ARCHIVUM MATHEMATICUM (BRNO)
Tomus 59 (2023), 223–230

EXISTENCE OF BLOW-UP SOLUTIONS FOR A DEGENERATE PARABOLIC-ELLIPTIC KELLER–SEGEL SYSTEM WITH LOGISTIC SOURCE

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Abstract. This paper deals with existence of finite-time blow-up solutions to a degenerate parabolic–elliptic Keller–Segel system with logistic source. Recently, finite-time blow-up was established for a degenerate Jäger–Luckhaus system with logistic source. However, blow-up solutions of the aforementioned system have not been obtained. The purpose of this paper is to construct blow-up solutions of a degenerate Keller–Segel system with logistic source.

1. Introduction and main result

In this paper we consider the quasilinear degenerate Keller–Segel system with logistic source,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u^m - \chi\nabla\cdot(u\nabla v) + \lambda u - \mu u^\kappa\,, & x\in\Omega,\ t>0\,,\\ 0 = \Delta v - v + u\,, & x\in\Omega,\ t>0\,,\\ \dfrac{\partial u^m}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0\,, & x\in\partial\Omega,\ t>0\,,\\ u(x,0) = u_0(x)\,, & x\in\Omega\,, \end{cases} \tag{1.1}$$

where $\Omega := B_R(0)\subset\mathbb{R}^n$ $(n\geq 3)$ be a ball with some $R>0$; $m\geq 1$, $\chi>0$, $\lambda>0$, $\mu>0$ and $\kappa>1$; ν is the outward normal vector to $\partial\Omega$; $u_0\in L^\infty(\Omega)$ is nonnegative and radially symmetric. This system describes a situation such that a cellular slime moves towards higher concentrations of the chemical substance.

In the case $m=1$, Winkler [10] obtained initial data leading to finite-time blow-up under a smallness condition for $\kappa>1$ in three- or higher-dimensional cases. In the case $m\in\left[1,2-\frac{2}{n}\right)$, for the system such that the diffusion term is replaced with $\nabla\cdot((u+1)^{m-1}\nabla u)$, Black, Fuest and Lankeit showed that solutions blow up in finite time under the condition that $\kappa<1+\min\left\{\frac{(m-1)n+1}{2(n-1)},\frac{n-2-(m-1)n}{n(n-1)}\right\}$ in [1, Theorem 1.2 (ii)]. On the other hand, a difficulty is caused in (1.1) by the degenerate diffusion term Δu^m because in the case of nondegenerate diffusion

2020 *Mathematics Subject Classification*: primary 35B44; secondary 35K65, 92C17.
Key words and phrases: degenerate Keller–Segel system, logistic source.
This research was supported by JSPS KAKENHI Grant Number JP22J11193.
Received August 9, 2022, accepted November 20, 2022. Editor Š. Nečasová.
DOI: 10.5817/AM2023-2-223

classical solutions can be considered, whereas in the case of degenerate diffusion classical solutions are not always obtained. In such circumstances, it had not been clear whether blow-up of solutions to (1.1) occurs.

Regarding this difficulty, existence of blow-up solutions was recently established in [8] for the following Jäger–Luckhaus system with $\varepsilon = 0$,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta(u+\varepsilon)^m - \chi\nabla\cdot(u\nabla v) + \lambda u - \mu u^\kappa\,, & x \in \Omega,\ t > 0\,, \\ 0 = \Delta v - \overline{M}(t) + u\,, & x \in \Omega,\ t > 0\,, \end{cases}$$

where $\overline{M}(t) := \frac{1}{|\Omega|}\int_\Omega u(x,t)\,dx$. This system was studied in [1, 3, 7, 9]; in the case $m = 1$ and $\varepsilon = 0$, finite-time blow-up was shown under smallness conditions for κ in the three- and higher-dimensional cases in [1, 9] (in the case $\overline{M}(t) = v$, see [10]); these conditions were improved in [3]; in the case $m \neq 1$, the condition $\kappa < \min\left\{2, \frac{n}{2}\right\}$ in [3] was generalized to the condition that $\kappa < \min\left\{2, (2-m)\frac{n}{2}\right\}$ if $m \geq 0$ or $\kappa < \min\{2, n\}$ if $m < 0$ in [7]. After that, in the case of degenerate diffusion ($\varepsilon = 0$), finite-time blow-up solutions was constructed in a framework of weak solutions in [8].

