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EXISTENCE OF SOLUTIONS FOR A CLASS OF FIRST ORDER BOUNDARY VALUE PROBLEMS

Amirouche Mouhous[a], Svetlin Georgiev Georgiev[b], and Karima Mebarki[c]

Abstract. In this work, we are interested in the existence of solutions for a class of first order boundary value problems (BVPs for short). We give new sufficient conditions under which the considered problems have at least one solution, one nonnegative solution and two non trivial nonnegative solutions, respectively. To prove our main results we propose a new approach based upon recent theoretical results. The results complement some recent ones.

1. Introduction

In this paper we investigate the existence of solutions of the following first order differential equation

$$x' = f(t,x)\,, \quad t \in [a,b]\,, \tag{1.1}$$

subject to the boundary conditions

$$Mx(a) + Rx(b) = 0\,, \tag{1.2}$$

where M, $R \in \mathbb{R}$, $M + R \neq 0$, $a < b < \infty$ are given constants and

(H1): $f \in \mathcal{C}([a,b] \times \mathbb{R})$, $|f(t,x)| \leq \sum\limits_{j=1}^{k} a_j(t)|x|^{p_j}$, $(t,x) \in [a,b] \times \mathbb{R}$, $a_j \in \mathcal{C}([a,b])$, $0 \leq a_j \leq A$ on $[a,b]$, $p_j \geq 0$, $j \in \{1,\dots,k\}$.

The first-order BVPs arise in many applications of science, engineering and technology (see [1, Chapter 1]). Thanks to these applications, more theoretical studies of the subject can be developed, including: solvability, uniqueness, positivity and multiplicity of solutions. For the recent developments involving existence of solutions to BVPs for first order differential equations, we can refer to [3, 6, 7, 8, 9, 10, 12].

In this article we propose a new approach to ensure the existence of solutions for the first-order, two BVP (1.1)–(1.2). Our method involves new fixed-point theorems for the sum of two operators. The problem (1.1)–(1.2) one can consider as a scalar-valued analogue of the problem in [9]. The scalar-valued analogues of

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the conditions used in [9] are as follows:

(**C1**) there exist nonnegative constants α and K so that

$$|f(t,x)| \le \alpha x f(t,x) + K\,, \quad (t,x) \in [a,b] \times \mathbb{R}\,, \quad \text{and} \quad \left|\frac{M}{R}\right| \le 1\,,$$

(**C2**) there exist nonnegative constants α and K so that

$$|f(t,x)| \le -\alpha x f(t,x) + K\,, \quad (t,x) \in [a,b] \times \mathbb{R} \quad \text{and} \quad \left|\frac{R}{M}\right| \le 1\,,$$

(**C3**) there exists a $\mathcal{C}^1$ function $V \colon \mathbb{R} \to [0,\infty)$ and nonnegative constants α and K so that

$$|f(t,x)| \le \alpha V'(x) f(t,x) + K\,, \quad (t,x) \in [a,b] \times \mathbb{R} \quad \text{and} \quad V(x(a)) \ge V(x(b))\,,$$

(**C4**) there exists a $\mathcal{C}^1$ function $V : \mathbb{R} \to [0,\infty)$ and nonnegative constants α and K so that

$$|f(t,x)| \le -\alpha V'(x) f(t,x) + K\,, \quad (t,x) \in [a,b] \times \mathbb{R} \quad \text{and} \quad V(x(a)) \le V(x(b))\,.$$

Note that the conditions (C1), (C2), (C3), (C4) in the scalar-valued case are different from the condition $(H1)$. Moreover, in [9], there is an additional restriction $\left|\frac{M}{R}\right| \le 1 \quad (\left|\frac{R}{M}\right| \le 1)$ on R and M. Thus, we can consider our main result as a complementary result to these of [9] in the scalar-valued case. Moreover, our main results are valid in the case when $R = 0$. Thus, our main results can be applied for the classical initial value problems of first-order ODEs whenever f satisfies $(H1)$.

The plan of this paper is as follows. In the next section, we recall some notations, definitions, and auxiliary results that we need throughout this paper. In Section 3, we prove our main results about existence and multiplicity of solutions for the problem (1.1)–(1.2). In Section 4, a concluding remarks are given. An example is given in Section 5 in order to illustrate our obtained results.

2. Preliminary results

In this section, we will give some preliminary results needed to prove our main results. To prove the existence of at least one solution to the problem (1.1)–(1.2), we will use the following fixed point theorem for a sum of two operators.

Theorem 2.1. *Let $\varepsilon > 0$, $\rho > 0$, E be a Banach space and $X = \{x \in E : \|x\| \le \rho\}$. Let also, $Tx = -\varepsilon x$, $x \in X$, $S \colon X \to E$ is continuous, $(I - S)(X)$ resides in a compact subset of E and*

$$\{x \in E : x = \lambda(I - S)x\,, \quad \|x\| = \rho\} = \emptyset \tag{2.1}$$

for any $\lambda \in \left(0, \frac{1}{\varepsilon}\right)$. Then there exists a $x^* \in X$ so that

$$Tx^* + Sx^* = x^*.$$

Theorem 2.1 will be used to prove Theorem 3.5 and Theorem 3.8 and its proof can be found in [4] and [5].

In the sequel, we are concerned with the existence of multiple positive fixed points for the sum of an expansive mapping and a completely continuous one.

