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L_p INEQUALITIES FOR THE GROWTH OF POLYNOMIALS WITH RESTRICTED ZEROS

Nisar A. Rather, Suhail Gulzar, and Aijaz A. Bhat

Abstract. Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree at most n which does not vanish in the disk $|z| < 1$, then for $1 \le p < \infty$ and $R > 1$, Boas and Rahman proved

$$\|P(Rz)\|_p \le \big(\|R^n + z\|_p \,/\, \|1+z\|_p \big) \, \|P\|_p \,.$$

In this paper, we improve the above inequality for $0 \le p < \infty$ by involving some of the coefficients of the polynomial $P(z)$. Analogous result for the class of polynomials $P(z)$ having no zero in $|z| > 1$ is also given.

1. Background & main results

Let $\mathcal{P}_n$ denote the space of all polynomials of degree at most n over the field of complex numbers. Let $P \in \mathcal{P}_n$ and $Q(z) = z^n \overline{P(1/\overline{z})}$ then $|Q(z)| = |P(z)|$ for $|z| = 1$. This implies $|Q(z)| \le \|P(z)\|_\infty$ for $|z| = 1$, where

$$\|P(z)\|_\infty := \max_{|z|=1} |P(z)| \,.$$

This further implies, by using maximum modulus theorem, that $|Q(z)| \le \|P(z)\|_\infty$ for $|z| \le 1$, or equivalently, $|z^n \overline{P(1/\overline{z})}| \le \|P(z)\|_\infty$. If we take $z = e^{i\theta}/R$ where $\theta \in [0, 2\pi)$ and $R \ge 1$, we get $|(e^{in\theta}/R^n)\overline{P(Re^{i\theta})}| \le \|P(z)\|_\infty$. Hence, the growth estimate for $|P(z)|$ over a large circle $|z| = R$ in comparison with its maximum modulus over the unit circle $|z| = 1$ is given by

$$\|P(Rz)\|_\infty \le R^n \, \|P(z)\|_\infty \,, \qquad R \ge 1 \,. \tag{1.1}$$

The equality in (1.1) holds if and only if $P(z) = az^n$, $a \ne 0$. That is, if some of the zeros of $P(z)$ are not at the origin then inequality (1.1) becomes strict. In this regard, for $P \in \mathcal{P}_n$ having no zero in $|z| < 1$, Ankeny and Rivlin [1] proved that

$$\|P(Rz)\|_\infty \le \frac{R^n + 1}{2} \, \|P(z)\|_\infty \,, \qquad R > 1 \,. \tag{1.2}$$

The bound in (1.2) is attained for those polynomials $P(z)$ of degree n which have all their zeros on the circle $|z| = 1$.

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For $P \in \mathcal{P}_n$, the Mahler measure and Hardy-norm of $P(z)$ are respectively defined as

$$\|P(z)\|_0 := \exp\Big(\frac{1}{2\pi}\int_0^{2\pi} \log\left|P(e^{i\theta})\right| d\theta\Big),$$

and

$$\|P(z)\|_p := \Big(\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\Big)^{1/p}, \qquad 0 < p < \infty\,.$$

Note that the limiting cases $\lim_{p\to 0^+}\|P(z)\|_p = \|P(z)\|_0$ and $\lim_{p\to\infty}\|P(z)\|_p = \|P(z)\|_\infty$ are not difficult to verify. For $1 \le p \le \infty$, $\|.\|_p$ is a norm. However, for $0 \le p < 1$, $\|.\|_p$ is not a norm (see [8]).

Under the hypothesis $P \in \mathcal{P}_n$ and $P(z) \neq 0$ for $|z| < 1$, the following variant of inequality (1.2) in L_p-norm was obtained by Boas and Rahman [4] for $p \ge 1$.

$$\|P(Rz)\|_p \le \frac{\|z + R^n\|_p}{\|z+1\|_p}\|P(z)\|_p\,, \quad R > 1\,. \tag{1.3}$$

Equality in (1.3) holds for $P(z) = az^n + b, |a| = |b| \neq 0$. One can obtain (1.2) by letting $p \to \infty$ in (1.3). Later, Rahman and Schmeisser [7] showed that the inequality (1.3) remains valid for $0 \le p < 1$ as well.