In contrast, for the degenerate Keller–Segel system with logistic source there is no result on blow-up. The purpose is to prove existence of blow-up solutions to (1.1) in a framework of weak solutions under the same condition as in [1, Theorem 1.2 (ii)]. Referring to the method in [8], we introduce *moment solutions* as follows.

Definition 1.1. Let $T \in (0, \infty]$. A pair (u, v) of nonnegative and radially symmetric functions defined on $\Omega \times (0, T)$ is called a *moment solution* of (1.1) on $[0, T)$ if

(i) $u \in C^0_{\mathrm{w}-\star}([0,T); L^\infty(\Omega)) \cap L^\infty_{\mathrm{loc}}([0,T); L^\infty(\Omega))$,
$u^m \in L^2(0,T; H^1(\Omega))$ if $T < \infty$; $u^m \in L^2_{\mathrm{loc}}([0,T); H^1(\Omega))$ if $T = \infty$,
$v \in L^\infty_{\mathrm{loc}}([0,T); H^1(\Omega))$,

(ii) for all $\varphi \in L^2(0,T; H^1(\Omega)) \cap W^{1,1}(0,T; L^2(\Omega))$ with $\operatorname{supp}\varphi(x,\cdot) \subset [0,T)$ (a.a. $x \in \Omega$),

$$\begin{aligned} &\int_0^T \int_\Omega (\nabla u^m \cdot \nabla\varphi - \chi u \nabla v \cdot \nabla\varphi - (\lambda u - \mu u^\kappa)\varphi - u\varphi_t)\,dxdt \\ &\quad = \int_\Omega u_0(x)\varphi(x,0)\,dx, \\ &\int_0^T \int_\Omega (\nabla v \cdot \nabla\varphi + v\varphi - u\varphi)\,dxdt = 0, \end{aligned}$$

(iii) (u, v) satisfies the following moment inequality:

$$\phi(t) - \phi(0) \geq K\int_0^t \phi^2(\tau)\,d\tau \quad \text{for all } t \in (0, T),$$

where

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s)w(s,t)\,ds \quad \text{for } t \in (0,T),$$

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u(\rho,t)\,d\rho \quad \text{for } s \in [0,R^n] \text{ and } t \in (0,T)$$

with some $s_0 \in (0,R^n)$, $\gamma \in (0,1)$ and $K = K(R,m,\chi,\mu,\kappa,\gamma,s_0) > 0$.

We next define *maximal moment solutions*, which are ensured by Zorn's lemma as in the proof of [6, Lemma 2.4].

Definition 1.2. Define the set $\mathcal{S}$ as

$$\mathcal{S} := \{(T,u,v) \mid T \in (0,\infty],\ (u,v) \text{ is a moment solution of (1.1) on } [0,T)\},$$

which is not empty as shown in the proof of Theorem 1.3, with the order relation $\preceq$ given by

$$(T_1,u_1,v_1) \preceq (T_2,u_2,v_2) \ :\Longleftrightarrow\ T_1 \le T_2,\ u_2|_{(0,T_1)} = u_1,\ v_2|_{(0,T_1)} = v_1\,.$$

Then Zorn's lemma assures some maximal element $(T_{\max},u,v) \in \mathcal{S}$, and (u,v) is called a *maximal moment solution* of (1.1) on $[0,T_{\max})$.

Now we state the main theorem, in which (1.2) is the same condition in [1, Theorem 1.2 (ii)].