Definition 2.2. Let X and Y be real Banach spaces. A mapping $T\colon X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that

$$\|Tx - Ty\|_Y \geq h\|x - y\|_X \,,$$

for any $x,\, y \in X$.

Let E be a real Banach space.

Definition 2.3. A closed, convex set $\mathcal{P}$ in E is said to be cone if

(1) $\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$,

(2) $x,\, -x \in \mathcal{P}$ implies $x = 0$.

Every cone $\mathcal{P}$ defines a partial ordering $\leq$ in E defined by:

$$x \leq y \quad \text{if and only if} \quad y - x \in \mathcal{P}\,.$$

Definition 2.4. A mapping $K\colon E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

In the sequel, we give an extension of the Leray-Schauder boundary condition. First, we present our result for the completely continuous mappings. Next, we extend it to the class of a completely continuous mapping perturbed by an expansive one by considering that the set X is a cone.

Lemma 2.5. *Let X be a closed convex subset of a Banach space E and $U \subset X$ a bounded open subset with $0 \in U$. Assume that $A\colon \overline{U} \to X$ is a completely continuous mapping without fixed point on ∂U and there exists $\varepsilon > 0$ small enough such that*

$$Ax \neq \lambda x \quad \textit{for all} \quad x \in \partial U \quad \textit{and} \quad \lambda \geq 1 + \varepsilon\,.$$

Then the fixed point index $i\,(A, U, X) = 1$.

Lemma 2.5 will be the basis of the Theorem 2.8, which we will use to prove Theorem 3.10. Its proof can be found in [4].

In the sequel, $\mathcal{P}$ will refer to a cone in a Banach space $(E, \|\cdot\|)$, Ω is a subset of $\mathcal{P}$, and U is a bounded open subset of $\mathcal{P}$, and $\mathcal{P}^* = \mathcal{P}\backslash\{0\}$. Assume that $S\colon \overline{U} \to E$ is a completely continuous mapping and $T\colon \Omega \to E$ is an expansive one with constant $h > 1$. By Lemma ([11, Lemma 2.1]), the operator $(I - T)^{-1}$ is $(h-1)^{-1}$-Lipschitzian on $T(\Omega)$. Suppose that

$$S(\overline{U}) \subset (I - T)(\Omega)\,,$$

and

$$Tx + Sx \neq x\,, \ \text{ for all } \ x \in \partial U \cap \Omega\,.$$

Then $(I - T)^{-1}Sx \neq x$, for all $x \in \partial U$ and the mapping $(I - T)^{-1}S\colon \overline{U} \to \mathcal{P}$ is completely continuous. So the fixed point index $i\,(I - T)^{-1}S, U, \mathcal{P})$ is well defined. Thus we put

$$i_*\,(T + S, U \cap \Omega, \mathcal{P}) = \begin{cases} i\,((I - T)^{-1}S, U, \mathcal{P})\,, & \text{if} \quad U \cap \Omega \neq \emptyset \\ 0\,, & \text{if} \quad U \cap \Omega = \emptyset\,. \end{cases}$$

Using the main properties of the fixed point index for strict set contractions, Djebali and Mebarki in [2], have discussed the properties of the generalized fixed point index i_*. The following lemma gives the computation of the index i_*. For details see [2].

Lemma 2.6. *Assume that $T\colon \Omega \to E$ is an expansive mapping with constant $h > 1$, $S\colon \overline{U} \to E$ is a completely continuous mapping and $S(\overline{U}) \subset (I - T)(\Omega)$. Suppose that $T + S$ has no fixed point on $\partial U \cap \Omega$. Then we have the following results:*

(1): *If $0 \in U$ and there exists $\varepsilon > 0$ small enough such that*

$$Sx \neq (I - T)(\lambda x) \quad \textit{for all} \quad \lambda \geq 1 + \varepsilon,\ x \in \partial U \quad \textit{and} \quad \lambda x \in \Omega\,,$$

then the fixed point index $i_*\left(T + S, U \cap \Omega, \mathcal{P}\right) = 1$.

(2): *If there exists $u_0 \in \mathcal{P}^*$ such that*

$$Sx \neq (I - T)(x - \lambda u_0), \quad \textit{for all} \quad \lambda > 0 \quad \textit{and} \quad x \in \partial U \cap (\Omega + \lambda u_0)\,,$$

then the fixed point index $i_*\left(T + S, U \cap \Omega, \mathcal{P}\right) = 0$.

Proof.

(1): The mapping $(I - T)^{-1}S\colon \overline{U} \to \mathcal{P}$ is completely continuous without fixed point on ∂U and it is readily seen that the following condition is satisfied

$$(I - T)^{-1}Sx \neq \lambda x \quad \text{for} \quad \text{all} \quad x \in \partial U \quad \text{and} \quad \lambda \geq 1 + \varepsilon\,.$$

Then, our claim follows from the definition of i_* and the Lemma 2.5.

(2): See [2, Proposition 2.16].

□

Remark 2.7. The result (1) in Lemma 2.6 is an extension of [2, Proposition 2.11].

Now we are able to present a multiple fixed point theorem. The proof rely on the results (1) and (2) of Lemma 2.6 producing the computation of the index i_*. The following result will be used to prove existence of at least two nonnegative solutions to the problem (1.1)–(1.2).