As an application of (1.3), A. Aziz [3] yielded the following L_p-norm version of Turán's inequality (see [9]) regarding lower estimate for $\|P'(z)\|_\infty$, when the polynomial $P(z)$ of degree n has its zeros in $|z| \le k$, $k \ge 1$.

$$\|P'(z)\|_\infty \ge \frac{n}{\|z + k^n\|_p}\|P(z)\|_p\,, \quad p \ge 1\,. \tag{1.4}$$

In this paper, we first show that the bound in (1.3) can be improved. The improvement is acheived by inflating the denominator in the right side of (1.3) with involving some coefficients of the polynomial. In fact, we prove the following result which constitutes a refinement of (1.3) and, therefore, of inequality (1.2) due to Ankeny and Rivlin [1] as well:

Theorem 1.1. *If $P(z) = \sum_{\nu=0}^n a_\nu z^\nu \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for each $R > 1$ and $0 \le p < \infty$,*

$$\|P(Rz)\|_p \le \frac{\|R^n + z\|_p}{\|\lambda(R) + z\|_p}\|P(z)\|_p \tag{1.5}$$

where

$$\lambda(R) = \Big(\frac{|a_n| + R|a_0|}{R|a_n| + |a_0|}\Big)\,. \tag{1.6}$$

The bound is sharp and equality in (1.5) *holds for $P(z) = az^n + b, |a| = |b| \neq 0$.*

Remark 1.1. Since all the zeros of $P(z) = \sum_{\nu=1}^n a_\nu z^\nu$ lie in $|z| \ge 1$, therefore, $|a_0| \ge |a_n|$. This implies $\lambda(R) \ge 1$ and hence it can be easily verified for $0 \le \theta < 2\pi$ that

$$\left|e^{i\theta} + \lambda(R)\right| \ge \left|e^{i\theta} + 1\right|\,,$$

which gives for each $p > 0$ that

$$\int_0^{2\pi} \left|e^{i\theta} + \lambda(R)\right|^p d\theta \geq \int_0^{2\pi} \left|e^{i\theta} + 1\right|^p d\theta\,.$$

Equivalently,

$$\|z + \lambda(R)\|_p \geq \|z + 1\|_p\ .$$

This shows that the inequality (1.5) refines inequality (1.3).

By letting $p \to \infty$ in (1.5) and noting that $\lambda(R) \geq 1$, we obtain the following refinement of inequality (1.2) due to Ankeny and Rivlin [1]:

Corollary 1.1. *If $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \in \mathcal{P}_n$ does not vanish in $|z| < 1$, then for each $R > 1$,*

$$\|P(Rz)\|_\infty \leq \left(\frac{R^n + 1}{\lambda(R) + 1}\right) \|P(z)\|_\infty\ , \tag{1.7}$$

where $\lambda(R)$ is given by (1.6). The inequality is sharp and equality in (1.7) holds for $P(z) = az^n + b$, $|a| = |b| \neq 0$.

Next, consider a polynomial $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ of degree n having all its zeros in $|z| \leq 1$, then the polynomial $P^*(z) = z^n\overline{P(1/\bar{z})}$ of degree at most n has no zero in $|z| < 1$. Applying Theorem 1.1 to $P^*(z)$ with $R = 1/r$, we get for $0 \leq p < \infty$ and $r < 1$,

$$\|P^*(z/r)_p\| \leq \frac{\|1/r^n + z\|_p}{\|\lambda(1/r) + z\|_p} \|P^*(z)\|_p$$

where

$$\lambda(1/r) = \left(\frac{|a_0| + (1/r)|a_n|}{(1/r)|a_0| + |a_n|}\right) = \left(\frac{r|a_0| + |a_n|}{|a_0| + r|a_n|}\right) = \mu(r)\,.$$

By using the fact that $|P(e^{i\theta})| = |P^*(e^{i\theta})|$, $|P^*(e^{i\theta}/r)| = |P(re^{i\theta})|/r^n$ and $\|r^n z + 1\|_p = \|r^n + z\|_p$, we obtain the following result:

Corollary 1.2. *If $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for each $r < 1$ and $0 \leq p < \infty$,*

$$\|P(rz)\|_p \leq \frac{\|r^n + z\|_p}{\|\mu(r) + z\|_p} \|P(z)\|_p \tag{1.8}$$

where

$$\mu(r) = \left(\frac{r|a_0| + |a_n|}{|a_0| + r|a_n|}\right).$$

The result is sharp and equality in (1.8) holds for $P(z) = az^n + b$, $|a| = |b| \neq 0$.

