \cdot \mathrm{Ric}$ is of course linear in f but is also invariant under metric rescaling: if $\overline{g} = \lambda^2 g$ for some nonzero real number λ, then $\overline{\nabla}^2 f = \lambda^{-2}\overline{\nabla}^2 f$ (this comes from the rescaling of the gradient) and $\overline{\mathrm{Ric}} = \lambda^{-2}\mathrm{Ric}$. Let us denote by

$$W(M^n, g) := \left\{ f \in C^\infty(M, \mathbb{R}) \mid \nabla^2 f = -f \cdot \mathrm{Ric} \right\}$$

the real vector space of all smooth functions satisfying (1) on (M^n, g).

Lemma 2.1 below corresponds to [8, Lemma 2.1] expanded with claims 3, 7 and 8.

Lemma 2.1. *Let (M^n, g) be any connected Riemannian manifold carrying a smooth real-valued function f satisfying* (1) *on M.*

1. *The gradient vector field ∇f of f w.r.t. g satisfies*

$$\mathrm{Ric}(\nabla f) = \frac{S}{2}\nabla f + \frac{f}{4}\nabla S\,. \tag{2}$$

2. *There exists a real constant μ such that*

$$f\Delta f + 2|\nabla f|^2 = \mu\,. \tag{3}$$

3. *The identity*

$$f|\mathrm{Ric}|^2 = \frac{fS^2}{2} - \frac{1}{4}\langle \nabla f, \nabla S\rangle + \frac{f}{4}\Delta S \tag{4}$$

   holds on M*.*

4. *If* $n > 2$ *and* f *is everywhere positive or negative, then* f *solves* (1) *if and only if, setting* $u := \frac{1}{2-n}\ln|f|$*, the metric* $\overline{g} := e^{2u}g$ *satisfies* $\overline{\mathrm{ric}} = (\overline{\Delta}u)\overline{g} - (n-2)(n-3)du \otimes du$ *on* M *and in that case* $\overline{\Delta}u = -\frac{\mu}{n-2}e^{2(n-3)u}$*. In particular, if* $n = 3$*, the existence of a positive solution* f *to* (1) *is equivalent to* $(M, f^{-2}g)$ *being Einstein with scalar curvature* $-3\overline{\Delta}\ln|f|$*.*
5. *If* M *is closed and* f *is everywhere positive or negative, then* f *is constant on* M*.*
6. *If nonempty, the vanishing set* $N_0 := f^{-1}(\{0\})$ *of* f *is a scalar-flat totally geodesic hypersurface of* M*.*
7. *For any* $x \in M$ *and all* $X, Y \in T_xM$*, the identity*

$$R_{X,Y}\nabla f = -X(f)\mathrm{Ric}(Y) + Y(f)\mathrm{Ric}(X) - f\left((\nabla_X \mathrm{Ric})Y - (\nabla_Y \mathrm{Ric})X\right) \tag{5}$$

   holds on M*. As a consequence, at any critical point of* f*, the Ricci-tensor must be Codazzi.*

8. *The dimension of* $W(M^n, g)$ *is always at most* $n+1$*.*
9. *If furthermore* M *is Einstein or* 2*-dimensional, then* M *is Ricci-flat or* $n = 2$ *and in that case* M *has constant curvature. In particular, when* (M^2, g) *is complete, there exists a nonconstant function* f *satisfying* (1) *if and only if, up to rescaling the metric, the manifold* (M^2, g) *is isometric to either the round sphere* $\mathbb{S}^2$ *and* f *is a nonzero eigenfunction associated to the first positive Laplace eigenvalue; or to flat* $\mathbb{R}^2$ *or cylinder* $\mathbb{S}^1 \times \mathbb{R}$ *and* f *is an affine-linear function; or to the hyperbolic plane* $\mathbb{H}^2$ *and* f *is a solution to the Tashiro equation* $\nabla^2 f = f \cdot \mathrm{Id}$*.*
10. *If* S *is constant, then outside the set of critical points of* f*, the vector field* $\nu := \frac{\nabla f}{|\nabla f|}$ *is geodesic. Moreover, assuming* (M^n, g) *to be also complete,*
    - (a) *if* $S > 0$*, then up to rescaling the metric as well as* f*, we may assume that* $S = 2$ *and that* $\mu = f\Delta f + 2|\nabla f|^2 = 2$ *on* M*, in which case the function* f *has* 1 *as maximum and* -1 *as minimum value and those are the only critical values of* f*;*
    - (b) *if* $S = 0$ *and* f *is nonconstant, then* (M^n, g) *is Ricci-flat, in particular it is isometric to* $(\mathbb{R} \times \Sigma, dt^2 \oplus g_\Sigma)$ *for some complete Ricci-flat Riemannian manifold* (Σ, g_Σ) *and, up to reparametrization, the function* f *is given by* $f(t,x) = t$ *for all* $(t,x) \in \mathbb{R} \times \Sigma$*;*
    - (c) *if* $S < 0$*, then up to rescaling the metric, we may assume that* $S = -2$ *on* M*, in which case one of the following holds:*
        - (i) *if* $\mu > 0$*, then up to rescaling* f *we may assume that* $\mu = 2$*, in which case* f *has no critical value and* $f(M) = \mathbb{R}$*, in particular* M *is noncompact;*

(ii) *if $\mu = 0$, then f has no critical value and empty vanishing set and, up to changing f into $-f$, we have $f(M) = (0, \infty)$, in particular M is noncompact;*

(iii) *if $\mu < 0$, then up to rescaling f we may assume that $\mu = -2$, in which case f has a unique critical value, which, up to changing f into $-f$, can be assumed to be a minimum; moreover, $f(M) = [1, \infty)$, in particular M is noncompact.*

Proof. The proof of statement 1. follows that of [18, Lemma 4]. On the one hand, we take the codifferential of $\nabla^2 f$ and obtain, choosing a local orthonormal basis $(e_j)_{1 \leq j \leq n}$ of TM and using the Weitzenböck formula for 1-forms:

$$
\begin{aligned}
\delta \nabla^2 f &= -\sum_{j=1}^{n} \left(\nabla_{e_j} \nabla^2 f \right)(e_j) \\
&= -\sum_{j=1}^{n} \left(\nabla_{e_j} \nabla_{e_j} \nabla f - \nabla_{\nabla_{e_j} e_j} \nabla f \right) \\
&= \nabla^* \nabla (\nabla f) \\
&= \Delta(\nabla f) - \mathrm{Ric}(\nabla f)\,. \qquad (6)
\end{aligned}
$$

On the other hand, by (1) and the formula $\delta \mathrm{Ric} = -\frac{1}{2} \nabla S$,

$$
\begin{aligned}
\delta \nabla^2 f &= \delta \left(-f \cdot \mathrm{Ric} \right) \\
&= \mathrm{Ric}(\nabla f) - f \cdot \delta \mathrm{Ric} \\
&= \mathrm{Ric}(\nabla f) + \frac{f}{2} \nabla S\,.
\end{aligned}
$$

Comparing both identities, we deduce that $\Delta(\nabla f) = 2\mathrm{Ric}(\nabla f) + \frac{f}{2} \nabla S$. But identity (1) also gives

$$
\Delta f = -\mathrm{tr}\left(\nabla^2 f \right) = fS\,, \qquad (7)
$$

so that $\Delta(\nabla f) = \nabla(\Delta f) = \nabla(fS) = S \nabla f + f \nabla S$ and therefore $\mathrm{Ric}(\nabla f) = \frac{S}{2} \nabla f + \frac{f}{4} \nabla S$, which is (2).

By (1) and (2), we have

$$
\begin{aligned}
2\nabla(|\nabla f|^2) &= 4 \nabla^2_{\nabla f} f \\
&= -4f \cdot \mathrm{Ric}(\nabla f) \\
&= -4f \cdot \left(\frac{S}{2} \nabla f + \frac{f}{4} \nabla S \right) \\
&= -2Sf \nabla f - f^2 \nabla S \\
&= -\nabla(Sf^2) \\
&\overset{(7)}{=} -\nabla(f \Delta f)\,,
\end{aligned}
$$

from which (3) follows.

Taking the codifferential of (2), we obtain on the one hand, using $\delta\mathrm{Ric} = -\frac{1}{2}\nabla S$:

$$\begin{aligned}\delta(\mathrm{Ric}\nabla f) &= \langle \delta\mathrm{Ric}, \nabla f\rangle - \langle \mathrm{Ric}, \nabla^2 f\rangle \\ &\overset{(1)}{=} -\frac{1}{2}\langle \nabla S, \nabla f\rangle + f|\mathrm{Ric}|^2.\end{aligned}$$

On the other hand, the codifferential of the r.h.s. of (2) is given by

$$\begin{aligned}\delta\Big(\frac{S}{2}\nabla f + \frac{f}{4}\nabla S\Big) &= -\frac{1}{2}\langle \nabla S, \nabla f\rangle + \frac{S}{2}\Delta f - \frac{1}{4}\langle \nabla f, \nabla S\rangle + \frac{f}{4}\Delta S \\ &= -\frac{3}{4}\langle \nabla f, \nabla S\rangle + \frac{S^2 f}{2} + \frac{f}{4}\Delta S\,.\end{aligned}$$

Comparing both identities yields (4).

If f vanishes nowhere, then up to changing f into $-f$, we may assume that $f > 0$ on M. Writing f as $e^{(2-n)u}$ for some real-valued function u (that is, $u = \frac{1}{2-n}\ln f$), the Ricci-curvatures (as $(0,2)$-tensor fields) ric and $\overline{\mathrm{ric}}$ of (M, g) and $(M, \overline{g} = e^{2u} g)$ respectively are related as follows:

$$\overline{\mathrm{ric}} = \mathrm{ric} + (2-n)(\nabla du - du \otimes du) + (\Delta u - (n-2)|du|_g^2)g\,. \tag{8}$$

But $\nabla df = (n-2)^2 f \cdot du \otimes du + (2-n) f \cdot \nabla du$ and the Laplace operators Δ of (M, g) and $\overline{\Delta}$ of $(M, \overline{g})$ are related via $\overline{\Delta} v = e^{-2u} \cdot (\Delta v - (n-2) g(du, dv))$ for any function v, so that

$$\begin{aligned}\overline{\mathrm{ric}} &= \mathrm{ric} + \frac{1}{f}\nabla df - (n-2)^2 du \otimes du + (n-2) du \otimes du + (\overline{\Delta} u)\overline{g} \\ &= \mathrm{ric} + \frac{1}{f}\nabla df - (n-2)(n-3) du \otimes du + (\overline{\Delta} u)\overline{g}\,.\end{aligned}$$

As a consequence, f satisfies (1) if and only if $\overline{\mathrm{ric}} = (\overline{\Delta} u)\overline{g} - (n-2)(n-3) du \otimes du$ holds on M. Moreover,

$$\begin{aligned}f\Delta f + 2|df|_g^2 &= f \cdot \big(-(n-2)^2 f |du|_g^2 - (n-2) f \Delta u\big) + 2(n-2)^2 f^2 |du|_g^2 \\ &= -(n-2) f^2 \cdot \big(\Delta u - (n-2)|du|_g^2\big) \\ &= -(n-2) f^2 \cdot e^{2u} \cdot \overline{\Delta} u \\ &= -(n-2) e^{2(2-n)u} \cdot e^{2u} \cdot \overline{\Delta} u \\ &= -(n-2) e^{2(3-n)u} \cdot \overline{\Delta} u\,,\end{aligned}$$

in particular (3) yields $\overline{\Delta} u = -\frac{\mu}{n-2} e^{2(n-3)u}$. In dimension 3, we notice that $\overline{\Delta} u = \frac{\overline{S}}{3}$. This shows statement 4.

If f vanishes nowhere, then again we may assume that $f > 0$ on M. Since M is closed, f has a minimum and a maximum. At a point x where f attains its maximum, we have $\mu = f(x)(\Delta f)(x) + 2|\nabla_x f|^2 = f(x)(\Delta f)(x) \geq 0$. In the same way, $\mu = f(y)(\Delta f)(y) \leq 0$ at any point y where f attains its minimum. We deduce that $\mu = 0$ which, by integrating the identity $f\Delta f + 2|\nabla f|^2 = \mu$ on M, yields $df = 0$. This shows statement 5.

The first part of statement 6. is the consequence of the following very general fact [14, Prop. 1.2], that we state and reprove here for the sake of completeness: if some smooth real-valued function f satisfies $\nabla^2 f = fq$ for some quadratic form q on M, then the subset $N_0 = f^{-1}(\{0\})$ is – if nonempty – a totally geodesic smooth hypersurface of M. First, it is a smooth hypersurface because of $d_x f \neq 0$ for all $x \in N_0$: namely if $c\colon \mathbb{R} \to M$ is any geodesic with $c(0) = x$, then the function $y := f \circ c$ satisfies the second order linear ODE $y'' = \langle \nabla^2_{\dot{c}} f, \dot{c}\rangle = q(\dot{c}, \dot{c}) \cdot y$ on $\mathbb{R}$ with the initial condition $y(0) = 0$; if $d_x f = 0$, then $y'(0) = 0$ and hence $y = 0$ on $\mathbb{R}$, which would imply that $f = 0$ on M by geodesic connectedness, contradiction. To compute the shape operator W of N_0 in M, we define $\nu := \frac{\nabla f}{|\nabla f|}$ to be a unit normal to N_0. Then for all $x \in N_0$ and $X \in T_x M$,

$$
\begin{aligned}
\nabla^M_X \nu &= X\left(\frac{1}{|\nabla f|}\right) \cdot \nabla f + \frac{1}{|\nabla f|} \cdot \nabla^M_X \nabla f \\
&= -\frac{X\left(|\nabla f|^2\right)}{2|\nabla f|^3} \cdot \nabla f + \frac{1}{|\nabla f|} \cdot \nabla^M_X \nabla f \\
&= \frac{1}{|\nabla f|} \cdot \left(\nabla^2_X f - \langle \nabla^2_X f, \nu\rangle \cdot \nu\right) ,
\end{aligned} \tag{9}
$$

in particular $W_x = -(\nabla\nu)_x = 0$ because of $\left(\nabla^2 f\right)_x = f(x) q_x = 0$. This shows that N_0 lies totally geodesically in M.

Now recall Gauß equations for Ricci curvature: for every $X \in TN_0$,

$$\mathrm{Ric}_{N_0}(X) = \mathrm{Ric}(X)^T - R^M_{X,\nu}\nu + \mathrm{tr}_g(W) \cdot WX - W^2 X,$$

where $\mathrm{Ric}(X)^T = \mathrm{Ric}(X) - \mathrm{ric}(X,\nu)\nu$ is the component of the Ricci curvature that is tangential to the hypersurface N_0. As a straightforward consequence, if S_{N_0} denotes the scalar curvature of N_0,

$$S_{N_0} = S - 2\mathrm{ric}(\nu,\nu) + \left(\mathrm{tr}_g(W)\right)^2 - |W|^2 .$$

Here, $W = 0$ and $\mathrm{Ric}(\nu) = \frac{S}{2}\nu$ along N_0 because N_0 lies totally geodesically in M, so that

$$S_{N_0} = S - 2\mathrm{ric}(\nu,\nu) = S - S = 0 .$$

This proves N_0 to be scalar-flat and statement 6.