Theorem 1.3. *Let $m \in \left[1, 2-\frac{2}{n}\right)$, $\chi > 0$, $\lambda > 0$, $\mu > 0$ and $\kappa > 1$. Assume that*

$$\kappa < 1 + \min\left\{\frac{(m-1)n+1}{2(n-1)}, \frac{n-2-(m-1)n}{n(n-1)}\right\}. \tag{1.2}$$

Then for all $M_0 > 0$ and $L > 0$ there exist $\sigma_0 > 0$, $\eta_0 \in (0,M_0)$ and $r_\star \in (0,R)$ with the following property: If

$$u_0 \in L^\infty(\Omega) \text{ is nonnegative and radially symmetric} \tag{1.3}$$

and

$$\int_\Omega u_0(x)\,dx = M_0 \quad \text{and} \quad \int_{B_{r_\star}(0)} u_0(x)\,dx \ge M_0 - \eta_0 \tag{1.4}$$

as well as

$$u_0(x) \le L|x|^{-p} \quad \text{for a.a. } x \in \Omega\,, \tag{1.5}$$

where $p := \frac{n(n-1)}{(m-1)n+1} + \sigma_0$, then there exists a moment solution of (1.1) on $[0,T_{\max})$ which blows up at $T_{\max} < \infty$ in the sense that

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty\,.$$

In order to prove Theorem 1.3, we will construct a moment solution. To this end, we derive a moment inequality for a solution of a problem approximate to (1.1). The key to obtaining the inequality is to establish a pointwise estimate for an approximate solution (Lemma 2.1).

2. Proof of Theorem 1.3

To show finite-time blow-up of solutions to (1.1), for the present we focus on the following approximate problem:

$$
\begin{cases}
\dfrac{\partial u_\varepsilon}{\partial t} = \Delta(u_\varepsilon+\varepsilon)^m - \chi\nabla\cdot(u_\varepsilon\nabla v_\varepsilon) + \lambda u_\varepsilon - \mu u_\varepsilon^\kappa\,, & x\in\Omega,\ t>0\,,\\
0 = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon\,, & x\in\Omega,\ t>0\,,\\
\dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{\partial v_\varepsilon}{\partial \nu} = 0\,, & x\in\partial\Omega,\ t>0\,,\\
u_\varepsilon(x,0) = u_{0\varepsilon}(x)\,, & x\in\Omega\,,
\end{cases}
\tag{2.1}
$$

where $\varepsilon\in(0,1)$, and $u_{0\varepsilon} := (\rho_\varepsilon * \overline{u_0})|_{\overline{\Omega}}$ with

$$
\overline{u_0}(x) := \begin{cases} u_0(x) & \text{if } x\in\Omega\,,\\ 0 & \text{otherwise,}\end{cases}
$$

$$
\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\left(\int_{\mathbb{R}^n}\rho(y)\,dy\right)^{-1}\rho\left(\frac{x}{\varepsilon}\right)\,,\quad \rho(x) := \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x|<1\,,\\ 0 & \text{if } |x|\geq 1\,.\end{cases}
$$

We note that the solution $(u_\varepsilon, v_\varepsilon)$ of (2.1) on $[0,T_\varepsilon)$ is obtained by a standard fixed point argument (see e.g. [11]), where T_ε is the maximal existence time for the solution $(u_\varepsilon, v_\varepsilon)$. We know that ρ_ε is nonnegative and radially symmetric. Thus, for the initial data u_0 satisfying (1.3), $u_{0\varepsilon}$ is nonnegative and radially symmetric. Moreover, we see that $u_{0,\varepsilon}\to u_0$ in $L^1(\Omega)$ as $\varepsilon\searrow 0$ and that on passing to a subsequence if necessary, $u_{0,\varepsilon}\to u_0$ a.a. $x\in\Omega$ as $\varepsilon\searrow 0$. Furthermore, as in [5, Section 2.2] and [8, Lemmas 2.2 and 2.3], we can find $T_0>0$ and $K_0>0$ such that for all $\varepsilon\in(0,1)$,

$$
T_0\leq T_\varepsilon \quad\text{and}\quad \sup_{t\in(0,T_0)}\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\leq K_0\,. \tag{2.2}
$$