Our next result is another application of Theorem 1.1 which constitutes a refinement of inequality (1.4).

Theorem 1.2. *If $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \geq 1$ then for each $0 \leq p < \infty$,*

$$\|P'(z)\|_\infty \geq \frac{n}{\|k^n + z\|_p} \frac{\|\phi(k) + z\|_p}{\|1 + z\|_p} \|P(z)\|_p \tag{1.9}$$

where

$$\phi(k) = \frac{|a_0| + k^{n+1}|a_n|}{k|a_0| + k^n|a_n|}. \tag{1.10}$$

The result is sharp and equality in (1.9) *holds for $P(z) = az^n + b, |a| = |b| \neq 0$.*

Remark 1.2. Since all the zeros of $P(z)$ are in $|z| \leq k$, then $|a_n|k^n \geq |a_0|$. This implies that $\phi(k) \geq 1$. In view of this fact, it is not difficult to see that $\|\phi(k) + z\|_p \geq \|1 + z\|_p$. This proves our assertion that the Theorem 1.2 is a refinement of inequality (1.4).

The following result is obtained by letting $p \to \infty$ in the Theorem 1.2.

Corollary 1.3. *If $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \geq 1$ then for each $0 \leq p < \infty$,*

$$\|P'(z)\|_\infty \geq \frac{n}{1 + k^n} \frac{1 + \phi(k)}{2} \|P(z)\|_\infty$$

where $\phi(k)$ is given by (1.10)*. The result is best possible as shown by $P(z) = az^n + b$, $|a| = |b| \neq 0$.*

2. Lemmas

We require following lemmas to prove our theorems.

Lemma 2.1. *Let a, b be complex numbers independent of α, where α is real. Then for each $p > 0$,*

$$\int_0^{2\pi} \left|a + be^{i\alpha}\right|^p d\alpha = \int_0^{2\pi} \left||a| + |b|e^{i\alpha}\right|^p d\alpha$$

Using periodicity, it is easy to verify the lemma, so we omit the details.

Definition 1. Let $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ be an $(n+1)$-tuple of complex numbers, Arestov [2] called the operator Λ_γ, which sends a polynomial $P(z) = \sum_{\nu=1}^{n} a_\nu z^\nu$ to $\Lambda_\gamma P(z) = \sum_{\nu=1}^{n} \gamma_\nu a_\nu z^\nu$, admissible if it preserves one of the following properties:

- $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$
- $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \geq 1\}$.

For admissible operator, Arestov [2] proved the following result:

Lemma 2.2. *Let $\phi(x) = \psi(\log x)$ where ψ is a convex non-decreasing function on $\mathbb{R}$. Then for all polynomials $P(z)$ of degree at most n and each admissible operator Λ_γ,*

$$\int_0^{2\pi} \phi\left(\left|\Lambda_\gamma P(e^{i\theta})\right|\right) d\theta \leq \int_0^{2\pi} \phi\left(c(\gamma, n)\left|P(e^{i\theta})\right|\right) d\theta,$$

where $c(\gamma, n) = \max(|\gamma_0|, |\gamma_n|)$.

Definition 2 ([6, pp. 36])**.** Let f and g be analytic in $|z| < 1$, we say that the function f is said to be subordinate to g, if there exists a function w analytic in $|z| < 1$ with $w(0) = 0$ and $|w(z)| < 1$ for $|z| < 1$ such that

$$f(z) = g(w(z)) \qquad (|z| < 1)\,.$$

Lemma 2.3 ([6, pp. 36])**.** *Let f and g are analytic for $|z| \leq 1$ such that f is subordinate to g. In addition, if g is univalent in the same disc, then for each $p > 0$, we have*

$$\int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \leq \int_0^{2\pi} |g(e^{i\theta})|^p \, d\theta\,.$$