As for claim 7., a straightforward consequence of (1) is that, at every $x \in M$ and for all $X, Y \in T_x M$, we have

$$
\begin{aligned}
R_{X,Y}\nabla f &= [\nabla_X, \nabla_Y]\nabla f - \nabla^2_{[X,Y]} f \\
&= -X(f)\mathrm{Ric}(Y) + Y(f)\mathrm{Ric}(X) - f\left((\nabla_X \mathrm{Ric})Y - (\nabla_Y \mathrm{Ric})X\right) ,
\end{aligned}
$$

which is identity (5). In particular, because 0 cannot be a critical value of f by statement 6., the Ricci-tensor of (M^n, g) must be Codazzi at every critical point of f. This proves claim 7.

Statement 8., which can be found in [14, Prop. 1.1], is a further consequence of the general fact mentioned above that any $f \in W(M^n, g)$ is uniquely determined by its value as well as the value of its gradient at a given point. This implies that, given any $x \in M$, the linear map

$$\begin{aligned} W(M^n, g) &\longrightarrow \mathbb{R} \times T_x M \\ f &\longmapsto (f(x), (\nabla f)(x)) \end{aligned}$$

is injective, which proves claim 8. Note that the upper bound $n+1$ for $\dim(W(M^n, g))$ is obviously attained when $(M^n, g) = (\mathbb{R}^n, \mathrm{can})$ is the flat Euclidean space.

Statement 9. can be considered as standard. In dimension 2, $\mathrm{Ric} = \frac{S}{2}\mathrm{Id} = K\mathrm{Id}$, where K is the Gauß curvature of (M^2, g). But we also know that $\mathrm{Ric}(\nabla f) = \frac{S}{2}\nabla f + \frac{f}{4}\nabla S = K\nabla f + \frac{f}{2}\nabla K$. Comparing both identities and using the fact that $\{f \neq 0\}$ is dense in M leads to $\nabla K = 0$, that is, M has constant Gauß curvature. Up to rescaling the metric as well as f, we may assume that $S, \mu \in \{-2, 0, 2\}$. If M^2 is complete with constant $S > 0$ (hence $K = 1$) and f is nonconstant, then $\mu > 0$ so that, by Obata's solution to the equation $\nabla^2 f + f \cdot \mathrm{Id}_{TM} = 0$, the manifold M must be isometric to the round sphere of radius 1 and the function f must be a nonzero eigenfunction associated to the first positive eigenvalue of the Laplace operator on $\mathbb{S}^2$, see [20, Theorem A]. If M^2 is complete and has vanishing curvature, then its universal cover is the flat $\mathbb{R}^2$ and the lift $\tilde{f}$ of f to $\mathbb{R}^2$ must be an affine-linear function of the form $\tilde{f}(x) = \langle a, x\rangle + b$ for some nonzero $a \in \mathbb{R}^2$ and some $b \in \mathbb{R}$; since the only possible nontrivial quotients of $\mathbb{R}^2$ on which $\tilde{f}$ may descend are of the form $\mathbb{R}/_{\mathbb{Z}\cdot\check{a}} \times \mathbb{R}$ for some nonzero $\check{a} \in a^\perp$, the manifold M itself must be either flat $\mathbb{R}^2$ or such a flat cylinder. If M^2 is complete with constant $S < 0$, then f satisfies the Tashiro equation $\nabla^2 f = f \cdot \mathrm{Id}_{TM}$. But then Y. Tashiro proved that (M^2, g) must be isometric to the hyperbolic plane of constant sectional curvature -1, see e.g. [23, Theorem 2 p.252], see also [15, Theorem G]. Note that the functions f listed above on $\mathbb{S}^2$, $\mathbb{R}^2$, $\mathbb{S}^1 \times \mathbb{R}$ or $\mathbb{H}^2$ obviously satisfy (1).

If (M^n, g) is Einstein with $n \geq 3$, then it has constant scalar curvature S and $\mathrm{Ric} = \frac{S}{n} \cdot \mathrm{Id}$. But again the identity $\mathrm{Ric}(\nabla f) = \frac{S}{2}\nabla f + \frac{f}{4}\nabla S = \frac{S}{2}\nabla f$ yields $n = 2$ unless $S = 0$ and thus M is Ricci-flat. Therefore, $n = 2$ is the only possibility for non-Ricci-flat Einstein M. This shows statement 9.

If S is constant, then $\mathrm{Ric}(\nabla f) = \frac{S}{2}\nabla f$. As a consequence, $\nabla^2_{\nabla f} f = -f\mathrm{Ric}(\nabla f) = -\frac{Sf}{2}\nabla f$. But, as already observed in e.g. [22, Prop. 1], away from its vanishing set, the gradient of f is a pointwise eigenvector of its Hessian if and only if the vector field $\nu = \frac{\nabla f}{|\nabla f|}$ is geodesic, see (9) above. Assuming furthermore (M^n, g) to be complete, we can rescale as before f and g such that $S, \mu \in \{-2, 0, 2\}$. In case $S > 0$ and hence $S = 2$, necessarily $\mu > 0$ holds and thus $\mu = 2$. But then $f^2 + |\nabla f|^2 = 1$, so that the only critical points of f are those where $f^2 = 1$, which by $f^2 \leq 1$ shows that the only critical points of f are those where $f = \pm 1$ and hence where f takes a maximum or minimum value. Outside critical points of f, we may consider the function $y := f \circ \gamma \colon \mathbb{R} \to \mathbb{R}$, where $\gamma \colon \mathbb{R} \to M$ is a maximal integral curve of the geodesic vector field ν. Then y satisfies $y' = |\nabla f| \circ \gamma > 0$ and $y(t)^2 + y'(t)^2 = 1$, so that $y' = \sqrt{1 - y^2}$ and therefore there exists some $\phi \in \mathbb{R}$ such that

$$y(t) = \cos(t + \phi) \qquad \forall\, t \in \mathbb{R}\,.$$

Since that function obviously changes sign and 0 is not a critical value of f, we can already deduce that f changes sign, in particular $N_0 = f^{-1}(\{0\})$ is nonempty. Moreover, the explicit formula for y shows that f must have critical points, which are precisely those where cos reaches its minimum or maximum value. This shows statement 10a.

In case $S = 0$, we have $\mathrm{Ric} = 0$ by (4) since f is assumed to be nonconstant. This proves statement 10b.

In case $S < 0$ and thus $S = -2$, there are still three possibilities for μ:

- If $\mu > 0$, then $\mu = 2$ and (3) becomes $-f^2 + |\nabla f|^2 = 1$, hence f has no critical point. If γ is any integral curve of the normalised gradient vector field $\nu = \frac{\nabla f}{|\nabla f|}$, then the function $y := f \circ \gamma$ satisfies the ODEs $y' = \sqrt{1+y^2}$, therefore $y(t) = \sinh(t+\phi)$ for some real constant ϕ. In particular, $f(M) = \mathbb{R}$ and M cannot be compact.
- If $\mu = 0$, then (3) becomes $f^2 = |\nabla f|^2$. But since no point where f vanishes can be a critical point by the fifth statement, f has no critical point and therefore must be of constant sign. Up to turning f into $-f$, we may assume that $f > 0$ and thus $f = |\nabla f|$. Along any integral curve γ of $\nu = \frac{\nabla f}{|\nabla f|}$, the function $y := f \circ \gamma$ satisfies $y' = y$ and hence $y(t) = C \cdot e^t$ for some positive constant C. This shows $f(M) = (0, \infty)$, in particular M cannot be compact.
- If $\mu < 0$, then $\mu = -2$ and (3) becomes $-f^2 + |\nabla f|^2 = -1$. As a consequence, because of $f^2 = 1 + |\nabla f|^2 \geq 1$, the function f has constant sign and hence we may assume that $f \geq 1$ up to changing f into $-f$. In particular, the only possible critical value of f is 1, which is an absolute minimum of f. If γ is any integral curve of the normalised gradient vector field $\nu = \frac{\nabla f}{|\nabla f|}$, which is defined at least on the set of regular points of f, then the function $y := f \circ \gamma$ satisfies the ODEs $y' = \sqrt{y^2 - 1}$, therefore $y(t) = \cosh(t + \phi)$ for some real constant ϕ. Since that function has an absolute minimum, it must have a critical point. It remains to notice that $f(M) = [1, \infty)$ and thus that M cannot be compact.

This shows statement 10c. □

Next we give a closer look at the case where the scalar curvature of (M^n, g) is constant.

Theorem 2.2. *Let (M^n, g) be any connected Riemannian manifold carrying a nonzero smooth real-valued function f satisfying (1) on M. Assume the scalar curvature S of (M^n, g) to be constant and nonvanishing. Up to rescaling the metric g on M it may be assumed that $S = 2\varepsilon$ for some $\varepsilon \in \{\pm 1\}$.*
Then the following holds.

1. *Every regular level hypersurface $N_c := f^{-1}(\{c\})$ of f must have vanishing scalar curvature and its Ricci-tensor be given by $\mathrm{Ric}_{N_c} = -\frac{f}{|\nabla f|^2}(\nabla_{\nabla f}\mathrm{Ric})$.*
2. *If either $n = 3$ or both $n \geq 4$ and Ric is assumed to be nonnegative when $\varepsilon = 1$ resp. nonpositive when $\varepsilon = -1$, then the Ricci-tensor has pointwise 2 eigenvalues, ε with multiplicity 2 and 0 with multiplicity $n - 2$.*

3. *If $n = 3$, the manifold (M^3, g) must be isometric to either $S^2(\varepsilon)\times\mathbb{R}$ or $S^2(\varepsilon)\times S^1$ with product Riemannian metric, where $S^2(\varepsilon)$ is the simply-connected complete surface of constant curvature $\varepsilon \in \{\pm 1\}$; and f must be the trivial extension to M of a solution of the Obata resp. Tashiro equation on $S^2(1) = \mathbb{S}^2$ (if $\varepsilon = 1$) resp. $S^2(-1) = \mathbb{H}^2$ (if $\varepsilon = -1$).*

Proof. We look at the Gauß equations for Ricci and scalar curvature along each $N_c := f^{-1}(\{c\})$ for any regular value c of f. Denoting $W = -\nabla\nu = \frac{f}{|\nabla f|}\mathrm{Ric}^T = \frac{f}{|\nabla f|}\mathrm{Ric}$ the Weingarten-endomorphism-field of N_c in M, where Ric^T is the pointwise orthogonal projection of Ric onto TN_c, we have $\mathrm{tr}(W) = \frac{f}{|\nabla f|}\cdot\frac{S}{2}$ by $\mathrm{Ric}(\nu) = \frac{S}{2}\nu$. As a consequence, we have, for all $X \in TN_c$:

$$\begin{aligned}\mathrm{Ric}(X) &= \mathrm{Ric}(X)^T\\ &= \mathrm{Ric}_{N_c}(X) + W^2X - \mathrm{tr}(W)WX + R_{X,\nu}\nu\\ &= \mathrm{Ric}_{N_c}(X) + \frac{f^2}{|\nabla f|^2}\left(\mathrm{Ric}^2(X) - \frac{S}{2}\mathrm{Ric}(X)\right) + R_{X,\nu}\nu.\end{aligned}$$

But we can compute the curvature term $R_{X,\nu}\nu$ explicitly from (5): for any $X \in TN_c$,

$$\begin{aligned}R_{X,\nu}\nu &= -\frac{X(f)}{|\nabla f|}\mathrm{Ric}(\nu) + \frac{\nu(f)}{|\nabla f|}\mathrm{Ric}(X) - \frac{f}{|\nabla f|}\left((\nabla_X\mathrm{Ric})\nu - (\nabla_\nu\mathrm{Ric})X\right)\\ &= \mathrm{Ric}(X) - \frac{f}{|\nabla f|}\Big(\nabla_X(\underbrace{\mathrm{Ric}\nu}_{\frac{S}{2}\nu}) - \mathrm{Ric}(\nabla_X\nu)\Big) + \frac{f}{|\nabla f|}(\nabla_\nu\mathrm{Ric})X\\ &= \mathrm{Ric}(X) + \frac{f}{|\nabla f|}\left(\frac{S}{2}\mathrm{Id} - \mathrm{Ric}\right)(WX) + \frac{f}{|\nabla f|}(\nabla_\nu\mathrm{Ric})X\\ &= \mathrm{Ric}(X) + \frac{f^2}{|\nabla f|^2}\left(\frac{S}{2}\mathrm{Ric}(X) - \mathrm{Ric}^2(X)\right) + \frac{f}{|\nabla f|}(\nabla_\nu\mathrm{Ric})X\,, \qquad (10)\end{aligned}$$

so that, with $(\nabla_\nu\mathrm{Ric})(\nu) = \nabla_\nu(\mathrm{Ric}(\nu)) - \mathrm{Ric}(\nabla_\nu\nu) = \nabla_\nu(\frac{S}{2}\nu) = 0$ on M, we obtain

$$\mathrm{Ric}_{N_c} = -\frac{f}{|\nabla f|}\cdot\nabla_\nu\mathrm{Ric}\,,$$

as claimed in statement 1. That identity has important consequences. First, choosing a local o.n.b. $(e_j)_{1\le j\le n-1}$ of TN_c,

$$\begin{aligned}S_{N_c} &= \sum_{j=1}^{n-1}\langle\mathrm{Ric}_{N_c}(e_j), e_j\rangle\\ &= -\frac{f}{|\nabla f|}\cdot\sum_{j=1}^{n-1}\langle(\nabla_\nu\mathrm{Ric})(e_j), e_j\rangle\\ &= -\frac{f}{|\nabla f|}\cdot\mathrm{tr}\left(\nabla_\nu\mathrm{Ric}\right)\,,\end{aligned}$$

because of $(\nabla_\nu \mathrm{Ric})(\nu) = 0$, so that

$$S_{N_c} = -\frac{f}{|\nabla f|} \cdot \mathrm{tr}(\nabla_\nu \mathrm{Ric}) = -\frac{f}{|\nabla f|} \cdot \nu(\mathrm{tr}(\mathrm{Ric})) = -\frac{f}{|\nabla f|} \cdot \nu(S) = 0\,.$$

Therefore, each level hypersurface N_c is scalar-flat. This concludes the proof of statement 1. We turn to 2. Because of S being constant, we already know by (2) that, outside its vanishing set, the gradient vector field ∇f of f is a pointwise eigenvector for the Ricci tensor associated to the eigenvalue $\frac{S}{2} = \varepsilon$. Writing the Ricci tensor as $\mathrm{Ric} = \varepsilon \nu^\flat \otimes \nu + \mathrm{Ric}^T$, where Ric^T is a pointwise symmetric endomorphism of $\nu^\perp \subset TM$, we deduce from (4) and the fact that $\{f \neq 0\}$ is dense in M that

$$|\mathrm{Ric}^T|^2 = \frac{S^2}{4} = 1 \tag{11}$$

on $\{\nabla f \neq 0\}$. Since $\mathrm{tr}(\mathrm{Ric}^T) = \frac{S}{2} = \varepsilon$, identity (11) implies that, at every point outside the critical set, the set of possible eigenvalues of Ric^T stands in one-to-one correspondence with the sphere $\mathbb{S}^{n-3}$ of dimension $n-3$, seen as the unit sphere in the $(n-2)$-dimensional space $\nu^\perp$. If $n = 3$, then this means that Ric^T has pointwise the eigenvalues ε and 0, each of multiplicity one, on the regular set of f. If $n \geq 4$, we assume furthermore that $\mathrm{Ric} \geq 0$ when $\varepsilon = 1$ and $\mathrm{Ric} \leq 0$ when $\varepsilon = -1$. In that case, (11) implies that Ric^T has exactly one eigenvalue that is equal to ε and that all other eigenvalues vanish, at least on $\{\nabla f \neq 0\}$. To sum up, the Ricci tensor of (M^n, g) has at each point of $\{\nabla f \neq 0\} \subset M$ the eigenvalues ε of multiplicity 2 and 0 of multiplicity $n-2$ respectively. Note that both eigendistributions of the Ricci-tensor are smooth since they have constant rank. Furthermore, the critical set $\{\nabla f = 0\}$ of f must have empty interior, otherwise the Ricci tensor would vanish identically on that interior by (1) and the fact that 0 is not a critical value of f. But this would contradict the fact that the scalar curvature S of (M^n, g) is assumed to be constant and nonvanishing. Therefore, Ric has actually ε and 0 as eigenvalues with multiplicities 2 and $n-2$ respectively on all of M. This proves 2.