Theorem 2.8. *Let U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T\colon \Omega \to E$ is an expansive mapping with constant $h > 1$, $S\colon \overline{U}_3 \to E$ is a completely continuous mapping and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i): *$Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,*

(ii): *there exists $\varepsilon > 0$ small enough such that $Sx \neq (I - T)(\lambda x)$, for all $\lambda \geq 1 + \varepsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$,*

(iii): *$Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.*

Then $T+S$ has at least two non-zero fixed points x_1, $x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \quad \textit{and} \quad x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \quad \textit{and} \quad x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega\,.$$

Proof. If $Sx = (I-T)x$ for $x \in \partial U_2 \cap \Omega$, then we get a fixed point $x_1 \in \partial U_2 \cap \Omega$ of the operator $T+S$. Suppose that $Sx \neq (I-T)x$ for any $x \in \partial U_2 \cap \Omega$. Without loss of generality, assume that $Tx + Sx \neq x$ on $\partial U_1 \cap \Omega$ and $Tx + Sx \neq x$ on $\partial U_3 \cap \Omega$, otherwise the result is obvious. By Lemma 2.6, we have

$$i_* \left(T+S, U_1 \cap \Omega, \mathcal{P}\right) = i_* \left(T+S, U_3 \cap \Omega, \mathcal{P}\right) = 0$$

and

$$i_* \left(T+S, U_2 \cap \Omega, \mathcal{P}\right) = 1\,.$$

The additivity property of the index yields

$$i_* \left(T+S, (U_2 \setminus \overline{U}_1) \cap \Omega, \mathcal{P}\right) = 1 \quad \text{and} \quad i_* \left(T+S, (U_3 \setminus \overline{U}_2) \cap \Omega, \mathcal{P}\right) = -1\,.$$

Consequently, by the existence property of the index, $T+S$ has at least two fixed points $x_1 \in (U_2 \setminus U_1) \cap \Omega$ and $x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$. □

3. Main results

In [9], it is proved that the BVP (1.1)–(1.2) is equivalent to the following integral equation

$$x(t) = \int_a^t f\big(s, x(s)\big) ds - \frac{R}{M+R} \int_a^b f\big(s, x(s)\big) ds, \quad t \in [a,b]\,. \tag{3.1}$$

Let $E = \mathcal{C}([a,b])$ be endowed with the maximum norm

$$\|x\| = \max_{t \in [a,b]} |x(t)|\,.$$

For $x \in E$, define the operator

$$S_1 x(t) = \int_a^t f\big(s, x(s)\big) ds - \frac{R}{M+R} \int_a^b f\big(s, x(s)\big) ds - x(t)\,, \quad t \in [a,b]\,.$$

By (3.1), it follows that if $x \in E$ satisfies the equation $S_1 x = 0$, then it is a solution to the BVP (1.1)–(1.2). Fix $B > 0$ arbitrarily.

Lemma 3.1. *Suppose that $(H1)$ holds. For any $x \in E$ with $\|x\| \leq B$, we have*

$$\|S_1 x\| \leq A\Big(1 + \Big|\frac{R}{M+R}\Big|\Big)(b-a) \sum_{j=1}^k B^{p_j} + B\,.$$

Proof. We have

$$\begin{aligned}|S_1x(t)| &= \Big|\int_a^t f\big(s,x(s)\big)ds - \frac{R}{M+R}\int_a^b f\big(s,x(s)\big)ds - x(t)\Big|\\
&\le \int_a^b |f\big(s,x(s)\big)|ds + \Big|\frac{R}{M+R}\Big|\int_a^b |f\big(s,x(s)\big)|ds + |x(t)|\\
&\le \Big(1+\Big|\frac{R}{M+R}\Big|\Big)\int_a^b \sum_{j=1}^k a_j(s)|x(s)|^{p_j}ds + B\\
&\le A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^k B^{p_j} + B\,, \quad t\in[a,b]\,,\end{aligned}$$

whereupon

$$\|S_1x\| \le A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^k B^{p_j} + B\,.$$

This completes the proof. □

Let $g \in \mathcal{C}([a,b])$ be positive except at a finite number of points on $[a,b]$ and

$$C = \int_a^b g(t)\,dt\,. \tag{3.2}$$

For $x \in E$, define the operator

$$S_2x(t) = \int_a^t g(\tau)S_1x(\tau)\,d\tau\,, \quad t\in[a,b]\,.$$

Lemma 3.2. *Suppose* $(H1)$. *If* $x\in E$ *satisfies the integral equation*

$$S_2x(t) = 0\,, \quad t\in[a,b]\,, \tag{3.3}$$

then x *is a solution to the* BVP (1.1)–(1.2).

Proof. We differentiate the equation (3.3) with respect to t and we get

$$g(t)S_1x(t) = 0\,, \quad t\in[a,b]\,,$$

whereupon

$$S_1x(t) = 0\,, \quad t\in[a,b]\,.$$

This completes the proof. □

Lemma 3.3. *Suppose that* $(H1)$ *hold. Let* $x\in E$ *be such that* $\|x\|\le B$. *Then*

$$\|S_2x\| \le C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^k B^{p_j} + B\Big)\,.$$

Proof. Using Lemma 3.1, we arrive at

$$\begin{aligned}|S_2x(t)| &= \Big|\int_a^t g(\tau)S_1x(\tau)\,d\tau\Big| \\ &\leq C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k}B^{p_j}+B\Big), \quad t\in[a,b]\,.\end{aligned}$$

Hence,

$$\|S_2x\| \leq C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k}B^{p_j}+B\Big).$$

This completes the proof. □

3.1. **Existence of at least one solution.** Suppose that $\epsilon>0$, $B>0$ be such that

(H2): $C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum\limits_{j=1}^{k}B^{p_j}+B\Big)<B.$

(H3): $\epsilon\Big(B+C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum\limits_{j=1}^{k}B^{p_j}+B\Big)\Big)\leq B,$

where C is the constant which appears in (3.2). In the last section, we will give an example for the constants ϵ, a, b, M, R, A, B, C and functions f, g that satisfy $(H1)$–$(H3)$.