3. Proof of the main results

Proof of Theorem 1.1. Let $P^*(z) = z^n \overline{P(1/\bar{z})}$, then $|P^*(z)| = |P(z)|$ for $|z| = 1$. Since $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ does not vanish in $|z| < 1$, then the function $f(z) = P^*(z)/P(z)$ is analytic inside and on $|z| = 1$. By a generalized form of Schwarz lemma [5, pp. 167],

$$|f(z)| \leq \frac{|z| + |f(0)|}{1 + |f(0)||z|} \qquad \text{for} \quad |z| < 1\,.$$

That is,

$$|P^*(z)| \leq \Big(\frac{|a_0||z| + |a_n|}{|a_0| + |a_n||z|}\Big)|P(z)| \quad \text{for} \quad |z| < 1\,.$$

Replacing z by $1/\bar{z}$ and noting that $P(z) = z^n \overline{P^*(1/\bar{z})}$, we get

$$|P(z)| \leq \left(\frac{|a_0| + |a_n||z|}{|a_0||z| + |a_n|}\right)|P^*(z)| \quad \text{for} \quad |z| > 1\,. \tag{3.1}$$

Setting $z = Re^{i\theta}$ in (3.1), we obtain for $0 \leq \theta < 2\pi$ and $R > 1$,

$$|P(Re^{i\theta})| \leq \left(\frac{R|a_n| + |a_0|}{|a_n| + R|a_0|}\right)|P^*(Re^{i\theta})|\,.$$

Equivalently, for $|z| = 1$ and $R > 1$, we have

$$\lambda(R)|P(Rz)| \leq |R^n P(z/R)|\,, \tag{3.2}$$

where $\lambda(R) = (|a_n| + R|a_0|)/(|a_0| + R|a_n|)$. Again, since all the zeros of $P(z)$ lie in $|z| \geq 1$, therefore, by Viète's formulae, $|a_0| \geq |a_n|$. This implies $\lambda(R) = (|a_n| + R|a_0|)/(|a_0| + R|a_n|) \geq 1$ and hence from (3.2), we get

$$|P(Rz)| \leq |R^n P(z/R)| \quad \text{for} \quad |z| = 1 \quad \text{and} \quad R > 1\,.$$

Moreover, the polynomial $R^n P(z/R)$ does not vanish in $|z| \leq 1$ for every $R > 1$. Hence, it follows by the maximum modulus principle that

$$|P(Rz)| < |R^n P(z/R)| \quad \text{for} \quad |z| < 1\,.$$

By Rouche's theorem, the polynomial

$$P(Rz)+e^{i\alpha}R^nP(z/R)=\sum_{\nu=0}^{n}\left(R^\nu+e^{i\alpha}R^{n-\nu}\right)a_\nu z^\nu$$

does not vanish in $|z|<1$. Therefore, the operator Λ_γ defined by

$$\begin{aligned}\Lambda_\gamma P(z):&=P(Rz)+e^{i\alpha}R^nP(z/R)\\ &=(R^n+e^{i\alpha})a_nz^n+\cdots+(1+R^ne^{i\alpha})\end{aligned}$$

is an admissible operator. Applying Lemma 2.2 with $\phi(x)=x^p$ and $p\in(0,\infty)$, we obtain for each $p>0$, $R>1$ and α real,

$$\int_0^{2\pi}\left|P(Re^{i\theta})+e^{i\alpha}R^nP(e^{i\theta}/R)\right|^p\,d\theta\le\left|R^n+e^{i\alpha}\right|^p\int_0^{2\pi}\left|P(e^{i\theta})\right|^p\,d\theta\,.$$

This gives,

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi}\left|P(Re^{i\theta})+e^{i\alpha}R^nP(e^{i\theta}/R)\right|^p d\theta\,d\alpha\\ &\qquad\le\int_0^{2\pi}\left|R^ne^{i\alpha}+1\right|^p\,d\alpha\int_0^{2\pi}\left|P(e^{i\theta})\right|^p\,d\theta\,.\end{aligned}\tag{3.3}$$