It remains to show that, when $n = 3$, both eigendistributions of the Ricci tensor of (M^3, g) are actually parallel. Let η be a unit eigenvector of Ric associated to the eigenvalue ε and e_3 be a unit eigenvector of Ric associated to the eigenvalue 0; since both Ric-eigenvalues are constant and distinct and Ric is smooth, η and e_3 exist globally along N_c, no need of analyticity. In dimension 3 again, because $S_{N_c} = 0$ yields $\mathrm{Ric}_{N_c} = 0$ and thus $\nabla_\nu \mathrm{Ric} = 0$, the vector fields η and e_3 can actually be defined everywhere on the regular set of f using parallel transport along ν-geodesics. Moreover, because the eigenvalue 0 of the Ricci-tensor has multiplicity 1 on all of M as we showed above, the vector field e_3 can be defined globally on M.

We show that $\nabla e_3 = 0$, i.e. e_3 is parallel on the dense open subset $\{\nabla f \neq 0\}$ and hence on M. First, because of $\nabla_\nu \mathrm{Ric} = 0$, $\ker(\mathrm{Ric}) = \mathbb{R} e_3$ and $|e_3| = 1$, we have $\nabla_\nu e_3 \in \ker(\mathrm{Ric}) \cap e_3^\perp = \{0\}$ i.e., $\nabla_\nu e_3 = 0$. Next, following from the identity

$$0 = \frac{1}{2}\nabla S = -\delta \mathrm{Ric} = (\nabla_\eta \mathrm{Ric})\eta + (\nabla_{e_3} \mathrm{Ric}) e_3 + \underbrace{(\nabla_\nu \mathrm{Ric})}_{0}\, \nu\,,$$

we have $(\nabla_\eta \mathrm{Ric})\eta = -(\nabla_{e_3}\mathrm{Ric})e_3$. Here we notice that

$$(\nabla_\eta \mathrm{Ric})\eta = \varepsilon\nabla_\eta\eta - \mathrm{Ric}(\nabla_\eta\eta) = \varepsilon(\nabla_\eta\eta - \langle\nabla_\eta\eta,\nu\rangle\nu)$$

and, with $\nabla_{e_3}\nu = -We_3 = \frac{f}{|\nabla f|}\mathrm{Ric}(e_3) = 0$, that

$$(\nabla_{e_3}\mathrm{Ric})e_3 = -\mathrm{Ric}(\nabla_{e_3}e_3) = -\varepsilon\langle\nabla_{e_3}e_3,\eta\rangle\eta\,.$$

Therefore,

$$\begin{aligned}
0 &= \varepsilon\langle\nabla_\eta\eta,\eta\rangle \\
&= \langle(\nabla_\eta \mathrm{Ric})\eta,\eta\rangle \\
&= -\langle(\nabla_{e_3}\mathrm{Ric})e_3,\eta\rangle \\
&= \varepsilon\langle\nabla_{e_3}e_3,\eta\rangle\,.
\end{aligned}$$

Since $\langle\nabla_{e_3}e_3,\nu\rangle = -\langle e_3,\nabla_{e_3}\nu\rangle = 0$ and $\langle\nabla_{e_3}e_3,e_3\rangle = 0$, it can be deduced that $\nabla_{e_3}e_3 = 0$.

Analogously,

$$\begin{aligned}
0 &= -\varepsilon\langle\nabla_{e_3}e_3,\eta\rangle\langle\eta,e_3\rangle \\
&= \langle(\nabla_{e_3}\mathrm{Ric})e_3,e_3\rangle \\
&= -\langle(\nabla_\eta \mathrm{Ric})\eta,e_3\rangle \\
&= -\varepsilon\langle\nabla_\eta\eta,e_3\rangle\,,
\end{aligned}$$

so that $\langle\nabla_\eta e_3,\eta\rangle = 0$. Again, because $\langle\nabla_\eta e_3,e_3\rangle = 0 = \langle\nabla_\eta e_3,\nu\rangle$, it can be deduced that $\nabla_\eta e_3 = 0$. To sum up, we obtain $\nabla e_3 = 0$ i.e., the vector field e_3 is parallel on $M\setminus\{\nabla f = 0\}$ and hence on M. As a consequence, the holonomy group of M splits locally, therefore the universal cover of M is isometric to the Riemannian product $\Sigma\times\mathbb{R}$ of some complete surface Σ with $\mathbb{R}$. Moreover, using formula (10) for $X=\eta$ and taking into account that $\nabla_\nu\mathrm{Ric} = 0$, we obtain

$$R_{\eta,\nu}\nu = \left(1+\frac{Sf^2}{2|\nabla f|^2}\right)\cdot\mathrm{Ric}(\eta) - \frac{f^2}{|\nabla f|^2}\cdot\mathrm{Ric}^2(\eta) = \mathrm{Ric}(\eta) = \varepsilon\eta\,,$$

so that $K(\eta,\nu) = \langle R_{\eta,\nu}\nu,\eta\rangle = \varepsilon|\eta|^2 = \varepsilon$. Therefore, the distribution $\mathrm{Span}(\eta,\nu)\to M$ integrates to a surface of constant curvature $\varepsilon\in\{\pm1\}$. Thus $\Sigma = S^2(\varepsilon)$, which is the simply-connected complete surface with curvature $\varepsilon\in\{\pm1\}$. In case $\varepsilon = 1$, the lift $\tilde f$ of f to $\mathbb{S}^2\times\mathbb{R}$ is constant along the $\mathbb{R}$-factor and satisfies the equation $(\nabla^{\mathbb{S}^2})^2 f = -f\cdot\mathrm{Id}$, which is exactly the equation characterizing the eigenfunctions associated to the first positive Laplace eigenvalue [20, Theorem A]. Furthermore, the isometry group of $\mathbb{S}^2\times\mathbb{R}$ embeds into the product group of both isometry groups of $\mathbb{S}^2$ and $\mathbb{R}$ and the first factor must be trivial since $\tilde f$, as the restriction of a linear form from $\mathbb{R}^3$ onto $\mathbb{S}^2$, is not invariant under $\{\pm\mathrm{Id}\}$. Therefore, M is isometric to either $\mathbb{S}^2\times\mathbb{R}$ or to $\mathbb{S}^2\times\mathbb{S}^1$ and in both cases f is the trivial extension of an eigenfunction associated to the first positive Laplace eigenvalue on $\mathbb{S}^2$. In case $\varepsilon = -1$, the lift $\tilde f$ of f to $\mathbb{H}^2\times\mathbb{R}$ is constant along the $\mathbb{R}$-factor and satisfies the equation $(\nabla^{\mathbb{H}^2})^2 f = f\cdot\mathrm{Id}$, which is exactly the Tashiro equation. Since the isometry group of $\mathbb{H}^2\times\mathbb{R}$ embeds into the product group of both isometry groups of $\mathbb{H}^2$ and $\mathbb{R}$ and the first factor must be trivial since $\tilde{f}$ has no nontrivial symmetry [23, Theorem 2 p.252], we can deduce as above that M is isometric to either $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{S}^1$ and f is the trivial extension of a solution to the Tashiro equation on $\mathbb{H}^2$. This proves statement 3 and concludes the proof of Theorem 2.2. $\square$

Next we look at manifolds with *harmonic* curvature tensor. Recall that, by definition, the Riemann curvature tensor R of (M^n, g) is harmonic if and only if $\delta R = 0$ holds on M. By the first and second Bianchi identities, we have, for all $X, Y, Z \in T_xM$ at some $x \in M$:

$$(\delta R)(X, Y, Z) = (\nabla_Y \mathrm{Ric})(Z, X) - (\nabla_Z \mathrm{Ric})(Y, X).$$

As a consequence, $\delta R = 0$ at some $x \in M$ is equivalent to

$$(\nabla_X \mathrm{Ric})(Y) - (\nabla_Y \mathrm{Ric})(X) = 0$$

for all $X, Y \in T_xM$ i.e., to Ric being a *Codazzi*-tensor at x. A 3-dimensional Riemannian manifold has harmonic curvature if and only if it is conformally flat and has constant scalar curvature. In dimension $n \geq 4$, a Riemannian manifold has harmonic curvature if and only if it has harmonic Weyl tensor W, that is, $\delta W = 0$ holds on M, and constant scalar curvature. For instance, any conformally flat manifold with constant scalar curvature has harmonic curvature tensor. We refer to [2, Sec. 16.4] for more details about harmonic curvature.

Theorem 2.3. *Let (M^n, g) be any connected complete Riemannian manifold carrying a nonzero smooth real-valued function f satisfying* (1) *on M. If the Riemann curvature tensor of (M^n, g) is harmonic, then either (M^n, g) is Ricci-flat or, up to rescaling the metric g, the manifold (M^n, g) is isometric to the Riemannian product $S^2(\varepsilon) \times \Sigma^{n-2}$, where $S^2(\varepsilon)$ is the simply-connected complete surface of constant curvature $\varepsilon \in \{\pm 1\}$ and Σ^{n-2} is a Ricci-flat manifold. Moreover, f is the trivial extension to M of a solution of the Obata resp. Tashiro equation on $\mathbb{S}^2$ (if $\varepsilon = 1$) resp. $\mathbb{H}^2$ (if $\varepsilon = -1$).*

Proof. First recall that, if $\delta R = 0$ holds on M – or, equivalently, if Ric is a Codazzi-tensor – then the scalar curvature S of (M^n, g) must be constant: given any pointwise o.n.b. $(e_j)_{1 \leq j \leq n}$ of TM and $X \in TM$, we have

$$\begin{aligned} X(S) &= X\left(\mathrm{tr}(\mathrm{Ric})\right) \\ &= \mathrm{tr}\left(\nabla_X \mathrm{Ric}\right) \\ &= \sum_{j=1}^n (\nabla_X \mathrm{Ric})(e_j, e_j) \\ &= \sum_{j=1}^n (\nabla_{e_j} \mathrm{Ric})(X, e_j) \\ &= -(\delta \mathrm{Ric})(X) \\ &= \frac{X(S)}{2}, \end{aligned}$$

so that necessarily $dS = 0$ holds on M. Since the scalar curvature S is assumed to be non-identically vanishing, we may assume up to rescaling g that $S = 2\varepsilon$ with $\varepsilon \in \{\pm 1\}$.

For any $s \in \mathbb{N}$, we denote by (a_s) the assertion $\mathrm{tr}(\mathrm{Ric}^s) = 2\varepsilon^s$ and by (b_s) the assertion $\delta(\mathrm{Ric}^s) = 0$. We show that, since the Ricci-tensor is assumed to be Codazzi, both (a_s) and (b_s) are true.

First, we have that, for every s, (b_s) implies (a_{s+1}): namely, as a consequence of $\mathrm{Ric}(\nabla f) = \varepsilon \nabla f$ (see (2)),

$$(\nabla_X \mathrm{Ric}^s)(\nabla f) = -f(\varepsilon^s \mathrm{Ric} X - \mathrm{Ric}^{s+1} X)$$

for every $X \in TM$. This yields, in a pointwise o.n.b. $(e_j)_{1 \le j \le n}$ of TM,

$$\begin{aligned}\delta(\mathrm{Ric}^s)(\nabla f) &= -\sum_{j=1}^n (\nabla_{e_j} \mathrm{Ric}^s)(e_j, \nabla f)\\ &= f(\varepsilon^s S - \mathrm{tr}(\mathrm{Ric}^{s+1}))\\ &= f(2\varepsilon^{s+1} - \mathrm{tr}(\mathrm{Ric}^{s+1}))\,.\end{aligned}$$

Therefore, if $\delta(\mathrm{Ric}^s) = 0$, then $\mathrm{tr}(\mathrm{Ric}^{s+1}) = 2\varepsilon^{s+1}$. This shows the claim. Note that here we have not used the property that Ric is a Codazzi-tensor.

Second, we have, under the condition that Ric is Codazzi, that $(b_s) \Rightarrow (b_{s+1})$. Namely assuming (b_s), assertion (a_{s+1}) must hold true from the previous claim. Therefore, for every $X \in TM$,

$$\sum_{j=1}^n (\nabla_X \mathrm{Ric})(e_j, \mathrm{Ric}^s e_j) = \mathrm{tr}(\nabla_X \mathrm{Ric} \circ \mathrm{Ric}^s) = \frac{1}{s+1} X\left(\left(\mathrm{tr}\left(\mathrm{Ric}^{s+1}\right)\right)\right) = 0\,.$$

Now using the fact that the Ricci-tensor is Codazzi, we compute

$$\begin{aligned}0 &= \sum_{j=1}^n (\nabla_X \mathrm{Ric})(e_j, \mathrm{Ric}^s e_j)\\ &= \sum_{j=1}^n (\nabla_{e_j} \mathrm{Ric})(\mathrm{Ric}^s e_j, X)\\ &= \sum_{j=1}^n ((\nabla_{e_j} \mathrm{Ric}^{s+1})(e_i), X) - \mathrm{Ric}(((\nabla_{e_j} \mathrm{Ric}^s)(e_j), X)\\ &= -(\delta \mathrm{Ric}^{s+1})(X)\end{aligned}$$

using again (b_s). We deduce that (b_{s+1}) is true.