In order to establish a moment inequality, an estimate for u_ε is a cornerstone. In a degenerate Jäger–Luckhaus system with logistic source the key is radial monotonicity of an approximate solution (see [8, Lemma 2.7]). However, in our case it is difficult to obtain this property due to the structure of the second equation in (2.1). For this reason, instead of monotonicity, based on [10, Lemma 3.3] and [1, lemma 5.2], we show a pointwise estimate for u_ε.

Lemma 2.1. *Let $m\in\left[1,2-\frac{2}{n}\right)$, $\chi>0$, $\lambda>0$, $\mu>0$, $\kappa>1$, $M_0>0$ and $L>0$. Moreover, for any $\sigma_0>0$, set $p := \frac{n(n-1)}{(m-1)n+1}+\sigma_0$ and assume that u_0 satisfies (1.3), (1.5) and $\int_\Omega u_0(x)\,dx = M_0$ and that there exist $T_0>0$ and $K_0>0$ fulfilling (2.2). Then there exist $\varepsilon_0\in(0,1)$ and $L_1>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$,*

$$
u_\varepsilon(x,t)\leq L_1|x|^{-p} \tag{2.3}
$$

for all $x\in\Omega$ and $t\in(0,T_0)$.

Proof. Putting $\widetilde{u}_\varepsilon(x,t) := e^{-\lambda t}u_\varepsilon(x,t)$, we can derive from (2.1) that

$$\begin{cases} \dfrac{\partial \widetilde{u}_\varepsilon}{\partial t} \le \nabla\cdot(m(e^{\lambda t}\widetilde{u}_\varepsilon+\varepsilon)^{m-1}\nabla\widetilde{u}_\varepsilon - \chi\widetilde{u}_\varepsilon\nabla v_\varepsilon)\,, & x\in\Omega,\ t>0,\\ (m(e^{\lambda t}\widetilde{u}_\varepsilon+\varepsilon)^{m-1}\nabla\widetilde{u}_\varepsilon - \chi\widetilde{u}_\varepsilon\nabla v_\varepsilon)\cdot\nu = 0\,, & x\in\partial\Omega,\ t>0\,,\\ \widetilde{u}_\varepsilon(x,0) = u_{0\varepsilon}(x)\,, & x\in\Omega\,. \end{cases} \tag{2.4}$$

Next, let $\sigma_0 > 0$. We can take $\xi > 0$ small enough and $\varepsilon_0 \in (0,1)$ such that $u_{0,\varepsilon} \le u_0 + \xi$ for a.a. $x \in \Omega$ and all $\varepsilon \in (0,\varepsilon_0)$. By virtue of this inequality, (1.5) and the fact that $|x| \le R$, it follows that

$$u_{0,\varepsilon} \le L|x|^{-p} + \xi R^p|x|^{-p} = (L+\xi R^p)|x|^{-p} \tag{2.5}$$

for all $x \in \Omega$ and $\varepsilon \in (0,\varepsilon_0)$. Also, from the condition $\int_\Omega u_0\,dx = M_0$, we obtain that

$$\int_\Omega u_{0,\varepsilon}\,dx \le M_0 + \xi|\Omega| =: \widetilde{M_0} \tag{2.6}$$

for all $\varepsilon \in (0,\varepsilon_0)$. On the other hand, integrating the first equation in (2.1) over Ω, we infer that

$$\frac{d}{dt}\int_\Omega u_\varepsilon\,dx = \lambda\int_\Omega u_\varepsilon\,dx - \mu\int_\Omega u_\varepsilon^\kappa\,dx \le \lambda\int_\Omega u_\varepsilon\,dx\,,$$

which ensures that

$$\int_\Omega u_\varepsilon\,dx \le e^{\lambda t}\int_\Omega u_{0,\varepsilon}\,dx \le e^{\lambda T_0}\widetilde{M_0} \tag{2.7}$$