Now for every real α and $r\ge t\ge 1$, we have

$$\left|r+e^{i\alpha}\right|\ge\left|t+e^{i\alpha}\right|\,,$$

which implies for each $p>0$,

$$\int_0^{2\pi}\left|r+e^{i\alpha}\right|^p\,d\alpha\ge\int_0^{2\pi}\left|t+e^{i\alpha}\right|^p\,d\alpha\,.$$

If $|P(Re^{i\theta})|\ne 0$, we take $r=|R^nP(e^{i\theta}/R)|/|P(Re^{i\theta})|$ and $t=\lambda(R)$, then by (3.2), $r\ge t\ge 1$ and we get by using Lemma 2.1,

$$\begin{aligned}\int_0^{2\pi}\left|P(Re^{i\theta})+e^{i\alpha}R^nP(e^{i\theta}/R)\right|^p\,d\alpha&=\left|P(Re^{i\theta})\right|^p\int_0^{2\pi}\left|1+e^{i\alpha}\frac{R^nP(e^{i\theta}/R)}{P(Re^{i\theta})}\right|^p\,d\alpha\\ &=\left|P(Re^{i\theta})\right|^p\int_0^{2\pi}\left|e^{i\alpha}+\frac{R^nP(e^{i\theta}/R)}{P(Re^{i\theta})}\right|^p\,d\alpha\\ &=\left|P(Re^{i\theta})\right|^p\int_0^{2\pi}\left|e^{i\alpha}+\left|\frac{R^nP(e^{i\theta}/R)}{P(Re^{i\theta})}\right|\right|^p\,d\alpha\\ &\ge\left|P(Re^{i\theta})\right|^p\int_0^{2\pi}\left|e^{i\alpha}+\lambda(R)\right|^p\,d\alpha\,.\end{aligned}$$

For $|P(Re^{i\theta})|=0$, this inequality is trivially true. Using this inequality in (3.3), we obtain for each $p>0$, $R>1$ and α real,

$$\begin{aligned}&\int_0^{2\pi}\left|e^{i\alpha}+\lambda(R)\right|^pd\alpha\int_0^{2\pi}\left|P(Re^{i\theta}\right|^p\,d\theta\\ &\qquad\le\int_0^{2\pi}\left|R^n+e^{i\alpha}\right|^p d\alpha\int_0^{2\pi}\left|P(e^{i\theta})\right|^p\,d\theta\,.\end{aligned}$$

Equivalently for each $p > 0$,

$$\|P(Rz)\|_p \leq \frac{\|R^n + z\|_p}{\|\lambda(R) + z\|_p} \|P\|_p \,, \tag{3.4}$$

which proves the desired result for $p > 0$. To prove the result for $p = 0$, we simply let $p \to 0+$ in 3.4. This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. By hypothesis, all the zeros of $P(z)$ lie in $|z| \leq k$ where $k \geq 1$. Therefore, all the zeros of $F(z) = P(kz)$ are in $|z| \leq 1$ and consequently, the zeros of polynomial $G(z) = z^n \overline{F(1/\overline{z})}$ are outside $|z| < 1$. If $z_1, z_2, \ldots, z_n$ are the zeros of $G(z)$, then $|z_j| \geq 1$, $j = 1, 2, \ldots, n$ and

$$\frac{zG'(z)}{G(z)} = \sum_{j=1}^{n} \frac{z}{z - z_j} \,,$$

or,

$$\mathrm{Re}\frac{e^{i\theta}G'(e^{i\theta})}{G(e^{i\theta})} = \sum_{j=1}^{n} \mathrm{Re}\frac{e^{i\theta}}{e^{i\theta} - z_j} \leq \sum_{j=1}^{n} \frac{1}{2} = \frac{n}{2}$$

for the points $e^{i\theta}$, $0 \leq \theta < 2\pi$, other than the zeros of $G(z)$. This implies

$$\left|\frac{e^{i\theta}G'(e^{i\theta})}{nG(e^{i\theta})}\right| \leq \left|1 - \frac{e^{i\theta}G'(e^{i\theta})}{nG(e^{i\theta})}\right|$$

for the points $e^{i\theta}$, $0 \leq \theta < 2\pi$, which are not the zeros of $G(z)$. Equivalently,