Since (a_s) and (b_s) are satisfied for $s = 1$, we deduce that they are satisfied for all $s \in \mathbb{N}$. From the Newton identities, it can be deduced that the Ricci tensor must have pointwise the eigenvalues ε and 0, the former of multiplicity 2 and the latter of multiplicity $n - 2$. Therefore, we get the pointwise orthogonal decomposition $TM = \ker(\mathrm{Ric} - \varepsilon \mathrm{Id}) \oplus \ker(\mathrm{Ric})$.

It remains to show that both eigendistributions of the Ricci-tensor are parallel. Let $X, Y \in \ker(\mathrm{Ric} - \varepsilon \mathrm{Id})$ and $Z \in \ker(\mathrm{Ric})$. Then the scalar product with Y in the formula $(\nabla_X \mathrm{Ric})Z = (\nabla_Z \mathrm{Ric})X$ allows to get on the one hand

$$g((\nabla_X \mathrm{Ric})Z, Y) = -g(\mathrm{Ric}(\nabla_X Z), Y) = -\varepsilon g(\nabla_X Z, Y)\,,$$

and on the other hand

$$\begin{aligned} g((\nabla_Z \mathrm{Ric})X, Y) &= \varepsilon g(\nabla_Z X, Y) - g(\mathrm{Ric}(\nabla_Z X), Y) \\ &= \varepsilon g(\nabla_Z X, Y) - g(\nabla_Z X, \mathrm{Ric}Y) \\ &= 0\,. \end{aligned}$$

Thus, we deduce that $0 = g(\nabla_X Z, Y) = -g(\nabla_X Y, Z)$. Hence $\nabla_X Y \in \ker(\mathrm{Ric} - \varepsilon\mathrm{Id})$ and therefore the distribution $\ker(\mathrm{Ric} - \varepsilon\mathrm{Id})$ is parallel. The same computations can be done for the distribution $\ker(\mathrm{Ric})$. This straightforwardly implies that both eigendistributions $\ker(\mathrm{Ric} - \varepsilon\mathrm{Id})$ and $\ker(\mathrm{Ric})$ are parallel and therefore integrable and totally geodesic. By the de Rham theorem, M splits locally as the Riemannian product of a surface and an $n-2$-dimensional submanifold. Moreover, the Ricci-curvature – which is the Gauß-curvature – of the surface that is pointwise tangent to the distribution $\ker(\mathrm{Ric} - \varepsilon\mathrm{Id})$ is ε and the submanifold that is pointwise tangent to $\ker(\mathrm{Ric})$ is Ricci-flat, see e.g. [2, Thm. 1.100]. Therefore the universal cover of M is isometric to the Riemannian product $S^2(\varepsilon) \times \tilde{\Sigma}$ of the simply-connected complete surface with curvature $\varepsilon \in \{-1, 0, 1\}$ with some simply-connected Ricci-flat manifold $\tilde{\Sigma}$. The rest of the proof is analogous to that of Theorem 2.2.3. This concludes the proof of Theorem 2.3. $\square$

3. Examples in warped product form

We look for examples of warped products $(M, g) := (M_1 \times M_2, g_1 \oplus \varphi^2 g_2)$ for some smooth positive function φ on M_1, where (M_1, g_1) and (M_2, g_2) are connected Riemannian manifolds. We make the ansatz $f(x_1, x_2) := f_1(x_1)f_2(x_2)$ for all $(x_1, x_2) \in M$ where f_1 and f_2 are smooth real-valued functions on M_1 and M_2 respectively. We look for necessary and sufficient conditions for f to satisfy (1) on (M, g).

Proposition 3.1. *Let $(M^n, g) := (M_1^{n_1} \times M_2^{n_2}, g_1 \oplus \varphi^2 g_2)$ be a connected Riemannian warped product, where $\varphi \in C^\infty(M_1, \mathbb{R}_+^\times)$. For any two functions $f_i \in C^\infty(M_i, \mathbb{R})$, $i = 1, 2$, let $f := \pi_1^* f_1 \cdot \pi_2^* f_2$ i.e., $f(x_1, x_2) = f_1(x_1)f_2(x_2)$ for all $(x_1, x_2) \in M$. Then f solves $\nabla^2 f = -f \cdot \mathrm{Ric}$ on (M, g) if and only if one of the following occurs:*

(a) *The function $\frac{f_1}{\varphi}$ is constant on M_1, in which case it can be assumed up to rescaling f that $f_1 = \varphi$. Then $\mu_1(f_1) := (n_2 - 2)|\nabla^{M_1} f_1|_1^2 - f_1 \Delta^{M_1} f_1$ is constant on M_1 and f_1, f_2 solve*

$$(n_2 - 1)(\nabla^{M_1})^2 f_1 = f_1 \cdot \mathrm{Ric}_{M_1} \tag{12}$$

$$(\nabla^{M_2})^2 f_2 = f_2 \cdot (\mu_1(f_1)\mathrm{Id}_{TM_2} - \mathrm{Ric}_{M_2}) \tag{13}$$

respectively.

(b) *The function* f_2 *is constant on* M_2*, in which case* f_1 *solves*

$$(\nabla^{M_1})^2 f_1 = -f_1 \cdot \Big(\mathrm{Ric}_{M_1} - \frac{n_2}{\varphi}(\nabla^{M_1})^2\varphi\Big) \tag{14}$$

on M_1*, the function* $-\frac{\varphi}{f_1}g_1(\nabla^{M_1}f_1, \nabla^{M_1}\varphi) + (n_2-1)|\nabla^{M_1}\varphi|_1^2 - \varphi\Delta^{M_1}\varphi$ *is constant on each connected component of* $M_1 \setminus f_1^{-1}(\{0\})$ *and the manifold* (M_2, g_2) *is Einstein with scalar curvature equal to*

$$n_2\Big(-\frac{\varphi}{f_1}g_1(\nabla^{M_1}f_1, \nabla^{M_1}\varphi) + (n_2-1)|\nabla^{M_1}\varphi|_1^2 - \varphi\Delta^{M_1}\varphi\Big).$$

Proof. First, we have $\nabla f = f_2\nabla f_1 + f_1\nabla f_2 = f_2\nabla^{M_1}f_1 + \frac{f_1}{\varphi^2}\nabla^{M_2}f_2$, where $\nabla^{M_i}f_i$ denotes the g_i-gradient of f_i on (M_i, g_i). Recall Koszul's formula, valid for any tangent vector fields X, Y, Z on some Riemannian manifold (M, g):

$$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2}\Big\{&X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &+ g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)\Big\}. \end{aligned} \tag{15}$$

It can be deduced from (15) that, for any $X_i, Y_i, Z_i \in \Gamma(\pi_i^* TM_i)$, we have

$$\nabla_{X_1}Y_1 = \nabla^{M_1}_{X_1}Y_1 \tag{16}$$

$$\nabla_{X_1}Y_2 = \partial_{X_1}Y_2 + \frac{X_1(\varphi)}{\varphi}Y_2 \tag{17}$$

$$\nabla_{X_2}Y_1 = \partial_{X_2}Y_1 + \frac{Y_1(\varphi)}{\varphi}X_2 \tag{18}$$

$$\nabla_{X_2}Y_2 = \nabla^{M_2}_{X_2}Y_2 - \frac{1}{\varphi}g(X_2,Y_2)\nabla^{M_1}\varphi. \tag{19}$$

As a first consequence,

$$\begin{aligned} \nabla^2_{X_1}f &= f_2\nabla_{X_1}\nabla^{M_1}f_1 + \frac{X_1(f_1)\varphi^2 - 2f_1X_1(\varphi)\varphi}{\varphi^4}\nabla^{M_2}f_2 + \frac{f_1}{\varphi^2}\nabla_{X_1}\nabla^{M_2}f_2 \\ &= f_2(\nabla^{M_1})^2_{X_1}f_1 + \frac{X_1(f_1)\varphi - 2f_1X_1(\varphi)}{\varphi^3}\nabla^{M_2}f_2 \\ &\quad + \frac{f_1}{\varphi^2}\left(\underbrace{\partial_{X_1}\nabla^{M_2}f_2}_{0} + \frac{X_1(\varphi)}{\varphi}\nabla^{M_2}f_2\right) \\ &= f_2(\nabla^{M_1})^2_{X_1}f_1 + \frac{X_1(f_1)\varphi - f_1X_1(\varphi)}{\varphi^3}\nabla^{M_2}f_2 \\ &= f_2(\nabla^{M_1})^2_{X_1}f_1 + \frac{1}{\varphi}X_1(\frac{f_1}{\varphi})\nabla^{M_2}f_2. \end{aligned} \tag{20}$$

Similarly,

$$\begin{aligned}
\nabla^2_{X_2} f &= X_2(f_2)\nabla^{M_1} f_1 + f_2 \nabla_{X_2}\nabla^{M_1} f_1 + \frac{f_1}{\varphi^2}\nabla_{X_2}\nabla^{M_2} f_2 \\
&= X_2(f_2)\nabla^{M_1} f_1 + f_2\Big(\underbrace{\partial_{X_2}\nabla^{M_1} f_1}_{0} + \frac{g_1(\nabla^{M_1} f_1, \nabla^{M_1}\varphi)}{\varphi} X_2\Big) \\
&\quad + \frac{f_1}{\varphi^2}\Big((\nabla^{M_2})^2_{X_2} f_2 - \frac{1}{\varphi} g(X_2, \nabla^{M_2} f_2)\nabla^{M_1}\varphi\Big) \\
&= \frac{f_1}{\varphi^2}(\nabla^{M_2})^2_{X_2} f_2 + X_2(f_2)\Big(\nabla^{M_1} f_1 - \frac{f_1}{\varphi}\nabla^{M_1}\varphi\Big) \\
&\quad + \frac{f_2}{\varphi} g_1(\nabla^{M_1} f_1, \nabla^{M_1}\varphi) X_2 \\
&= \frac{f_1}{\varphi^2}(\nabla^{M_2})^2_{X_2} f_2 + X_2(f_2)\varphi\nabla^{M_1}\Big(\frac{f_1}{\varphi}\Big) + \frac{f_2}{\varphi} g_1(\nabla^{M_1} f_1, \nabla^{M_1}\varphi) X_2 \,. \qquad (21)
\end{aligned}$$

Independently, by [2, Prop. 9.106], we have

$$\mathrm{Ric}(X_1) = \mathrm{Ric}_{M_1}(X_1) - \frac{n_2}{\varphi}(\nabla^{M_1})^2_{X_1}\varphi \qquad (22)$$

$$\mathrm{Ric}(X_2) = \frac{1}{\varphi^2}\mathrm{Ric}_{M_2}(X_2) + \Big(\frac{\Delta^{M_1}\varphi}{\varphi} - (n_2 - 1)\frac{|\nabla^{M_1}\varphi|_1^2}{\varphi^2}\Big) X_2 \,. \qquad (23)$$

Therefore, f satisfies (1) on (M, g) if and only if the following system of equations holds, for all $(X_1, X_2) \in TM$:

$$\begin{cases} l_1(X_1) &= r_1(X_1) \\ l_2(X_2) &= r_2(X_2) \end{cases},$$

where

$$\begin{aligned}
l_1(X_1) &= f_2(\nabla^{M_1})^2_{X_1} f_1 + \frac{1}{\varphi} X_1\Big(\frac{f_1}{\varphi}\Big)\nabla^{M_2} f_2 \\
r_1(X_1) &= -f_1 f_2 \cdot \Big(\mathrm{Ric}_{M_1}(X_1) - \frac{n_2}{\varphi}(\nabla^{M_1})^2_{X_1}\varphi\Big) \\
l_2(X_2) &= \frac{f_1}{\varphi^2}(\nabla^{M_2})^2_{X_2} f_2 + X_2(f_2)\varphi\nabla^{M_1}\Big(\frac{f_1}{\varphi}\Big) + \frac{f_2}{\varphi} g_1(\nabla^{M_1} f_1, \nabla^{M_1}\varphi) X_2 \\
r_2(X_2) &= -f_1 f_2 \cdot \Big(\frac{1}{\varphi^2}\mathrm{Ric}_{M_2}(X_2) + \Big(\frac{\Delta^{M_1}\varphi}{\varphi} - (n_2 - 1)\frac{|\nabla^{M_1}\varphi|_1^2}{\varphi^2}\Big) X_2\Big) \,.
\end{aligned}$$

Both equations imply that $d\left(\frac{f_1}{\varphi}\right) \otimes df_2 = 0$, that is, that $\frac{f_1}{\varphi}$ is constant on M_1 or f_2 is constant on M_2.