Let $\widetilde{X}$ be the set of all equi-continuous families in E. Let also,

$$X=\{x\in\widetilde{X}:\|x\|\leq B\}\,.$$

For $x\in E$, define the operators

$$\begin{aligned}Tx(t)&=-\epsilon x(t)\,,\\ Sx(t)&=(1+\epsilon)x(t)+\epsilon S_2x(t)\,,\quad t\in[a,b]\,.\end{aligned}$$

By Lemma 3.2, it follows that any fixed point of the operator $T+S$ is a solution to the BVP (1.1), (1.2).

Lemma 3.4. *Suppose that* $(H1)$*–*$(H3)$ *hold. For* $x\in X$*, we have*

$$\|(I-S)x\|\leq B\quad\textit{and}\quad\|((1+\epsilon)I-S)x\|<\epsilon B\,.$$

Proof. By Lemma 3.3, we get

$$\begin{aligned}\|(I-S)x\| &= \|-\epsilon x-\epsilon S_2x\| \\ &\leq \epsilon\|x\|+\epsilon\|S_2x\| \\ &\leq \epsilon\Big(B+C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k}B^{p_j}+B\Big)\Big) \\ &\leq B\end{aligned}$$

and

$$\begin{aligned}\|((1+\epsilon)I-S)x\| &= \|\epsilon S_2 x\| \\ &= \epsilon\|S_2 x\| \\ &\le \epsilon\, C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k} B^{p_j}+B\Big) \\ &< \epsilon\, B\,.\end{aligned}$$

This completes the proof. □

Our main result in this section is as follows.

Theorem 3.5. *Suppose that* $(H1)$–$(H3)$ *hold. Then the BVP* (1.1)–(1.2) *has at least one solution in* $\mathcal{C}^1([a,b])$.

Proof. By Lemma 3.4, it follows that $I-S\colon X\to X$ and it is continuous. Since the continuous map of equi-continuous families are equi-continuous families, we conclude that $(I-S)(X)$ resides in a compact subset of E. Now, assume that there is an $x\in\partial X$ and $\lambda\in\big(0,\frac{1}{\epsilon}\big)$ so that

$$\lambda(I-S)x = x$$

or

$$\frac{1}{\lambda}x = -\epsilon x-\epsilon S_2 x\,,$$

or

$$\Big(\frac{1}{\lambda}+\epsilon\Big)x = -\epsilon S_2 x = \big((1+\epsilon)I-S\big)x\,,$$

whereupon

$$\begin{aligned}\epsilon B < \Big(\frac{1}{\lambda}+\epsilon\Big)B &= \Big(\frac{1}{\lambda}+\epsilon\Big)\|x\| = \epsilon\|S_2 x\| \\ &= \|\big((1+\epsilon)I-S\big)x\| < \epsilon B\,.\end{aligned}$$

This is a contradiction. Hence and Theorem 2.1, it follows that the operator $T+S$ has a fixed point and the BVP (1.1)–(1.2) has at least one solution. □

3.2. **Existence of at least one nonnegative solution.** Below, suppose that $\epsilon>0$, $B>0$, $\widetilde{r}>0$ satisfy the following inequalities

$$\textbf{(H4)}\qquad \begin{cases} C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum\limits_{j=1}^{k} B^{p_j}+B\Big)+\max(B,\widetilde{r})\le \min(B,\widetilde{r})\,, \\ \epsilon\Big(B+C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum\limits_{j=1}^{k} B^{p_j}+B\Big)+\widetilde{r}\Big)\le B\,, \end{cases}$$

where C is the constant which appears in (3.2). In the last section, we will give an example for constants ϵ, C, B, A, a, b, $\widetilde{r}$ that satisfy $(H4)$. Let

$$X_1 = \{x \in \widetilde{X} : x \geq 0,\ \|x\| \leq B\}\,.$$

For $x \in E$, define the operator

$$\widetilde{S}x(t) = (1+\epsilon)x(t) + \epsilon S_2 x(t) - \epsilon\widetilde{r}\,, \quad t \in [a,b]\,.$$

Lemma 3.6. *Suppose that* $(H1)$ *holds. If* $x \in E$ *is a fixed point of the operator* $T + \widetilde{S}$*, then it satisfies the BVP* (1.1)–(1.2)*.*