for all $t \in (0,T_0)$. Moreover, we see from the second equation in (2.1) that

$$r^{n-1}(v_\varepsilon)_r = \int_0^r \rho^{n-1}v_\varepsilon\,d\rho - \int_0^r \rho^{n-1}u_\varepsilon\,d\rho \le \frac{1}{\omega_n}\left(\int_\Omega v_\varepsilon\,dx + \int_\Omega u_\varepsilon\,dx\right)$$

for all $r \in (0,R)$ and $t \in (0,T_\varepsilon)$, where $\omega_n := n|B_1(0)|$. Here, since we integrate the second equation in (2.1) over Ω to guarantee that

$$\int_\Omega u_\varepsilon\,dx = \int_\Omega v_\varepsilon\,dx\,,$$

the above inequality and (2.7) yields

$$r^{n-1}(v_\varepsilon)_r \le \frac{2}{\omega_n}e^{\lambda T_0}\widetilde{M_0} =: c_1$$

for all $r \in (0,R)$ and $t \in (0,T_0)$. Picking $\theta_0 > n$ so large satisfying $m-1 > \frac{1}{\theta_0} - \frac{1}{n}$ and $p = \frac{n(n-1)}{(m-1)n+1} + \sigma_0 > \frac{(n-1)}{(m-1)+\frac{1}{n}-\frac{1}{\theta_0}}$, we have

$$\begin{aligned}\int_\Omega |x|^{\theta_0(n-1)}|\nabla v_\varepsilon(x,t)|^{\theta_0}\,dx &= \omega_n\int_0^R r^{(\theta_0+1)(n-1)}|(v_\varepsilon)_r(\rho,t)|^{\theta_0}\,d\rho\\ &\le \frac{1}{n}\omega_n c_1^{\theta_0}R^n\end{aligned}$$

for all $t \in (0,T_0)$. From this inequality and (2.4)–(2.6) we therefore can apply [2, Theorem 1.1] to obtain (2.3). □

We next derive a moment inequality for an approximate solution of (2.1).

Lemma 2.2. *Let $m \in \left[1, 2-\frac{2}{n}\right)$, $\chi > 0$, $\lambda > 0$, $\mu > 0$ and $\kappa > 1$. Assume that (1.2) is satisfied and that there exist $T_0 > 0$ and $K_0 > 0$ fulfilling (2.2). Then for all $M_0 > 0$ and $L > 0$ there exist $\eta_0 \in (0, M_0)$ and $r_\star \in (0, R)$ which satisfy the following property: If u_0 satisfies (1.3)–(1.5) with some $\sigma_0 > 0$, then there exist $\varepsilon_0 \in (0,1)$ and $K > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$,*

$$\phi_\varepsilon(t) - \phi_\varepsilon(0) \ge K \int_0^t \phi_\varepsilon^2(\tau)\, d\tau \tag{2.8}$$

for all $t \in (0, T_0)$, where

$$\phi_\varepsilon(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s) w_\varepsilon(s,t)\, ds \quad \text{for } t \in (0, T_\varepsilon),$$

$$w_\varepsilon(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u_\varepsilon(\rho, t)\, d\rho \quad \text{for } s \in [0, R^n] \text{ and } t \in (0, T_\varepsilon)$$

with some $s_0 \in (0, R^n)$ and $\gamma \in (0,1)$.