Case $\frac{f_1}{\varphi}$ is constant on M_1: We may assume, up to rescaling f_2 and hence f, that $f_1 = \varphi$ holds on M_1. The above system of equations becomes equivalent to the following:

$$\begin{cases} (\nabla^{M_1})^2_{X_1} f_1 &= -f_1 \cdot \Big(\mathrm{Ric}_{M_1}(X_1) - \frac{n_2}{f_1}(\nabla^{M_1})^2_{X_1} f_1\Big) \\ \frac{1}{f_1}(\nabla^{M_2})^2_{X_2} f_2 + \frac{f_2|\nabla^{M_1} f_1|_1^2}{f_1} X_2 &= -f_1 f_2 \cdot T_2(X_2) \end{cases},$$

where $T_2(X_2) := \left(\frac{1}{f_1^2}\mathrm{Ric}_{M_2}(X_2) + \left(\frac{\Delta^{M_1} f_1}{f_1} - (n_2-1)\frac{|\nabla^{M_1} f_1|_1^2}{f_1^2}\right) X_2\right)$. Thus

$$\begin{cases} (1-n_2)(\nabla^{M_1})^2 f_1 & = -f_1 \cdot \mathrm{Ric}_{M_1} \\ (\nabla^{M_2})^2 f_2 & = -f_2 \cdot \mathrm{Ric}_{M_2} + f_2\left((n_2-2)|\nabla^{M_1} f_1|_1^2 - f_1 \Delta^{M_1} f_1\right) \mathrm{Id}_{TM_2} \end{cases}.$$

Since f_2 is assumed to be non-identically vanishing and the second identity above only depends on M_2, the factor $\mu_1(f_1) := (n_2-2)|\nabla^{M_1} f_1|_1^2 - f_1\Delta^{M_1} f_1$ must be constant on M_1. Actually we shall see later that, when $n_2 \geq 2$, this already follows from the equation for f_1.
Therefore, in case $f_1 = \varphi$, equation (1) for $f := \pi_1^* f_1 \cdot \pi_2^* f_2$ is equivalent to the function $(n_2-2)|\nabla^{M_1} f_1|_1^2 - f_1\Delta^{M_1} f_1 = \mu_1(f_1)$ begin constant on M_1 and

$$\begin{cases} (n_2-1)(\nabla^{M_1})^2 f_1 & = f_1 \cdot \mathrm{Ric}_{M_1} \\ (\nabla^{M_2})^2 f_2 & = f_2 \cdot (\mu_1(f_1)\mathrm{Id}_{TM_2} - \mathrm{Ric}_{M_2}) \end{cases}.$$

Case f_2 is constant on M_2: Then $\nabla^2 f = -f \cdot \mathrm{Ric}$ on (M,g) is equivalent to the system

$$\begin{cases} (\nabla^{M_1})^2 f_1 & = -f_1 \cdot \left(\mathrm{Ric}_{M_1} - \frac{n_2}{\varphi}(\nabla^{M_1})^2\varphi\right) \\ \frac{g_1(\nabla^{M_1} f_1, \nabla^{M_1}\varphi)}{\varphi}\mathrm{Id}_{TM_2} & = -f_1 \cdot \left(\frac{1}{\varphi^2}\mathrm{Ric}_{M_2} + \left(\frac{\Delta^{M_1}\varphi}{\varphi} - (n_2-1)\frac{|\nabla^{M_1}\varphi|_1^2}{\varphi^2}\right)\mathrm{Id}_{TM_2}\right) \end{cases},$$

that is, assuming f_1 not to vanish identically on M_1,

$$\begin{cases} (\nabla^{M_1})^2 f_1 & = -f_1 \cdot \left(\mathrm{Ric}_{M_1} - \frac{n_2}{\varphi}(\nabla^{M_1})^2\varphi\right) \\ \mathrm{Ric}_{M_2} & = \left(-\frac{\varphi}{f_1} g_1(\nabla^{M_1} f_1, \nabla^{M_1}\varphi) + (n_2-1)|\nabla^{M_1}\varphi|_1^2 - \varphi\Delta^{M_1}\varphi\right) \cdot \mathrm{Id}_{TM_2} \end{cases},$$

the second equation holding on the dense open subset $M_1 \setminus f^{-1}(\{0\})$. The second of both above identities implies that the quantity

$$\mu_1' := \left(-\frac{\varphi}{f_1} g_1(\nabla^{M_1} f_1, \nabla^{M_1}\varphi) + (n_2-1)|\nabla^{M_1}\varphi|_1^2 - \varphi\Delta^{M_1}\varphi\right)$$

is constant on M_1 and that M_2 is Einstein with constant scalar curvature equal to $n_2\mu_1'$, whatever n_2 is. This concludes the proof of Proposition 3.1. □

Now we look at (1) on Riemannian products, where f is not assumed to be in product form.

Theorem 3.2. *Let $(M^n, g) = (M_1 \times M_2, g_1 \oplus g_2)$ for some connected Riemannian manifolds (M_1, g_1) and (M_2, g_2). Assume M to be non Ricci-flat i.e., that $\mathrm{Ric}_{M_1} \neq 0$ or $\mathrm{Ric}_{M_2} \neq 0$. W.l.o.g. let $\mathrm{Ric}_{M_2} \neq 0$. Then a function $f \in C^\infty(M,\mathbb{R})\setminus\{0\}$ satisfies (1) on (M^n, g) if and only if $\mathrm{Ric}_{M_1} = 0$, the function f only depends on M_2 and satisfies (1) on (M_2, g_2). As a consequence, the map $W(M_2, g_2) \longrightarrow W(M, g)$ extending a function trivially on the M_1-factor, is an isomorphism.*

Proof. First, we split pointwise $\nabla f = \nabla^{M_1} f + \nabla^{M_2} f$ according to the g-orthogonal splitting $T_{(x_1,x_2)}M = T_{x_1}M_1 \oplus T_{x_2}M_2$, for all $(x_1, x_2) \in M$. Using formulae (16) –

(19) and $\varphi = 1$, it can be deduced that, for all $X_1 \in TM_1$,

$$\begin{aligned}\nabla^2_{X_1} f &= \nabla_{X_1}(\nabla^{M_1} f) + \nabla_{X_1}(\nabla^{M_2} f)\\ &= \nabla^{M_1}_{X_1}(\nabla^{M_1} f) + \partial_{X_1}(\nabla^{M_2} f)\end{aligned}$$

and similarly, for all $X_2 \in TM_2$,

$$\begin{aligned}\nabla^2_{X_2} f &= \nabla_{X_2}(\nabla^{M_1} f) + \nabla_{X_2}(\nabla^{M_2} f)\\ &= \partial_{X_2}(\nabla^{M_1} f) + \nabla^{M_2}_{X_2}(\nabla^{M_2} f)\,.\end{aligned}$$

By (22) and (23), we obtain that f satisfies (1) on (M^n, g) if and only if, for all $(X_1, X_2) \in TM_1 \oplus TM_2$,

$$\nabla^{M_1}_{X_1}(\nabla^{M_1} f) + \partial_{X_1}(\nabla^{M_2} f) = -f\mathrm{Ric}_{M_1}(X_1) \tag{24}$$

$$\partial_{X_2}(\nabla^{M_1} f) + \nabla^{M_2}_{X_2}(\nabla^{M_2} f) = -f\mathrm{Ric}_{M_2}(X_2)\,. \tag{25}$$

It can be deduced that both $\partial_{X_1}(\nabla^{M_2} f) = 0$ and $\partial_{X_2}(\nabla^{M_1} f) = 0$, for all $(X_1, X_2) \in TM_1 \oplus TM_2$. But the first identity is equivalent to the existence of functions $a_1 \in C^\infty(M_1, \mathbb{R})$ and $a_2 \in C^\infty(M_2, \mathbb{R})$ such that $f(x_1, x_2) = a_1(x_1) + a_2(x_2)$ for all $(x_1, x_2) \in M$. Then the second identity is trivial and (25) is equivalent to

$$(\nabla^{M_2})^2 a_{2|_{x_2}} = -(a_1(x_1) + a_2(x_2))\mathrm{Ric}_{M_2|_{x_2}}$$

for all $(x_1, x_2) \in M_1 \times M_2$. But since the l.h.s. of the preceding inequality does not depend on M_1 and because of $\mathrm{Ric}_{M_2} \neq 0$, this implies a_1 is constant on M_1, therefore $a_1 + a_2 \in C^\infty(M_2, \mathbb{R})$ satisfies (1) on (M_2, g_2). But then (24) together with the assumption $f \neq 0$ forces $\mathrm{Ric}_{M_1} = 0$: choose a point $x_2 \in M_2$ where $f(x_2) \neq 0$. This concludes the proof. $\square$

Next we look for examples and partial classifications results for identities (12) and (13), which correspond to the case $f_1 = \varphi$. An obvious case is when $f_1 = \varphi$ are constant (and nonvanishing) on M_1. Then (M_1, g_1) must be Ricci-flat, $f(x_1, x_2) = f_2(x_2)$ for all $(x_1, x_2) \in M$ and, because of $\mu_1(f_1) = 0$ then, the function f_2 must satisfy (1) on (M_2, g_2). This is actually a consequence of Theorem 3.2 above. Therefore we obtain an already known example in that case, see introduction.

Proposition 3.3. *Let (M^n, g) be any connected Riemannian manifold.*

1. *Assume there exists an $f \in C^\infty(M, \mathbb{R}^\times_+)$ solving $\nabla^2 f = \frac{f}{m-1} \cdot \mathrm{Ric}$ on M for some integer $m \geq 2$. Then $\mu_1(f) := (m-2)|\nabla f|^2 + \frac{f^2 S}{m-1} = (m-2)|\nabla f|^2 - f\Delta f$ is constant on M and, if $m > 2$, then $\mathrm{Ric}(\nabla f) = -\frac{1}{(m-2)(m-1)}\nabla(f^2 S)$, where S is the scalar curvature of (M, g). Moreover, if $m > 2$, then f defines a $(0, n+m-1)$-Einstein metric on (M, g).*
2. *Assume there exists an $f \in C^\infty(M, \mathbb{R}^\times_+)$ solving $\nabla^2 f = f \cdot (\mu\mathrm{Id} - \mathrm{Ric})$ on M for some $\mu \in \mathbb{R}$. Then $\mathrm{Ric}(\nabla f) = -\frac{(n-1)\mu}{2}\nabla f + \frac{f}{4}\nabla S + \frac{S}{2}\nabla f$ and $\mu_2(f) := 2|\nabla f|^2 + f^2(S - (n+1)\mu) = 2|\nabla f|^2 + f\Delta f - \mu f^2$ is constant on M.*

3. *In case* $(M^n, g) = (M_1^{n_1} \times M_2^{n_2}, g_1 \oplus f_1^2 g_2)$ *for some* $f_1 \in C^\infty(M_1, \mathbb{R}_+^\times)$ *and* $f := \pi_1^* f_1 \cdot \pi_2^* f_2$ *for some* $f_2 \in C^\infty(M_2, \mathbb{R})$*, there are, for each* $n_1, n_2 \geq 1$ *examples of* (M_i, g_i, f_i) *for which* f *solves* (1).
4. *If* (M^n, g) *is closed and* $f \in C^\infty(M, \mathbb{R}_+^\times)$ *is such that* $\mu_1(f) := k|\nabla f|^2 - f\Delta f$ *is constant for some* $k \in \mathbb{R}$*, then* f *must be constant on* M *and therefore* $\mu_1(f)$ *must vanish. As a consequence, if there exists a nonzero* $f \in C^\infty(M, \mathbb{R}_+^\times)$ *solving* $\nabla^2 f = \frac{f}{m-1} \cdot \mathrm{Ric}$ *on some closed* M *and for some integer* $m \geq 2$*, then* f *must be constant and therefore* M *must be Ricci-flat.*

Proof. We first look at equation

$$\nabla^2 f = \frac{f}{m-1} \cdot \mathrm{Ric} \tag{26}$$

on M, for some integer $m \geq 2$. We first derive a few identities following from (26), see e.g. [18, Lemma 4]. We write down the proof for the sake of completeness. Namely, by (6), we know that

$$\begin{aligned}
\delta\left(\nabla^2 f\right) &= \Delta(\nabla f) - \mathrm{Ric}(\nabla f) \\
&= \nabla(\Delta f) - \mathrm{Ric}(\nabla f) \\
&= -\frac{1}{m-1}\nabla(fS) - \mathrm{Ric}(\nabla f) \\
&= -\frac{1}{m-1}\left(S\nabla f + f\nabla S\right) - \mathrm{Ric}(\nabla f)\,,
\end{aligned}$$

where, as above, $S := \mathrm{tr}(\mathrm{Ric})$ is the scalar curvature of (M, g) and where we have used $\Delta f = -\frac{fS}{m-1}$ tracing (26). But (26) also yields

$$\begin{aligned}
\delta\left(\nabla^2 f\right) &= \frac{1}{m-1}\left(-\mathrm{Ric}(\nabla f) + f\delta(\mathrm{Ric})\right) \\
&= \frac{1}{m-1}\Big(-\mathrm{Ric}(\nabla f) - \frac{f}{2}\nabla S\Big)\,,
\end{aligned}$$

so that, bringing both identities for $\delta\left(\nabla^2 f\right)$ together, we deduce that

$$\begin{aligned}
\frac{m-2}{m-1} \cdot \mathrm{Ric}(\nabla f) &= -\frac{1}{m-1}\Big(S\nabla f + \frac{f}{2}\nabla S\Big) \\
&= -\frac{1}{2(m-1)f} \cdot \nabla(f^2 S)\,. \qquad (27)
\end{aligned}$$

In case $m = 2$, we deduce that $\nabla(f^2 S) = 0$ i.e., that $f^2 S = -f\Delta f$ is constant on M. In case $m > 2$, we deduce that

$$\mathrm{Ric}(\nabla f) = -\frac{1}{2(m-2)f} \cdot \nabla(f^2 S)\,. \tag{28}$$

Still when $m > 2$, it follows that

$$\begin{aligned}\nabla\left(|\nabla f|^2\right) &= 2(\nabla)^2_{\nabla f} f\\ &= \frac{2f}{m-1}\mathrm{Ric}(\nabla f)\\ &\overset{(28)}{=} -\frac{1}{(m-2)(m-1)}\cdot\nabla(f^2S).\end{aligned}$$

Therefore, $\mu_1(f) := (m-2)|\nabla f|^2 + \frac{1}{m-1}f^2S = (m-2)|\nabla f|^2 - f\Delta f$ is constant on M. Note that this is also the case when $m = 2$ by the above remark. Note also that, when $m > 2$, identity (26) defines a so-called $(0, n+m-1)$-Einstein metric on (M, g) according to [14, 12]. By [5, Theorem 2.2], the existence of such a positive f is equivalent to the warped product $(M \times F, g \oplus f^2 g_F)$ being Ricci-flat, where (F, g_F) is an Einstein manifold of dimension $m-1$ and with $\mathrm{Ric}_F = \mu_1 \cdot \mathrm{Id}$, the constant μ_1 being given by $\mu_1 = (m-2)|\nabla f|^2 - f\Delta f = (m-2)|\nabla f|^2 + \frac{f^2S}{m-1}$, which is exactly the constant $\mu_1(f)$ described above, see also [18, Cor. 3]. This statement remains true when $m = 2$ and $\Delta f = 0$ (or equivalently $\mu_1(f) = 0$). This shows statement 1.

Next we look at

$$\nabla^2 f = f\cdot(\mu\mathrm{Id} - \mathrm{Ric}) \tag{29}$$

on M^n for some $\mu \in \mathbb{R}$ and $n \geq 2$. First and as before, a few identities can be deduced from (29). Namely, by (6), we know that

$$\begin{aligned}\delta\left(\nabla^2 f\right) &= \Delta(\nabla f) - \mathrm{Ric}(\nabla f)\\ &= \nabla(\Delta f) - \mathrm{Ric}(\nabla f)\\ &= \nabla(f(S - n\mu)) - \mathrm{Ric}(\nabla f)\\ &= (S - n\mu)\nabla f + f\nabla S - \mathrm{Ric}(\nabla f)\,,\end{aligned}$$

where we have used $\Delta f = f(S - n\mu)$ tracing (29). But (29) also yields

$$\begin{aligned}\delta\left(\nabla^2 f\right) &= -(\mu\nabla f - \mathrm{Ric}(\nabla f)) + f\delta\left(\mu\mathrm{Id} - \mathrm{Ric}\right)\\ &= -\mu\nabla f + \mathrm{Ric}(\nabla f) + \frac{f}{2}\nabla S\end{aligned}$$

so that, bringing both identities for $\delta\left(\nabla^2 f\right)$ together, we deduce that

$$\mathrm{Ric}(\nabla f) = -\frac{n-1}{2}\mu\nabla f + \frac{f}{4}\nabla S + \frac{S}{2}\nabla f\,. \tag{30}$$

It follows that

$$\begin{aligned}
\nabla\left(|\nabla f|^2\right) &= 2\nabla^2_{\nabla f} f \\
&= 2f(\mu\nabla f - \mathrm{Ric}(\nabla f)) \\
&\overset{(30)}{=} f\Big((n+1)\mu\nabla f - \frac{f}{2}\nabla S - S\nabla f\Big) \\
&= \frac{n+1}{2}\mu\nabla(f^2) - \frac{1}{2}\nabla(f^2 S) \\
&= \frac{1}{2}\nabla\left((n+1)\mu f^2 - f^2 S\right).
\end{aligned}$$

Therefore, $\mu_2(f) := 2|\nabla f|^2 + f^2(S-(n+1)\mu) = 2|\nabla f|^2 + f\Delta f - \mu f^2$ is constant on M. This proves statement 2.