$$|G'(e^{i\theta})| \leq |nG(e^{i\theta}) - e^{i\theta}G'(e^{i\theta})| \tag{3.5}$$

for the points $e^{i\theta}$, $0 \leq \theta < 2\pi$, which are not the zeros of $G(z)$. This inequality is also true, even if $e^{i\theta}$ is a zero of $G(z)$, it follows that

$$|G'(z)| \leq |nG(z) - zG'(z)| \quad \text{for} \quad |z| = 1 \,.$$

Since all the zeros of $F(z)$ are in $|z| \leq 1$, by the Gauss-Lucas theorem, the zeros of $F'(z)$ are also therein. This implies that the polynomial

$$z^{n-1}\overline{F'(1/\overline{z})} \equiv nG(z) - zG'(z)$$

does not vanish in $|z| < 1$. Therefore, in view of (3.5), we conclude that the rational function

$$w(z) = \frac{zG'(z)}{nG(z) - zG'(z)}$$

is analytic for $|z| \leq 1$ and $|w(z)| \leq 1$ for $|z| = 1$. Moreover, $w(0) = 0$. It follows that the function $1 + w(z)$ is subordinate to the univalent function $1 + z$ for $|z| \leq 1$. Hence, from Lemma 2.3, we obtain

$$\int_0^{2\pi} |1 + w(e^{i\theta})|^p d\theta \leq \int_0^{2\pi} |1 + e^{i\theta}|^p \, d\theta \qquad p > 0 \,. \tag{3.6}$$

Now,

$$1+w(z)=\frac{nG(z)}{nG(z)-zG'(z)} \quad \text{and} \quad |G'(z)|=|nG(z)-zG'(z)| \quad \text{for } |z|=1\,.$$

This gives for $|z|=1$,

$$n|G(z)|=|1+w(z)||nG(z)-zG'(z)|=|1+w(z)||G'(z)|\,. \tag{3.7}$$

Inequality (3.6) in conjunction with (3.7) gives for $p>0$,

$$n^p\int_0^{2\pi}|G(e^{i\theta})|^p d\theta \le \int_0^{2\pi}|1+e^{i\theta}|^p\,d\theta\left(\max_{|z|=1}|G'(z)|\right)^p\,. \tag{3.8}$$

As the polynomial $G(z)$ does not vanish in $|z|<1$, we can apply Theorem 1.1 to $G(z)$ with $R=k$ and obtain,

$$\int_0^{2\pi}|G(ke^{i\theta})|^p\,d\theta \le \frac{\|k^n+z\|_p}{\|\phi(k)+z\|_p}\int_0^{2\pi}|G(e^{i\theta})|^p\,d\theta \tag{3.9}$$

where

$$\phi(k)=\frac{|a_0|+k^{n+1}|a_n|}{k|a_0|+k^n|a_n|}\,.$$

Again, since $G(z)=z^n\overline{F(1/\overline{z})}=z^n\overline{P(k/\overline{z})}$, then it is not difficult to see that

$$|G(ke^{i\theta})|=k^n|P(e^{i\theta})| \quad \text{for} \quad 0\le\theta<2\pi\,.$$

Combining this equation with (3.8) and (3.9), we get:

$$nk^n\|P(z)\|_p \le \frac{\|1+z\|_p}{\|\phi(k)+z\|_p}\|k^n+z\|_p\max_{|z|=1}|G'(z)|\,. \tag{3.10}$$

Applying inequality (1.1) to the polynomial $G'(z)=kP'(kz)$, of degree at most $n-1$ and $k\ge 1$, we have

$$\max_{|z|=1}|G'(z)|=k\max_{|z|=1}|P'(kz)|\le k^n\max_{|z|=1}|P'(z)|\,. \tag{3.11}$$

By using inequality (3.11) in (3.10), we finally obtain

$$n\|P(z)\|_p \le \frac{\|1+z\|_p}{\|\phi(k)+z\|_p}\|k^n+z\|_p\|P'(z)\|_\infty\,.$$

This completes the proof. □

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P.G. Department of Mathematics,
University of Kashmir,
Hazratbal, Srinagar-190006, India
E-mail: `dr.narather@gmail.com`
`sgmattoo@gmail.com`
`Aijazb824@gmail.com`