Proof. Let us first put $p := \frac{n(n-1)}{(m-1)n+1} + \sigma_0$, where we choose $\sigma_0 > 0$ sufficiently small fulfilling that $\kappa < 1 + \min\left\{\frac{n}{2p}, \frac{n-2}{p} - (m-1)\right\}$. Furthermore, we select $\gamma \in \left(\max\left\{\frac{2p\kappa}{n}, 1 - \frac{2}{n} - \frac{p}{n}(m-1)\right\}, \min\left\{2 - \frac{4}{n} - \frac{2p}{n}(m-1), 1\right\}\right)$. Also, noting that $u_{0,\varepsilon} \to u_0$ in $L^1(\Omega)$ as $\varepsilon \searrow 0$, we fix $\xi_0 > 0$ small enough and pick $\varepsilon_0 \in (0,1)$ given by Lemma 2.1 satisfying

$$\int_\Omega u_{0,\varepsilon} \ge M_0 - \xi_0$$

for all $\varepsilon \in (0, \varepsilon_0)$. In order to obtain (2.8), we shall show that there exist $c_1 > 0$, $c_2 > 0$, $\theta \in (0,2)$ and $s_1 \in (0, R^n)$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $s_0 \in (0, s_1)$,

$$\phi_\varepsilon'(t) \ge c_1 s_0^{\gamma-3} \phi_\varepsilon^2(t) - c_2 s_0^{3-\gamma-\theta} \tag{2.9}$$

for all $t \in (0, T_0)$. By straightforward computations we have from (2.1) and the definitions of w_ε and ϕ_ε that

$$\begin{aligned}
\phi_\varepsilon'(t) \ge\ & mn^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s) \left(n(w_\varepsilon)_s + \varepsilon\right)^{m-1} (w_\varepsilon)_{ss}\, ds \\
& + n \int_0^{s_0} s^{-\gamma}(s_0 - s)(w_\varepsilon)_s w_\varepsilon\, ds - n \int_0^{s_0} s^{-\gamma}(s_0 - s)(w_\varepsilon)_s z_\varepsilon\, ds \\
& - n^{\kappa-1}\mu \int_0^{s_0} s^{-\gamma}(s_0 - s) \left\{ \int_0^s (w_\varepsilon)_s^\kappa\, d\sigma \right\} ds
\end{aligned}$$

for all $t \in (0, T_\varepsilon)$, where $z_\varepsilon(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} v_\varepsilon(\rho, t)\, d\rho$ for $s \in [0, R^n]$ and $t \in (0, T_\varepsilon)$. Here, we note that we can apply [1, Lemmas 3.5, 3.8 and 3.9] to the second, third and fourth terms on the right-hand side of the above inequality. Thus, in order to derive (2.9), it is sufficient to estimate the first term. To this end, we will find $c_3 > 0$ independent of ε such that

$$\left(n(w_\varepsilon)_s + \varepsilon\right)^m \le c_3 s^{-\frac{p}{n}(m-1)} (w_\varepsilon)_s + c_3 \tag{2.10}$$

for all $s \in (0, R^n)$ and $t \in (0, T_0)$, which is used after integration by parts in estimating the first term. By means of (2.3), it follows that for any $\varepsilon \in (0, \varepsilon_0)$, $w_\varepsilon(s,t) = \frac{1}{n} u_\varepsilon(s^{\frac{1}{n}}, t) \le c_4 s^{-\frac{p}{n}}$ for all $s \in (0, R^n)$ and $t \in (0, T_0)$, where $c_4 := \frac{L_1}{n}$. From this inequality and the fact that $s \le R^n$ as well as $\varepsilon < 1$, we have

$$\begin{aligned}(n(w_\varepsilon)_s + \varepsilon)^m &\le 2^{m-1}(n^m (w_\varepsilon)_s^m + \varepsilon^m) \\ &\le 2^{m-1} n^m c_4^{m-1} s^{-\frac{p}{n}(m-1)} (w_\varepsilon)_s + 2^{m-1}\end{aligned}$$

for all $s \in (0, R^n)$ and $t \in (0, T_0)$, which means that (2.10) holds. Therefore, by [1, Lemmas 3.5, 3.6 (i), 3.8, 3.9 and 3.11] we can take $c_5 > 0$, $c_6 > 0$, $\theta \in (0,2)$ and $s_1 \in (0, R^n)$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $s_0 \in (0, s_1)$,

$$\phi_\varepsilon'(t) \ge c_5 s_0^{\gamma-3} \phi_\varepsilon^2(t) - c_6 s_0^{3-\gamma-\theta}$$