As for statement 3, we look at different cases according to the values of n_2 and n_1.

Case $n_2 = 1$: Then (12) is equivalent to M_1 being Ricci-flat. Together with $f_1\Delta^{M_1} f_1 + |\nabla^{M_1} f_1|_1^2 = -\mu_1(f_1)$ being constant by Proposition 3.1, identity (13) is equivalent to $f_2'' = \mu_1(f_1)f_2$. Whatever the sign of $\mu_1(f_1)$, there exists a solution f_2 to that second-order linear ordinary differential equation on $\mathbb{R}$, which is periodic (and hence can be pulled down on a circle of suitable radius) if and only if $\mu_1(f_1) < 0$. As for f_1, a trivial family of examples in each dimension n_1 may be produced as follows. When $n_1 = 1$, the function f_1 solves the ordinary differential equation $-f_1 f_1'' + (f_1')^2 = -\mu_1(f_1)$, whose general solution is

$$f_1(t) = \begin{cases} a_1(t) & \text{if} \quad \mu_1(f_1) > 0 \\ b_1(t) & \text{if} \quad \mu_1(f_1) = 0 \\ c_1(t), d_1(t), e_1(t) & \text{if} \quad \mu_1(f_1) < 0 \end{cases}$$

where

$$\begin{aligned}
a_1(t) &:= A\cosh(A^{-1}\sqrt{\mu_1(f_1)}t + \phi) \\
b_1(t) &:= Ae^{\phi t} \\
c_1(t) &:= A\cos(A^{-1}\sqrt{-\mu_1(f_1)}t + \phi) \\
d_1(t) &:= \pm\sqrt{-\mu_1(f_1)}t + \phi \\
e_1(t) &:= A\sinh(A^{-1}\sqrt{-\mu_1(f_1)}t + \phi)
\end{aligned}$$

for real arbitrary constants A, ϕ with w.l.o.g. $A > 0$ (remember that $f_1 = \varphi > 0$ by assumption). Note that all solutions are defined on $\mathbb{R}$ but that, in case $\mu_1(f_1) < 0$, the function f_1 must change sign somewhere, which makes the solution f_1 only local then. Moreover, in case $\mu_1(f_1) \geq 0$, the solution f_1 – though positive on $\mathbb{R}$ – is not periodic and therefore cannot be pulled down on an $\mathbb{S}^1$. Obviously, each of the above f_1's can be trivially extended constantly in the other variables on $\mathbb{R}^{n_1}$ for every $n_1 \geq 1$.

It is important to note that, in the cases where $f_1 > 0$ on $\mathbb{R}$, corresponding to $\mu_1(f_1) \geq 0$ as we have seen above, the induced metric $ds^2 \oplus f_1(s)^2 dt^2$ on $\mathbb{R}^2$ is the hyperbolic one, for which we can anyway describe $W(M, g)$ explicitly.

Case $n_2 > 1$: When $n_2 \geq 2$ and $n_1 = 1$, equation (12) reduces to $f_1'' = 0$ on M_1, which has no positive solution on M_1 unless f_1 is constant or M_1 is a strict open subinterval of $\mathbb{R}$.

When $n_2 \geq 2$ and $n_1 = 2$, equation (12) is equivalent to $(\nabla^{M_1})^2 f_1 = f_1 \phi_1 \cdot \mathrm{Id}_{TM_1}$, where $\phi_1 := \frac{S_1}{2(n_2-1)}$. But by [23, Sec. 2], this implies that, on any open subset where f_1 has no critical point, (M_1^2, g_1) is locally isometric to $(\mathbb{R}^2, dt^2 \oplus \rho(t)^2 ds^2)$, where $\rho := \frac{u'}{u'(0)}$ and u is f_1 along the flow of its normalised gradient $\nu := \frac{\nabla^{M_1} f_1}{|\nabla^{M_1} f_1|_1}$. Moreover, along any integral curve γ of ν, which is a geodesic of (M_1, g_1) because of $\nabla^{M_1} f_1$ being a pointwise eigenvector of $(\nabla^{M_1})^2 f_1$, the function u must satisfy the following second-order ordinary differential equation: for any t in some nonempty open interval,

$$\begin{aligned}
u''(t) &= g_1((\nabla^{M_1})^2_{\dot\gamma(t)} f_1, \dot\gamma(t)) \\
&= \frac{(f_1 S_1) \circ \gamma(t)}{2(n_2 - 1)} \\
&= \Big(\frac{\mu_1(f_1)}{2f_1} - \frac{n_2 - 2}{2} \cdot \frac{|\nabla^{M_1} f_1|_1^2}{f_1}\Big) \circ \gamma(t) \\
&= -\frac{n_2 - 2}{2u(t)} u'(t)^2 + \frac{\mu_1(f_1)}{2u(t)} ,
\end{aligned}$$

that is,

$$u'' \cdot u + \frac{n_2 - 2}{2} (u')^2 = \frac{\mu_1(f_1)}{2} . \tag{31}$$

In the first special case where $\mu_1(f_1) = 0$, the general form of the solution u to (31) is $u(t) = (at + b)^{\frac{2}{n_2}}$ for real constants a, b with $a \neq 0$; assuming a and b to be positive, the maximal existence interval for u is $[-\frac{b}{a}, \infty)$, in particular no complete M_1 can exist unless f_1 has critical points.

In the second special case where $n_2 = 2$, the second-order ordinary differential equation (31) may be reduced to the first-order one

$$u' = \sqrt{\mu_1(f_1) \ln(u) + C}$$

for some real constant C. Note that this implies that u is constant when $n_2 = 2$ and $\mu_1(f_1) = 0$. If $\mu_1(f_1) > 0$, the maximal existence interval for u is of the form $]a, \infty[$, whereas if $\mu_1(f_1) < 0$, that interval is of the form $]-\infty, a[$ for some real a.

Conversely, let us assume u to be any positive solution with w.l.o.g. positive first derivative of (31) on some open interval I about 0. Consider the warped product $(M_1, g_1) := (I \times \Sigma, dt^2 \oplus \varphi(t)^2 ds^2)$ for $\Sigma = \mathbb{R}$ or $\mathbb{S}^1$, where $\varphi(t) := \frac{u'(t)}{u'(0)}$. Let $f(t, s) := u(t)$ for all $(t, s) \in M_1$. The above formulae (20) and (21) for the Hessian of f simplify to $\nabla^2_{\partial_t} f = u'' \cdot \partial_t$ and $\nabla^2_{\partial_s} f = \frac{u'\varphi'}{\varphi} \cdot \partial_s$. The identities (22) and (23) become $\mathrm{Ric} = -\frac{\varphi''}{\varphi} \cdot \mathrm{Id}_{TM}$. Taking into account that $\varphi = \frac{u'}{u'(0)}$, we have $\frac{u'\varphi'}{\varphi} = u''$, so that $\nabla^2 f = u'' \cdot \mathrm{Id}_{TM}$, as well as $\mathrm{Ric} = -\frac{u^{(3)}}{(n_2-1)u'} \cdot \mathrm{Id}_{TM}$ (recall that $n_2 = 2$ here). Therefore, $\nabla^2 f = \frac{f}{n_2-1} \cdot \mathrm{Ric}$ if and only if $u'' = -\frac{uu^{(3)}}{(n_2-1)u'}$ on I. But because $u'' = \frac{\mu_1(f_1)}{2u} - \frac{n_2-2}{2u}(u')^2$, we have

$$
\begin{aligned}
-\frac{uu^{(3)}}{(n_2-1)u'} &= -\frac{u}{(n_2-1)u'} \cdot \Big(\frac{\mu_1(f_1)}{2u} - \frac{n_2-2}{2} \cdot \frac{(u')^2}{u}\Big)' \\
&= -\frac{u}{(n_2-1)u'} \cdot \Big(-\frac{\mu_1(f_1)u'}{2u^2} - \frac{n_2-2}{2} \cdot \frac{2u'u''u-(u')^3}{u^2}\Big) \\
&= \frac{1}{n_2-1} \cdot \Big(\frac{\mu_1(f_1)}{2u} + \frac{n_2-2}{2} \cdot \frac{2u''u-(u')^2}{u}\Big) \\
&= \frac{1}{n_2-1} \cdot \Big(\frac{\mu_1(f_1)}{2u} + \frac{n_2-2}{2} \cdot \frac{\mu_1(f_1)-(n_2-2)(u')^2-(u')^2}{u}\Big) \\
&= \frac{1}{n_2-1} \cdot \Big(\frac{(n_2-1)\mu_1(f_1)}{2u} - \frac{(n_2-2)(n_2-1)(u')^2}{2u}\Big) \\
&= u'',
\end{aligned}
$$

so that (26) is satisfied on (M_1^2, g_1).

In the subcase where $n_2 = 2$, equation (13) is equivalent to $(\nabla^{M_2})^2 f_2 = f_2\phi_2 \cdot \mathrm{Id}_{TM_2}$, where $\phi_2 := \mu_1 - \frac{S_2}{2}$. Now (30) yields $\frac{S_2}{2}\nabla^{M_2} f_2 = \frac{S_2-\mu_1}{2}\nabla^{M_2} f_2 + \frac{f_2}{4}\nabla^{M_2} S_2$, that is, $f_2\nabla^{M_2} S_2 = 2\mu_1\nabla^{M_2} f_2$, which is equivalent to the existence of a real constant C such that

$$S_2 = 2\mu_1 \ln(|f_2|) + C$$

on each connected component of the dense open subset $M_2 \setminus f_2^{-1}(\{0\})$. Denoting $\mu_2 := \mu_2(f_2)$, it can be deduced that

$$
\begin{aligned}
|\nabla^{M_2} f_2|^2 &= \frac{\mu_2}{2} - \frac{f_2^2(S_2 - 3\mu_1)}{2} \\
&= \frac{\mu_2}{2} - \frac{f_2^2(2\mu_1 \ln(|f_2|) + C - 3\mu_1)}{2} \\
&= \frac{\mu_2}{2} + \left(\frac{3\mu_1 - C}{2} - \mu_1 \ln(|f_2|)\right) f_2^2.
\end{aligned}
$$

This gives rise to a first-order ordinary differential equation for $u(t) := f_2 \circ F_t^\nu$, where $(F_t^\nu)_t$ is the local flow of $\nu := \frac{\nabla^{M_2} f_2}{|\nabla^{M_2} f_2|_2}$ on some open subset of the regular set of f_2. Namely, [23, Sec. 2] again implies that, on any open subset where f_2 has no critical point and vanishes nowhere, (M_2^2, g_2) is locally isometric to $(\mathbb{R}^2, dt^2 \oplus \rho(t)^2 ds^2)$, where $\rho := \frac{u'}{u'(0)}$. Moreover, along any integral curve γ of ν, which is a geodesic of (M_2, g_2) because of $\nabla^{M_2} f_2$ being a pointwise eigenvector of $(\nabla^{M_2})^2 f_2$, the function u must satisfy the following first-order ordinary differential equation: for any t in some nonempty open interval,

$$u' = \left(\frac{\mu_2}{2} + \Big(\frac{3\mu_1 - C}{2} - \mu_1 \ln(|u|)\Big)u^2\right)^{\frac{1}{2}}. \tag{32}$$

Except in possibly very particular cases – e.g. when $\mu_1 = \mu_2 = C = 0$, in which u is constant – the maximal existence time for such a solution u to (32) is strictly contained in $\mathbb{R}$. Note also that, if u solves (32), then

$$\begin{aligned}
u'' &= \frac{1}{2}\left(\frac{\mu_2}{2} + \Big(\frac{3\mu_1 - C}{2} - \mu_1 \ln(|u|)\Big)u^2\right)^{-\frac{1}{2}} \cdot \big((3\mu_1 - C - 2\mu_1 \ln(|u|))uu' - \mu_1 u'u\big)\\
&= \left(\frac{\mu_2}{2} + \Big(\frac{3\mu_1 - C}{2} - \mu_1 \ln(|u|)\Big)u^2\right)^{-\frac{1}{2}} \cdot \Big(\mu_1 - \frac{C}{2} - \mu_1 \ln(|u|)\Big)uu'\\
&= (u')^{-1} \cdot \Big(\mu_1 - \frac{C}{2} - \mu_1 \ln(|u|)\Big)uu'\\
&= \Big(\mu_1 - \frac{C}{2} - \mu_1 \ln(|u|)\Big)u\,,
\end{aligned}$$

where $\mu_1 - \frac{C}{2} - \mu_1 \ln(|u|) = \mu_1 - \frac{S_2 \circ \gamma}{2}$ by the above identity for S_2.
This implies that, given any nowhere vanishing solution u to (32) on some open interval I about 0, the function $f(t,s) := u(t)$ solves

$$\nabla^2 f = u'' \cdot \mathrm{Id}_{TM} = \Big(\mu_1 - \frac{S}{2}\Big) \cdot \mathrm{Id}_{TM}$$

on $(M_2^2, g_2) := (I \times \Sigma, dt^2 \oplus (\frac{u'(t)}{u'(0)})^2 ds^2)$, where $\Sigma = \mathbb{R}$ or $\mathbb{S}^1$.