Proof. We have

$$\begin{aligned} x(t) &= Tx(t) + \widetilde{S}x(t) \\ &= -\epsilon x(t) + (1+\epsilon)x(t) + \epsilon S_2 x(t) - \epsilon\widetilde{r}\,, \quad t \in [a,b]\,, \end{aligned}$$

whereupon

$$0 = S_2 x(t) - \widetilde{r}\,, \quad t \in [a,b]\,.$$

We differentiate the last equation with respect to t and we get

$$g(t)S_1 x(t) = 0\,, \quad t \in [a,b]\,,$$

or

$$S_1 x(t) = 0\,, \quad t \in [a,b]\,.$$

This completes the proof. □

Lemma 3.7. *Suppose that* $(H1)$ *and* $(H4)$ *hold. Then* $I - \widetilde{S} \colon X_1 \to X_1$*,*

$$\|(I-\widetilde{S})x\| \leq B \quad \text{and} \quad \|((1+\epsilon)I - \widetilde{S})x\| < 2\epsilon B\,, \quad x \in X_1\,.$$

Proof. Take $x \in X_1$ arbitrarily. Then

$$(I-\widetilde{S})x = -\epsilon x - \epsilon S_2 x + \epsilon\widetilde{r}\,.$$

Since

$$\begin{aligned} \|-\epsilon x - \epsilon S_2 x\| &\leq \epsilon\|x\| + \epsilon\|S_2 x\| \\ &\leq \epsilon\Big(B + C\Big(A\Big(1 + \Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k} B^{p_j} + B\Big)\Big) \\ &\leq \epsilon\widetilde{r}\,, \end{aligned}$$

where we have used the first inequality of $(H4)$, we conclude that $(I-\widetilde{S})x \geq 0$. Next, using the second inequality of $(H4)$, we get

$$\begin{aligned}
\|(I-\widetilde{S})x\| &= \|-\epsilon x - \epsilon S_2 x + \epsilon \widetilde{r}\| \\
&\leq \epsilon\|x\| + \epsilon\|S_2 x\| + \epsilon\widetilde{r} \\
&\leq \epsilon\Big(B + C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k} B^{p_j} + B\Big) + \widetilde{r}\Big) \\
&\leq B\,.
\end{aligned}$$

Thus, $I-\widetilde{S}\colon X_1 \to X_1$. Moreover,

$$\begin{aligned}
\|((1+\epsilon)I-\widetilde{S})x\| &= \|-\epsilon S_2 x + \epsilon\widetilde{r}\| \\
&\leq \epsilon\|S_2 x\| + \epsilon\widetilde{r} \\
&\leq \epsilon\Big(C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k} B^{p_j} + B\Big) + \widetilde{r}\Big) \\
&< 2\epsilon B\,,
\end{aligned}$$

where we have used the second inequality of $(H4)$. This completes the proof. $\square$

Our main result in this section is as follows.

Theorem 3.8. *Suppose that* $(H1)$ *and* $(H4)$ *hold. Then the BVP* (1.1)–(1.2) *has at least one nonnegative solution in* $\mathcal{C}^1([a,b])$.

Remark 3.9. Note that, when we say that $u \in \mathcal{C}^1([a,b])$ is a nonnegative solution to the BVP (1.1)–(1.2), we have in mind that $u(t) \geq 0$ for any $t \in [a,b]$.

Proof. By Lemma 3.7, we have that $I-\widetilde{S}\colon X_1 \to X_1$ and it is continuous and $(I-\widetilde{S})(X_1)$ resides in a compact subset of E. Now, assume that there are an $x \in \partial X_1$ and $\lambda \in \big(0, \frac{1}{\epsilon}\big)$ so that

$$\lambda(I-\widetilde{S})x = x$$

or

$$\frac{1}{\lambda}x = (I-\widetilde{S})x = -\epsilon x - \epsilon S_2 x + \epsilon\widetilde{r}\,,$$

or

$$\Big(\frac{1}{\lambda}+\epsilon\Big)x = -\epsilon S_2 x + \epsilon\widetilde{r} = \big((I+\epsilon)I - \widetilde{S}\big)x\,.$$

Hence, applying Lemma 3.7, we get

$$\begin{aligned}
2\epsilon B < \Big(\frac{1}{\lambda}+\epsilon\Big)B &= \Big(\frac{1}{\lambda}+\epsilon\Big)\|x\| \\
&= \|((1+\epsilon)I-\widetilde{S})x\| < 2\epsilon B\,.
\end{aligned}$$

This is a contradiction. From here, applying Lemma 3.6 and Theorem 2.1, we get that the BVP (1.1)–(1.2) has at least one nonnegative solution. This completes the proof. □

3.3. **Existence of at least two nonnegative solutions.** Let $m > 0$ be large enough and A, r, L, R_1 be positive constants that satisfy the following inequalities

$$\textbf{(H5)} \qquad \begin{cases} r < L < R_1, \quad R_1 > \left(\frac{2}{5m} + 1\right)L\,, \\ C\Big(A\Big(1 + \Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum\limits_{j=1}^{k} R_1^{p_j} + R_1\Big) < \frac{L}{5}\,, \end{cases}$$

where C is the constant which appears in (3.2). Let $\epsilon > 0$, For $x \in E$, define the operators

$$T_1x(t) = (1 + m\epsilon)x(t) - \epsilon\frac{L}{10}\,,$$

$$S_3x(t) = -\epsilon S_2x(t) - m\epsilon x(t) - \epsilon\frac{L}{10}\,, \quad t \in [a,b]\,.$$

Note that any fixed point $x \in E$ of the operator $T + S_3$ is a solution to the problem (1.1)–(1.2).
Our main result in this section is as follows.

Theorem 3.10. *Suppose that* $(H1)$ *and* $(H5)$ *hold. Then the problem* (1.1)–(1.2) *has at least two nontrivial nonnegative solutions in* $\mathcal{C}^1([a,b])$.