for all $t \in (0, T_0)$. Furthermore, arguing as in [8, Proof of Proposition 2], we pick $\eta_0 \in (0, M_0)$ and $r_\star \in (0, R)$ such that for any u_0 satisfying (1.3)–(1.5), the inequality $\phi_\varepsilon'(t) \ge \frac{c_5}{2} s_0^{\gamma-3} \phi_\varepsilon^2(t)$ holds for all $\varepsilon \in (0, \varepsilon_0)$, $s_0 \in (0, s_1)$ and $t \in (0, T_0)$, which implies (2.8). □

We are now in the position to show Theorem 1.3.

Proof of Theorem 1.3. We can derive results similar to [8, Lemmas 2.4 and 2.5] since the second equation in (2.1) entails that $\Delta v_\varepsilon = v_\varepsilon - u_\varepsilon \ge -u_\varepsilon$. Thus, as in the proof of [4, Lemma 5.3] we can choose subsequence $\{u_{\varepsilon_k}\}$, $\{v_{\varepsilon_k}\}$ ($\varepsilon_k \to 0$ as $k \to \infty$) and nonnegative functions u, v such that $u \in L^\infty(0, T_0; L^\infty(\Omega))$, $u^m \in L^2(0, T_0; H^1(\Omega))$, $v \in L^\infty(0, T_0; W^{1,\infty}(\Omega))$ and

$$u_{\varepsilon_k} \to u \quad \text{weakly}^\star \text{ in } L^\infty(0, T_0; L^\infty(\Omega)), \tag{2.11}$$

$$u_{\varepsilon_k} \to u \quad \text{in } C^0([\delta, T_0]; L^q(\Omega)) \quad \text{for all } \delta \in (0, T_0) \text{ and } q \in [1, \infty), \tag{2.12}$$

$$\nabla (u_{\varepsilon_k} + \varepsilon)^m \to \nabla u^m \quad \text{weakly in } L^2(0, T_0; L^2(\Omega)), \tag{2.13}$$

$$\nabla v_{\varepsilon_k} \to \nabla v \quad \text{weakly}^\star \text{ in } L^\infty(0, T_0; L^\infty(\Omega)) \tag{2.14}$$

as $k \to \infty$. Moreover, thanks to Lemma 2.2, we can take the initial data u_0 leading to (2.8). Thus, by (2.11)–(2.14), we can show that (u, v) fulfills (i)–(iii) with $T = T_0$ in Definition 1.1 as in [8, Proof of Proposition 1]. Hence, from Definition 1.2 there exists a maximal moment solution (u, v) on $(0, T_{\max})$. In particular, we have

$$\phi(t) - \phi(0) \ge K \int_0^t \phi^2(\tau)\, d\tau$$

for all $t \in (0, T_{\max})$ with some $K > 0$. Putting $\Phi(t) := \int_0^t \phi^2(\tau)\, d\tau + \frac{\phi(0)}{K}$ for $t \in (0, T_{\max})$, we see that $\Phi \in C^0([0, T_{\max}) \cap C^1((0, T_{\max}))$ and from the above inequality that $\Phi'(t) \ge K^2 \Phi^2(t)$ for all $t \in (0, T_{\max})$, which yields

$$t \le \frac{1}{K^2} \left(-\frac{1}{\Phi(t)} + \frac{1}{\Phi(0)} \right) \le \frac{1}{K^2 \Phi(0)}$$

for all $t \in (0, T_{\max})$. This proves $T_{\max} \le \frac{1}{K^2 \Phi(0)} < \infty$. By an extension argument as in [8, Proof of Theorem 1.1] we can obtain $\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$, which concludes the proof. □

Acknowledgement. The author would like to thank Professor Tomomi Yokota for a lot of helpful comments on the manuscript which improve quality of this paper. Moreover, the author also would like to express thanks to the referees for their careful reading and helpful comments.

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