Still in the case where $n_2 = 2$, equation (26) has not been considered yet in the literature as far as we know. In the special subcase where $\mu_1 = 0$, which is equivalent to $S_1 = 0$, equation (26) can be rewritten under the form $(\nabla^{M_1})^2 f_1 = f_1 \cdot \mathrm{Ric}_{M_1} - (\Delta^{M_1} f_1) \cdot \mathrm{Id}$, which is the general form of an element of $\ker(L^*_{g_1})$ in [6] when the underlying manifold is scalar-flat. In case $\ker(L^*_{g_1}) \neq \{0\}$, the metric g_1 is called *static*. Although it is unclear whether a nonconstant positive solution f_1 to that equation can exist on a complete M_1, there is a noncomplete example: take the outer Schwarzschild manifold $(\mathbb{R}^3 \setminus \overline{B}_{\frac{m}{2}}, (1 + \frac{m}{2r})^4 \langle\cdot\,,\cdot\rangle)$ for some constant $m > 0$, where $r = r(x) = |x|$ in $\mathbb{R}^3$ and $f_1(x) = \frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}$, see [6, p.145]. In case M_1 is either closed, complete with nonnegative Ricci curvature or with so-called moderate volume growth, the function f_1 must be constant. The latter two are due to S.T. Yau [24, Cor. 1 p. 217] and to L. Karp [16, Theorem B] (see also [17, Sec. 3]) respectively, using only the harmonicity of f_1. As a consequence, if $n_1 = 2$ (and $n_2 = 2$), then there is no nonconstant solution f_1 (for $S_1 = 0$ implies $\mathrm{Ric}_{M_1} = 0$).

Case $n_2 > 2$ and $n_1 > 2$: Then (26) defines a so-called $(0, n_1 + n_2 - 1)$-Einstein metric on (M_1, g_1) according to [14, 12] as we noticed in statement 1. As for (13), it has not been considered either in the literature when $\mu_1 \neq 0$ – for $\mu_1 = 0$, it is already (1) on M_2. When $\mu_1 \neq 0$, we may take for $(M_2^{n_2}, g_2, f_2)$ the standard solution to the Obata resp. Tashiro equation on the n_2-dimensional simply-connected spaceform of sectional curvature $\frac{\mu_1(f_1)}{n_2 - 2}$, which are the only Einstein solutions to

(13) when $n_2 > 2$. This shows statement 3.

In the particular case where (M^n, g) is closed and $f \in C^\infty(M, \mathbb{R}_+^\times)$ is such that $\mu_1(f) := k|\nabla f|^2 - f\Delta f$ is constant for some $k \in \mathbb{R}$, we can mimic the proof of Lemma 2.1.5. First, we have $\mu_1(f) = 0$: it suffices to evaluate $\mu_1(f)$ at two points, one where $\min\limits_M(f)$ is attained and one where $\max\limits_M(f)$ is attained to obtain that $\mu_1(f)$ must be both nonpositive and nonnegative because of $f > 0$ and the opposite signs of the Laplace operator of f at a minimum and maximum respectively. Independently, we can integrate $\mu_1(f)$ over M and obtain

$$\mu_1(f) \cdot \mathrm{Vol}(M^n, g) = (k-1) \cdot \int_M |\nabla f|^2 \, d\mu_g \, .$$

Therefore, if $k \neq 1$, then f must be constant. If $k = 1$, the vanishing of $\mu_1(f)$ is equivalent to $\Delta f = \frac{|\nabla f|^2}{f} \geq 0$ on the closed manifold M, which with $\int_M \Delta f \, d\mu_g = 0$ shows that, again, $\nabla f = 0$ must hold on M, therefore f must also be constant on M. This proves statement 4 and concludes the proof of Proposition 3.3. □

In case the factor (M_1, g_1) of the warped product is complete, we show that actually the map f must be constant along M_1.

Theorem 3.4. *Let $f = \pi_1^* f_1 \cdot \pi_2^* f_2$ satisfy* (1) *on $(M^n, g) = (M_1 \times M_2, g_1 \oplus f_1^2 g_2)$ for some smooth positive function f_1 on M_1 and smooth function f_2 on M_2. Assume (M_1, g_1) to be complete and connected.*
Then f_1 must be constant on M_1, the manifold (M_1, g_1) must be Ricci-flat and f_2 must satisfy (1) *on (M_2, g_2). Therefore, the map $W(M_2, g_2) \longrightarrow W(M, g)$ extending any solution* (1) *to M is an isomorphism.*

Proof. In case $f_1 > 0$ on M_1 and for $f = \pi_1^* f_1 \cdot \pi_2^* f_2$ on $M_1 \times_{f_1^2} M_2$, the constants $\mu(f)$, $\mu_1(f_1)$ and $\mu_2(f_2)$ defined above are related as follows:

$$\begin{aligned}
\mu(f) &= f\Delta f + 2|\nabla f|^2 \\
&= f_1 f_2((\Delta f_1) f_2 + f_1 \Delta f_2) + 2|f_2(\nabla f_1) + f_1 \nabla f_2|^2 \\
&= f_1 f_2((\Delta^{M_1} f_1) f_2 + \frac{f_1}{f_1^2} \Delta^{M_2} f_2) + 2|f_2(\nabla^{M_1} f_1) + \frac{f_1}{f_1^2} \nabla^{M_2} f_2|^2 \\
&= f_1(\Delta^{M_1} f_1) f_2^2 + f_2(\Delta^{M_2} f_2) + 2 f_2^2 |\nabla^{M_1} f_1|_1^2 + 2|\nabla^{M_2} f_2|_2^2 \\
&= \left(f_1(\Delta^{M_1} f_1) + 2|\nabla^{M_1} f_1|_1^2\right) \cdot f_2^2 + f_2 \Delta^{M_2} f_2 + 2|\nabla^{M_2} f_2|_2^2 \\
&= \left(f_1(\Delta^{M_1} f_1) + 2|\nabla^{M_1} f_1|_1^2 + \mu_1(f_1)\right) \cdot f_2^2 \\
&\quad + f_2 \Delta^{M_2} f_2 + 2|\nabla^{M_2} f_2|_2^2 - \mu_1(f_1) f_2^2 \\
&= n_2 |\nabla^{M_1} f_1|_1^2 f_2^2 + \mu_2(f_2) \, .
\end{aligned}$$

This implies that, if $f \neq 0$ solves (1) and $\varphi = f_1 > 0$, then $|\nabla^{M_1} f_1|_1$ is constant on M_1. Note that this holds whether (M_1, g_1) is complete or not, i.e. whenever M_1 is connected. From now on assume (M_1, g_1) to be complete. By contradiction, if $|\nabla^{M_1} f_1|_1$ were a positive constant, then f_1 would have no critical point on M_1 and therefore the flow of the normalised gradient vector field $\nu_1 := \frac{\nabla^{M_1} f_1}{|\nabla^{M_1} f_1|_1}$ would define a diffeomorphism from M_1 to the product $\mathbb{R} \times \Sigma_1$ for some smooth level hypersurface Σ_1 of f_1; and f_1 would be a nonconstant affine linear function of $t \in \mathbb{R}$. But this would contradict $f_1 > 0$ on M_1. Therefore, $\nabla^{M_1} f_1 = 0$ must hold on M_1 i.e., f_1 must be constant on M_1. In turn, this implies that $\mu_1(f_1) = 0$, $\mathrm{Ric}_{M_1} = 0$ when $n_2 \geq 2$ (anyway $\mathrm{Ric}_{M_1} = 0$ when $n_2 = 1$ as we saw above) and that $f_2 \in W(M_2, g_2)$. Therefore, the function f is the trivial extension on M of $f_2 \in W(M_2, g_2)$. $\square$

4. Case where $\dim(W(M^n, g)) \geq 2$

In this section, we look at the particular case where (1) has a $k \geq 2$-dimensional space of solutions.

Theorem 4.1. *Let (M^n, g) be any connected complete Riemannian manifold. Assume that (1) has a $k \geq 2$-dimensional space of solutions. Then we have one of the following:*

1. *Case $k = 2$: the manifold (M^n, g) must be isometric to the Riemannian product $(M_1^{n-1} \times \mathbb{R}, g_1 \oplus dt^2)$ for some complete Ricci-flat manifold admitting no line (M_1^{n-1}, g_1). Moreover, the solutions of (1) on (M^n, g) are the affine linear functions of $t \in \mathbb{R}$ extended constantly along M_1.*
2. *Case $k > 2$: the manifold (M^n, g) must be isometric to the Riemannian product $(M_1^{n-k+1} \times M_2^{k-1}, g_1 \oplus g_2)$ for some complete Ricci-flat manifold admitting no line (M_1^{n-k+1}, g_1) and where (M_2^{k-1}, g_2) is either $\mathbb{S}^2, \mathbb{R}^2$ or $\mathbb{H}^2$ with standard metric of curvature $1, 0, -1$ (up to rescaling g) for $k = 3$ or is $\mathbb{R}^{k-1}$ with standard flat metric for $k > 3$. Moreover, the solutions of (1) on (M^n, g) are the solutions of the Obata resp. Tashiro equation on (M_2, g_2) extended constantly along M_1.*

Proof. We first assume M to be simply-connected. By [14, Theorem A], which can be applied since (1) is the particular case of the equation $\nabla^2 f = f \cdot q$ for some quadratic form q on TM, we already know that, if $k \geq 2$, then (M^n, g) must be isometric to the warped product $(M_1 \times M_2, g_1 \oplus f_1^2 g_2)$ for some smooth positive function f_1 on M_1, where (M_1^{n-k+1}, g_1) and (M_2^{k-1}, g_2) are complete [3, Lemma 7.2] simply-connected Riemannian manifolds and f_1 is a smooth positive function on M_1. Moreover, (M_2, g_2) must be a spaceform and any solution f of (1) is of the form $f = \pi_1^* f_1 \cdot \pi_2^* f_2$, where f_2 satisfies the Obata resp. Tashiro equation on (M_2, g_2) [14, Theorem B]. Taking the above considerations on solutions of (1) on warped products into account in case f_1 is the warping function, Theorem 3.4 can be applied and implies that f_1 is constant, that (M_1, g_1) is Ricci-flat and that $f_2 \in W(M_2, g_2)$. We look at different cases according to k:

1. Case $k = 2$: then we could conclude above that f_2 is an affine linear function of $t \in \mathbb{R}$. Since no nonconstant affine function can be periodic, any group action leaving invariant some nonconstant $f_2 \in W(M_2, g_2)$ must be trivial. Moreover, if (M_1, g_1) could be split off a line, then it would be isometric to $\Sigma_1 \times \mathbb{R}$ for

some smooth hypersurface Σ_1 of M_1; but then $M_1 \times \mathbb{R} \cong \Sigma_1 \times \mathbb{R}^2$ would carry a $k \geq 3$-dimensional space of solutions to (1), which would contradict $k = 2$. Therefore, (M_1, g_1) cannot contain any line.

2. Case $k > 2$: then we could conclude above that $f_2 \in W(M_2, g_2)$. If $k = 3$, then, up to rescaling g, the manifold (M_2, g_2) must be isometric to either $\mathbb{S}^2, \mathbb{R}^2$ or $\mathbb{H}^2$ with standard metric of constant curvature $1, 0, -1$ respectively; and $W(M_2, g_2)$ must consist of the solutions of the Obata resp. Tashiro equation on (M_2, g_2) as we saw in Lemma 2.1.9. Again, in case $M_2 = \mathbb{S}^2$ or $\mathbb{H}^2$, no group action on M_2 can leave any nonzero solution to (1) invariant on M_2. If $M_2 = \mathbb{R}^2$, then no nontrivial group action preserves the 3-dimensional space of affine linear functions on $\mathbb{R}^2$.
If $k > 3$, then, as a consequence of Lemma 2.1.9, the manifold (M_2, g_2) must be isometric to flat $\mathbb{R}^{k-1}$ and again no nontrivial group action preserves the k-dimensional space of affine linear functions on $\mathbb{R}^{k-1}$.
In both subcases, (M_1, g_1) cannot contain any line, otherwise $\dim(W(M^n, g)) \geq k + 1$.

In all cases, the only possible nontrivial group actions on $M_1 \times M_2$ is trivial along the M_2 factor. Thus, if M is not simply-connected, then M must be isometric to $M_1^{n-k+1} \times M_2^{k-1}$, where M_2 is a simply connected model space as above and M_1 is a complete Ricci-flat manifold having no line since its universal cover cannot contain any. Furthermore, every $f \in W(M, g)$ must be the trivial extension on $M_1 \times M_2$ of a solution $f_2 \in W(M_2, g_2)$. This concludes the proof of Theorem 4.1. $\square$

Note that, as a consequence of Theorem 4.1, if a complete (M^n, g) carries an $(n+1)$-dimensional space of solutions to (1) with $n \neq 2$, then (M^n, g) must be isometric to $\mathbb{R}^{n+1}$ with standard flat metric.

5. Homogeneous case

Next, we look at homogeneous manifolds carrying nontrivial solutions of (1).

Theorem 5.1. *Let (M^n, g) be any connected homogeneous Riemannian manifold. Assume the existence of a non-identically vanishing smooth function f on M satisfying* (1)*.*
Then one of the following holds:

1. *If the scalar curvature S of (M^n, g) vanishes and f is nonconstant, then (M^n, g) must be isometric to a flat manifold $\mathbb{R}^n/\Gamma$ for some discrete fixed-point free subgroup Γ of $\mathrm{O}(n) \ltimes \mathbb{R}^n$.*
2. *If $k := \dim(W(M^n, g)) = 2$, then (M^n, g) must be isometric to the Riemannian product $\mathbb{R}^{n-1}/\Gamma \times \mathbb{R}$ for some discrete fixed-point free and co-compact subgroup Γ of $\mathrm{O}(n-1) \ltimes \mathbb{R}^{n-1}$. In that case, the map $W(\mathbb{R}, dt^2) \longrightarrow W(M^n, g)$ extending any affine linear function trivially on the first factor is an isomorphism.*
3. *If $k = 3$, then up to rescaling g, the manifold (M^n, g) must be isometric to the Riemannian product $\mathbb{R}^{n-2}/\Gamma \times S^2(\varepsilon)$, where $S^2(\varepsilon)$ is the simply-connected*

complete surface of constant curvature $\varepsilon \in \{0, \pm 1\}$ *and* ${}^{\mathbb{R}^{n-2}}\!/\!_{\Gamma}$ *is a compact flat manifold. In that case, the map* $W(S^2(\varepsilon), g_{S^2(\varepsilon)}) \longrightarrow W(M^n, g)$ *extending any function trivially on the* Σ*-factor is an isomorphism.*

4. *If* $k \geq 4$*, then* (M^n, g) *must be isometric to the Riemannian product* ${}^{\mathbb{R}^{n-k+1}}\!/\!_{\Gamma} \times \mathbb{R}^{k-1}$*, where* ${}^{\mathbb{R}^{n-k+1}}\!/\!_{\Gamma}$ *is a compact flat manifold and* $\mathbb{R}^{k-1}$ *carries its standard Euclidean metric.*
5. *If* $k = 1$*, then unless* $W(M^n, g)$ *consists of constant functions,* $\mu(f) = 0$ *must hold for every* $f \in W$*. Moreover, the manifold* (M^n, g) *must be a one-dimensional extension of some homogeneous Riemannian manifold satisfying the particular conditions* (33) *below.*