Proof. Define

$$U_1 = \mathcal{P}_r = \{v \in \mathcal{P} : \|v\| < r\}\,,$$

$$U_2 = \mathcal{P}_L = \{v \in \mathcal{P} : \|v\| < L\}\,,$$

$$U_3 = \mathcal{P}_{R_1} = \{v \in \mathcal{P} : \|v\| < R_1\}\,,$$

$$R_2 = R_1 + \frac{C}{m}\Big(A\Big(1 + \Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k} R_1^{p_j} + R_1\Big) + \frac{L}{5m}\,,$$

$$\Omega = \overline{\mathcal{P}_{R_2}} = \{v \in \mathcal{P} : \|v\| \le R_2\}\,.$$

(1) For $v_1, v_2 \in \Omega$, we have

$$\|T_1v_1 - T_1v_2\| = (1 + m\epsilon)\|v_1 - v_2\|\,,$$

whereupon $T_1 \colon \Omega \to E$ is an expansive operator with a constant $1 + m\epsilon$.

(2) For $v \in \overline{\mathcal{P}_{R_1}}$, we get

$$\begin{aligned} \|S_3v\| &\le \epsilon\|S_2v\| + m\epsilon\|v\| + \epsilon\frac{L}{10} \\ &\le \epsilon\Big(C\Big(A\Big(1 + \Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k} R_1^{p_j} + R_1\Big) + mR_1 + \frac{L}{10}\Big)\,. \end{aligned}$$

Therefore $S_3(\overline{\mathcal{P}_{R_1}})$ is uniformly bounded. Since $S_3\colon \overline{\mathcal{P}_{R_1}} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}_{R_1}})$ is equi-continuous. Consequently $S_3\colon \overline{\mathcal{P}_{R_1}} \to X$ is a completely continuous mapping.

(3) Let $v_1 \in \overline{\mathcal{P}_{R_1}}$. Set

$$v_2 = v_1 + \frac{1}{m} S_2 v_1 + \frac{L}{5m}.$$

Note that $S_2 v_1 + \frac{L}{5} \geq 0$ on $[a,b]$. We have $v_2 \geq 0$ on $[a,b]$ and

$$\begin{aligned} \|v_2\| &\leq \|v_1\| + \frac{1}{m}\|S_2 v_1\| + \frac{L}{5m} \\ &\leq R_1 + \frac{C}{m}\left(A\left(1 + \left|\frac{R}{M+R}\right|\right)(b-a)\sum_{j=1}^{k} R_1^{p_j} + R_1\right) + \frac{L}{5m} \\ &= R_2 . \end{aligned}$$

Therefore $v_2 \in \Omega$ and

$$-\epsilon m v_2 = -\epsilon m v_1 - \epsilon S_2 v_1 - \epsilon \frac{L}{10} - \epsilon \frac{L}{10}$$

or

$$\begin{aligned} (I - T_1) v_2 &= -\epsilon m v_2 + \epsilon \frac{L}{10} \\ &= S_3 v_1 . \end{aligned}$$

Consequently $S_3(\overline{\mathcal{P}_{R_1}}) \subset (I - T_1)(\Omega)$.

(4) Assume that for any $u_0 \in \mathcal{P}^*$ there exist $\lambda > 0$ and $x \in \partial\mathcal{P}_r \cap (\Omega + \lambda u_0)$ or $x \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda u_0)$ such that

$$S_3 x = (I - T_1)(x - \lambda u_0).$$

Then

$$-\epsilon S_2 x - m\epsilon x - \epsilon \frac{L}{10} = -m\epsilon(x - \lambda u_0) + \epsilon \frac{L}{10}$$

or

$$-S_2 x = \lambda m u_0 + \frac{L}{5}.$$

Hence,

$$\|S_2 x\| = \left\|\lambda m u_0 + \frac{L}{5}\right\| > \frac{L}{5}.$$

This is a contradiction.

(5) Let $\varepsilon_1 = \frac{2}{5m}$. Assume that there exist $\lambda_1 \geq \varepsilon_1 + 1$ and $x_1 \in \partial\mathcal{P}_L$, $\lambda_1 x_1 \in \overline{\mathcal{P}_{R_2}}$ such that

$$S_3 x_1 = (I - T_1)(\lambda_1 x_1). \tag{3.4}$$

Since $x_1 \in \partial\mathcal{P}_L$ and $\lambda_1 x_1 \in \overline{\mathcal{P}_{R_2}}$, it follows that

$$\Big(\frac{2}{5m}+1\Big)L < \lambda_1 L = \lambda_1\|x_1\| \le R_2\,.$$

Moreover,

$$-\epsilon S_2 x_1 - m\epsilon x_1 - \epsilon\frac{L}{10} = -\lambda_1 m\epsilon x_1 + \epsilon\frac{L}{10}\,,$$

or

$$S_2 x_1 + \frac{L}{5} = (\lambda_1 - 1)m x_1\,.$$

From here,

$$2\frac{L}{5} > \Big\|S_2 x_1 + \frac{L}{5}\Big\| = (\lambda_1-1)m\|x_1\| = (\lambda_1-1)mL$$

and

$$\frac{2}{5m} + 1 > \lambda_1\,,$$

which is a contradiction.