Proof. If (M^n, g) has vanishing scalar curvature and f is nonconstant, then we already know from Lemma 2.1 that (M^n, g) must be Ricci-flat. But because any homogeneous Ricci-flat Riemannian manifold must be flat [1], actually (M^n, g) must be isometric to a flat manifold ${}^{\mathbb{R}^n}\!/\!_{\Gamma}$ for some discrete and necessarily fixed-point free subgroup Γ of $\mathrm{O}(n) \ltimes \mathbb{R}^n$. This shows statement 1.
If $\dim(W(M^n, g)) = k \geq 2$, then Theorem 4.1 implies that (M^n, g) must be isometric to the Riemannian product $M_1^{n-k+1} \times M_2^{k-1}$, where M_1^{n-k+1} is a Ricci-flat manifold containing no line and M_2^{k-1} is flat Euclidean space except when $k = 3$, in which case it is also allowed to be $\mathbb{S}^2$ or $\mathbb{H}^2$ with standard spherical resp. hyperbolic metric. Moreover, any solution to (1) must be the trivial extension to M of a standard solution on M_2. Now recall the following result, which is a combination of Lemma 5.6 and the first part of the proof of Theorem 5.7 in [12]; the latter can be applied because of $W(M^n, g)$ being invariant under isometry: in our notation, the isometries of $(M_1 \times M_2, g_1 \oplus g_2)$ are the maps of the form $h = (h_1, h_2)$, where h_1 and h_2 are isometries of (M_1, g_1) and (M_2, g_2) respectively. This already implies that, writing $M = {}^{G}\!/\!_{K}$, the group G when can be embedded into the direct product of two groups, the first one acting isometrically and transitively on M_1 and the second one acting transitively on M_2. In particular, (M_1, g_1) must itself be homogeneous. In turn, this implies that, being Ricci-flat, (M_1, g_1) must be flat, again by [1]. Therefore (M_1, g_1) must be isometric to ${}^{\mathbb{R}^{n-k+1}}\!/\!_{\Gamma}$ for some discrete fixed-point free subgroup Γ of $\mathrm{O}(n-k+1) \ltimes \mathbb{R}^{n-k+1}$. Since only *compact* flat manifolds have no line, the subgroup Γ must be co-compact i.e., M_1 must be compact. This shows statements 2, 3 and 4.

Let us now assume the space $W(M^n, g)$ of functions satisfying (1) to be one-dimensional on $M = {}^{G}\!/\!_{K}$. Then as in [12, Sec. 5] we consider the action of G on $W(M^n, g)$. Because the Ricci-tensor of M is isometry- and thus G-invariant, so is equation (1), i.e. for every f satisfying (1) and every $h \in G$, the function $f \circ L_{h^{-1}}$ also satisfies (1). But because of $\dim(W(M^n, g)) = 1$, there exists for a fixed nonzero $f \in W(M^n, g)$ and every $h \in G$ a nonzero constant C_h such that $f \circ L_{h^{-1}} = C_h \cdot f$. The map $G \to \mathbb{R}^\times$, $h \mapsto C_h$ is a Lie-group homomorphism and actually takes its values in $\{\pm 1\}$ if $\mu(f) \neq 0$ since, by invariance of $\mu(f)$ under isometry,

$$\mu(f) = \mu(f \circ L_{h^{-1}}) = \mu(C_h \cdot f) = C_h^2 \cdot \mu(f)$$

for every $h \in G$. Therefore, if $\mu(f) \neq 0$, then $C_h \in \{\pm 1\}$ for every $h \in G$. Now if M is connected as in the assumptions, then so can be assumed G (otherwise replace G by the connected component of the neutral element), in which case necessarily $C_h = 1$ holds for every $h \in G$ and therefore every $f \in W(M^n, g)$ is constant.

Therefore $\mu(f) = 0$ holds. As a consequence, $S = -2$ and f has no critical point on M, see Lemma 2.1.

Next we show that (M^n, g) must be the one-dimensional extension of some homogeneous Riemannian manifold N^{n-1} with Ricci-tensor having particular properties. Consider the subgroup H of G defined by

$$H := \{h \in G \mid C_h = 1\} ,$$

that is, H is the subgroup of all elements of G leaving a (thus any) function $f \in W(M^n, g)$ invariant. Since $C\colon G \to \mathbb{R}_+^\times$ is a nontrivial and therefore surjective Lie-group-homomorphism, $H = \ker(C)$ is a closed normal subgroup of G and of codimension 1. Moreover, fixing $f \in W(M^n, g) \setminus \{0\}$, we know from Lemma 2.1 that $f(M) = \mathbb{R}_+^\times = (0, \infty)$ since f can be expressed as an exponential function along any integral curve of its normalised gradient. We let $N := f^{-1}(\{1\})$, which is a smooth hypersurface of M. By definition, H leaves N invariant. Moreover, fixing some $x \in N$, any $h \in G$ with $L_h(x) = x$ must satisfy $C_h = 1$ and therefore lie in H. In other words, the isotropy group $H_x := \{h \in H \mid L_h(x) = x\}$ of x under the H-action must coincide with $K = G_x$. Independently, for any $y \in N$, there is an $h \in G$ such that $L_h(x) = y$; again, because of $f(x) = f(y) \neq 0$, necessarily $C_h = 1$ must hold, i.e. $h \in H$. This proves that the orbit $H \cdot x := \{L_h(x) \,|\, h \in H\}$ of x in N must be all of N and therefore $N = {}^{H}\!/_{K}$ is a H-homogeneous Riemannian manifold. As in the proof of [13, Theorem 5.1], we split the Lie algebra $\underline{G} = \underline{P} \oplus \underline{K}$ of G in an $\mathrm{Ad}_G(K)$-invariant and orthogonal way and let $\xi \in \underline{P} \cong TM$ be the vector corresponding to $\nu \in T^\perp N$. Note that, because of $C_{|_H} = 1$, the gradient of f and therefore also ν are preserved by the H-action, so that ξ makes sense. Actually, $\underline{P} = \mathbb{R}\xi \oplus \big((\mathbb{R}\xi)^\perp \cap \underline{P}\big)$ and $\underline{H} = \big((\mathbb{R}\xi)^\perp \cap \underline{P}\big) \oplus \underline{K}$, the splittings being orthogonal. Furthermore, the Lie-bracket of ξ in $\underline{G}$ preserves $\underline{H}$ because of H being a normal subgroup of G. This already proves that $G = H \ltimes \mathbb{R}$ and that (M, g) is the one-dimensional extension of the H-homogeneous space $(N^{n-1}, g_{|_N})$.

In that case, following [13], we fix some $\alpha \in \mathbb{R}^\times$ and let $D := \frac{1}{\alpha}[\xi, \cdot] = \frac{1}{\alpha}\mathcal{L}_\xi$, which is hence a derivation of $\underline{H}$. We denote by $\mathcal{S}$ and $\mathcal{A}$ the symmetric and skew-symmetric parts of D respectively seen as endomorphisms of TN, see [13, Eq. (2.1)]. Let $\mathcal{T} := -\nabla\xi$ denote the Weingarten map of N in M. Then by [13, Prop. 2.7] we have $\mathcal{T} = \alpha\mathcal{S}$ and $\nabla_\xi \mathcal{T} = -\alpha^2[\mathcal{S}, \mathcal{A}]$. Furthermore, [13, Lemma 2.9] implies that, for all $X, Y \in TN$,

$$\begin{cases} \mathrm{ric}(\xi, \xi) &= -\alpha^2 \mathrm{tr}(\mathcal{S}^2) \\ \mathrm{ric}(X, \xi) &= \alpha(\delta\mathcal{S})(X) \\ \mathrm{ric}(X, Y) &= \mathrm{ric}^N(X, Y) - (\alpha^2 \mathrm{tr}(\mathcal{S}))g(\mathcal{S}X, Y) - \alpha^2 g([\mathcal{S}, \mathcal{A}]X, Y) \end{cases}$$

Now writing $f(t) = e^t$, where t lies in the $\mathbb{R}$-factor of $G = H \ltimes \mathbb{R}$, we have $\nabla df = f dt^2 - f g(T\cdot, \cdot)$ which, together with $\nabla_\xi \xi = 0$, gives that identity (1) is equivalent to

$$\begin{cases} \alpha^2\mathrm{tr}(\mathcal{S}^2)(=\alpha^2|\mathcal{S}|^2) & =1 \\ \alpha(\delta\mathcal{S}) & =0 \\ -\alpha g(\mathcal{S}X,Y) & =-\mathrm{ric}^N(X,Y)+\alpha^2\mathrm{tr}(\mathcal{S})g(\mathcal{S}X,Y)+\alpha^2 g([\mathcal{S},\mathcal{A}]X,Y) \end{cases}$$

for all $X, Y \in TN$. In other words, (1) is equivalent to

$$\begin{cases} \alpha & =\frac{\epsilon}{|\mathcal{S}|} \\ \delta\mathcal{S} & =0 \\ \mathrm{Ric}_N & =\frac{1}{|\mathcal{S}|^2}\left(\left(\mathrm{tr}(\mathcal{S})+\epsilon|\mathcal{S}|\right)\mathcal{S}+[\mathcal{S},\mathcal{A}]\right) \end{cases} \tag{33}$$

for some $\epsilon \in \{\pm 1\}$. This shows statement 5 and completes the proof of Theorem 5.1. $\square$

Note that [21, Theorem 1.5] allows for some partial classification in case (M^n, g) is homogeneous, because in their notation our 2-tensor $q = -\mathrm{Ric}$ is preserved by the group action. Nevertheless, we point out that the results we obtain in Theorem 5.1 describe the underlying space as well as the space of solutions in a more detailed way according to the dimension.

The case where $\dim(W(M^n, g)) = 1$ could lead to new examples, see [13] and [11].

6. Kähler case

As in [5], we next consider the case where (M^n, g) is assumed to be Kähler.

Theorem 6.1. *Assume (M^{2n}, g, J) to be a complete Kähler manifold and let f be any nonconstant smooth real-valued function satisfying (1) on M. Then, up to rescaling g, the Kähler manifold (M^{2n}, g, J) is holomorphically isometric to $S^2(\varepsilon)\times\Sigma^{2n-2}$ for some Ricci-flat Kähler manifold Σ, where $S^2(\varepsilon) = \mathbb{S}^2$ if $\varepsilon = 1$, $\mathbb{H}^2$ if $\varepsilon = -1$ and either $\mathbb{R}^2$ or $\mathbb{R}\times\mathbb{S}^1$ if $\varepsilon = 0$; moreover, the Kähler structure is the product Kähler structure and f is the trivial extension to M of a solution to (1) on $S^2(\varepsilon)$.*

Proof. The first steps follow those in the proof of [5, Theorem 1.3]. Since the Ricci-tensor of (M, g, J) is J-invariant, so is the Hessian of f by (1), i.e. $\nabla^2 f\circ J = J\circ\nabla^2 f$. As a first consequence, the vector field $J\nabla f$ is a (real) holomorphic vector field on (M, g, J) and therefore its zeros – which are precisely the critical points of f – form a totally geodesic Kähler submanifold of M of dimension $2k < 2n$; in particular the regular set of f is dense in M. As a second consequence, the 2-form $g(\nabla^2 f\circ J\cdot\,,\cdot)$ may be rewritten $\frac{1}{2}\mathcal{L}_{\nabla f}\Omega$, where $\Omega := g(J\cdot\,,\cdot)$ is the Kähler form of (M, g, J). Therefore,

$$d\left(g(\nabla^2 f\circ J\cdot\,,\cdot)\right) = \frac{1}{2}d\left(\mathcal{L}_{\nabla f}\Omega\right) = \frac{1}{2}d\left(\nabla f\lrcorner d\Omega + d(\nabla f\lrcorner\Omega)\right) = 0\,,$$

i.e. $g(\nabla^2 f\circ J\cdot\,,\cdot)$ is a closed 2-form on M. But because the Ricci-form $g(\mathrm{Ric}\circ J\cdot\,,\cdot)$ is also closed on M, so is the 2-form $\frac{1}{f}g(\nabla^2 f\circ J\cdot\,,\cdot)$ on $\{f\neq 0\}$, again by (1). This implies $df\wedge\left(g(\nabla^2 f\circ J\cdot\,,\cdot)\right) = 0$ on $\{f\neq 0\}$ and therefore on M by density (recall that $f^{-1}(\{0\})$, if nonempty, is a totally geodesic hypersurface of (M, g)). In turn this implies the existence at each regular point of f of a linear form λ on $(\nabla f)^\perp$ such that, for every $X \perp \nabla f$,

$$\nabla^2_{JX} f = \lambda(X) \nabla f \,. \tag{34}$$

For $X = J\nabla f$, we obtain via (2) that ∇S is pointwise tangent to ∇f, i.e. there exists a function θ on M such that $\nabla S = \theta \nabla f$ on M (this holds true on the regular set of M and hence on M by density, taking into account that at every critical point both ∇f and ∇S vanish). For $X \in \{\nabla f, J\nabla f\}^\perp$, by J-invariance of $\nabla^2 f$ the r.h.s. of (34) must vanish whenever the basepoint is a regular point of f. In turn this implies $\mathrm{Ric}(X) = 0$ for all $X \in \{\nabla f, J\nabla f\}^\perp$ and at every regular point of f. Now because of $\mathrm{Ric}(\nabla f) = \left(\frac{S}{2} + \frac{f\theta}{4}\right)\nabla f$, the J-invariance of Ric and $\mathrm{Ric}_{|_{\{\nabla f, J\nabla f\}^\perp}} = 0$, we obtain

$$S = S + \frac{f\theta}{2} \,,$$

so that $\theta = 0$, first on the regular set of f and then on M by density, i.e. S is constant on M. This implies that both distributions $\mathrm{Span}(\nabla f, J\nabla f)$ and $\{\nabla f, J\nabla f\}^\perp$ are integrable and totally geodesic, the former one being the tangent bundle of a surface of curvature $\frac{S}{2}$ – which may be assumed to be ± 1 up to rescaling g in case $S \neq 0$ – and the latter the tangent bundle of a necessarily Ricci-flat Kähler manifold Σ. The rest of the proof is analogous to that of Theorem 2.2.3. ☐

7. Outlook

The equation (1) can be seen as a particular case of the more general equation

$$\nabla^2 f = -\frac{f}{m} \left(\mathrm{Ric} - \lambda \cdot \mathrm{Id}\right) \,, \tag{35}$$

where $\lambda \in \mathbb{R}$ and $m \in \mathbb{N}$ are parameters which are *a priori* allowed to take arbitrary values. Note that, for $m = n - 2$ and positive f, equation (35) is the same as equation (7) in [19, Lemma 2.1]. As in [11, 13, 14, 12], a much broader and richer family of geometries could be recovered from Equation (35). This is the object of future work.

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