Therefore all conditions of Theorem 2.8 hold. Hence, the problem (1.1)–(1.2) has at least two solutions u_1 and u_2 so that

$$\|u_1\| = L < \|u_2\| < R_1$$

or

$$r < \|u_1\| < L < \|u_2\| < R_1\,.$$

This completes the proof. $\square$

4. Concluding remarks

(1) The conditions (1.2) are general and they capture in particular the anti-periodic conditions corresponding to the case $M = N = 1$.

(2) Our main results in this paper and the results in [9] are complementary. Moreover, in [9], there is an additional restriction on the constants R and M ($\left|\frac{M}{R}\right| \le 1$ or $\left|\frac{R}{M}\right| \le 1$). Note that our main results also depend on the hypotheses $(H2)$-$(H5)$, where the conditions are controlled by the constants C, ϵ and B and the source term f does not depend on these constants.

(3) We obtained new sufficient conditions for the existence of at least one or two solutions of the BVP (1.1)–(1.2).

(4) New existence results of multiple non trivial nonnegative solutions are proved using recent fixed point theorems on cones in Banach spaces for the sum of two operators.

(5) We can clearly see that Theorem 2.8 led us to show the existence of multiple nonnegative nontrivial solutions (see Theorem 3.10) under weaker conditions compared with those used in Theorem 3.8, which guarantees the existence of only one nonnegative solution of the problem (1.1)–(1.2). Recall that Theorem 2.8 is a multiple fixed point theorem on cones and Theorem 2.1, used in the proof of Theorem 3.8, guarantees the existence of a fixed point in any ball of a Banach space. A reason that motivates the use of the theory of the fixed point on the cones when it comes to the search for nonnegative solutions.

(6) In this paper we investigated a class of boundary value problems for first order ODEs. The nonlinear term depends on the solution and may change sign, and it satisfies general polynomial growth conditions. We prove existence of at least one solution, one nonnegative solution and two nonnegative solutions in $\mathcal{C}^1([a,b])$ of the considered class of first order ODEs. The proof of the main results is based upon recent theoretical results, developed by the authors of this article and presented in Section 2.

(7) It is noted that Theorem 3.5, Theorem 3.8 and Theorem 3.10 can be generalized to the case where $f \in \mathcal{C}([a,b] \times \mathbb{R}^n, \mathbb{R}^n)$, $n > 1$. In this case, we will consider the space $E_1 = (\mathcal{C}([a,b]))^n$ endowed with the norm

$$\|x\|_1 = \max_{j \in \{1,\dots,n\}} \|x_j\|, \quad x = (x_1, \dots, x_n)\,.$$

The hypothesis $(H1)$ takes the form

(H1′): $f \in \mathcal{C}([a,b] \times \mathbb{R}, \mathbb{R}^n)$, $f = (f_1, \dots, f_n)$, $|f_i(t,x)| \leq \sum\limits_{j=1}^{k} a_{ji}(t)|x|^{p_{ji}}$, $(t,x) \in [a,b] \times \mathbb{R}^n$, $a_{ji} \in \mathcal{C}([a,b])$, $0 \leq a_{ji} \leq A$ on $[a,b]$, $p_{ji} \geq 0$, $j \in \{1, \dots, k\}$, $i \in \{1, \dots, n\}$,

the hypotheses $(H2)$–$(H5)$ will be the same.

(8) These theoretical results can be used to study other classes of BVP as well as some IVP in ODEs. For these aims, firstly has to be find an integral representation of the solutions of the considered IVPs/BVPs and using it to be defined the operators S_1, S_2, S, $\widetilde{S}$ and T and finally to be applied Theorem 2.1 and Theorem 2.8.

5. Examples

Consider the boundary value problem:

$$\begin{aligned} &x'(t) = (x(t))^2 + \frac{1}{1+t^2}\,(x(t))^4 + 1, \quad t \in [0,1]\,, \\ &2x(0) + x(1) = 0\,. \end{aligned} \tag{5.1}$$

Here

$$f(t,x) = x^2 + \frac{1}{1+t^2}x^4 + 1, \quad k = 3, \quad a_1(t) = 1, \quad a_2(t) = \frac{1}{1+t^2}, \quad t \in [0,1],$$

$$a = 0, \quad b = 1, \quad p_1 = 2, \quad p_2 = 4, \quad p_3 = 0, \quad M = 2, \quad R = 1\,.$$

Firstly, we will note that the scalar-valued case of the results in [9] are not applicable for the BVP (5.1). Here $\frac{R}{M} = \frac{1}{2} < 1$. Assume that there are nonnegative constants α and K so that

$$|f(t,x)| \le -\alpha x f(t,x) + K\,, \quad (t,x) \in [0,1] \times \mathbb{R}\,,$$

which is equivalent to

$$x^2 + \frac{1}{1+t^2}x^4 + 1 \le -\alpha x\Big(x^2 + \frac{1}{1+t^2}x^4 + 1\Big) + K\,, \quad (t,x) \in [0,1] \times \mathbb{R}\,,$$

or

$$(1+\alpha x)\Big(x^2 + \frac{1}{1+t^2}x^4 + 1\Big) \le K\,, \quad (t,x) \in [0,1] \times \mathbb{R}\,.$$

The last inequality is impossible because

$$\lim_{x\to\infty}(1+\alpha x)\Big(x^2 + \frac{1}{1+t^2}x^4 + 1\Big) = \infty\,,$$

i.e., $(C2)$ does not hold. Now, we will show that our main results are applicable for the BVP (5.1). We have $A = 1$. Take $B = 1$. We have

$$A\Big(1 + \Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k} B^{p_j} + B = \Big(1 + \frac{1}{3}\Big)(1+1+1) + 1 = 5\,.$$

Let

$$g(t) = \frac{2}{10^{10}}t\,, \quad t \in [0,1]\,.$$

Then

$$\int_0^1 g(t)\,dt = \frac{2}{10^{10}}\int_0^1 t\,dt = \frac{1}{10^{10}}\,.$$

Take $C = \epsilon = \frac{1}{10^{10}}$. Then

$$C\Big(A\Big(1 + \Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k} B^{p_j} + B\Big) = \frac{5}{10^{10}} < 1 = B$$

and

$$\epsilon\Big(B + C\Big(A\Big(1 + \Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k} B^{p_j} + B\Big)\Big) = \frac{1}{10^{10}}\Big(1 + \frac{5}{10^{10}}\Big) < 1 = B\,.$$

Thus, $(H1)$–$(H3)$ hold. Hence and Theorem 3.5, we conclude that the considered BVP has at least one solution.

Now, take $\widetilde{r} = \frac{3}{2}$. Then

$$C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k} B^{p_j}+B\Big)+B=\frac{5}{10^{10}}+1<\frac{3}{2}=\widetilde{r}\,,$$

$$\begin{aligned}\epsilon\Big(B+C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k} B^{p_j}+B\Big)+\widetilde{r}\Big)&\\ =\frac{1}{10^{10}}\Big(1+\frac{5}{10^{10}}+\frac{3}{2}\Big)\le 1=B\,,&\\ C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k} B^{p_j}+B\Big)+\widetilde{r}&\\ =\frac{5}{10^{10}}+\frac{3}{2}<2=2B\,.&\end{aligned}$$

So, $(H4)$ holds. Now, applying Theorem 3.8, we conclude that the BVP (5.1) has at least one nonnegative solution.

Let now

$$R_1=10,\quad L=5,\quad r=4,\quad m=10^{50}\,.$$

Then

$$r<L<R_1,\quad 10=R_1>\Big(\frac{2}{5\cdot 10^{50}}+1\Big)5=\Big(\frac{2}{5m}+1\Big)L$$

and

$$\begin{aligned}C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k} R_1^{p_j}+R_1\Big)&\\ =\frac{1}{10^{10}}\Big(\frac{4}{3}\cdot\big(10^2+10^3+1\big)+10\Big)<\frac{1}{10^5}<1=\frac{L}{5}\,.&\end{aligned}$$

So, $(H5)$ holds. Then, by Theorem 3.10, it follows that the BVP (5.1) has at least two nonnegative solutions.

Let now, $R=0$ and f, k, a_1, a_2, a, b, p_1, p_2, p_3, M, R_1, L, r, $\widetilde{r}$, m, C, ϵ and g be as above. Consider the IVP

$$\begin{aligned}&x'(t)=(x(t))^2+\frac{1}{1+t^2}\,(x(t))^4+1\,,\quad t\in[0,1]\,,\\ &x(0)=0\,.\end{aligned}\tag{5.2}$$

Then

$$A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k} B^{p_j}+B=1\cdot(1+1+1)+1=4$$

and

$$C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k}B^{p_j}+B\Big)=\frac{4}{10^{10}}<1=B$$

and

$$\epsilon\Big(B+C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k}B^{p_j}+B\Big)\Big)=\frac{1}{10^{10}}\Big(1+\frac{4}{10^{10}}\Big)$$
$$<1=B.$$

Thus, $(H1)$–$(H3)$ hold. Hence, we conclude that the IVP (5.2) has at least one solution.

Now, take $\widetilde{r}=\frac{3}{2}$. Then

$$C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k}B^{p_j}+B\Big)+B=\frac{4}{10^{10}}+1<\frac{3}{2}=\widetilde{r},$$

$$\epsilon\Big(B+C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k}B^{p_j}+B\Big)+\widetilde{r}\Big)$$
$$=\frac{1}{10^{10}}\Big(1+\frac{4}{10^{10}}+\frac{3}{2}\Big)\leq 1=B\,,$$

$$C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k}B^{p_j}+B\Big)+\widetilde{r}$$
$$=\frac{4}{10^{10}}+\frac{3}{2}<2=2B\,.$$

So, $(H4)$ holds we conclude that the IVP (5.2) has at least one nonnegative solution. Next,

$$C\Big(A\Big(1+\Big|\frac{R}{M+R}\Big|\Big)(b-a)\sum_{j=1}^{k}R_1^{p_j}+R_1\Big)$$

$$=\frac{1}{10^{10}}\big(1\cdot\big(10^2+10^3+1\big)+10\big)<\frac{1}{10^5}<1=\frac{L}{5}\,.$$

So, $(H5)$ holds. Then the IVP (5.2) has at least two nonnegative solutions.

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[a,c]Laboratory of Applied Mathematics, Faculty of Exact Sciences,
University of Bejaia, 06000 Bejaia, Algeria
E-mail: amirouche.mouhous@univ-bejaia.dz
karima.mebarki@univ-bejaia.dz

[b]Department of Differential Equations,
Faculty of Mathematics and Informatics,
University of Sofia, Sofia, Bulgaria
E-mail: svetlingeorgiev1@gmail.com