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SOLUTIONS OF RIEMANN–WEBER TYPE HALF-LINEAR DIFFERENTIAL EQUATION

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Abstract. We establish an asymptotic formula for a pair of linearly independent solutions of the subcritical Riemann–Weber type half-linear differential equation. We also complement the results of the author and M. Ünal, Acta Math. Hungar. **120** (2008), 147–163, where the equation was considered in the critical case.

1. Introduction

The classical linear Riemann–Weber differential equation is the equation

$$x'' + \Big(\frac{1}{4t^2} + \frac{\mu}{t^2\log^2 t}\Big)x = 0\,, \tag{1}$$

where μ is a real parameter. The transformation $x = \sqrt{t}y$ followed by the change of independent variable $s = \log t$ transforms (1) into the differential equation

$$\frac{d^2}{ds^2}y + \frac{\mu}{s^2}y = 0$$

which is explicitly solvable with a solution in the form $y(s) = s^\lambda$, λ being the root of the quadratic equation $\lambda^2 - \lambda + \mu = 0$. Consequently, (1) is conditionally oscillatory with respect to the parameter μ with the oscillation constant $\tilde\mu = \frac14$. More precisely, (1) is oscillatory if and only if $\mu > \frac14$.

The above mentioned classical result which can be found in standard monographs on differential equations, e.g. [9, Chapter XI], offers possibilities to extend to various generalizations of (1). One of the possible directions is to consider the half-linear Riemann–Weber differential equation, see [8], which is the equation of the form

$$\big(\Phi(x')\big)' + \left[\frac{\gamma_p}{t^p} + \frac{\mu}{t^p\log^2 t}\right]\Phi(x) = 0, \tag{2}$$

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where $\Phi(x) = |x|^{p-2}x$, $p > 1$, is the odd power function and $\gamma_p = \left(\frac{p-1}{p}\right)^p$. Obviously, if $p = 2$, then (2) reduces to (1).

Equation (2) is a special case of the general half-linear differential equation

$$L[x] := \big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0 \tag{3}$$

with continuous functions r, c and $r(t) > 0$. Oscillation theory of this equation is relatively deeply developed, see [1, 4, 7, 10], and it is known that this theory is very similar to that of the Sturm–Liouville second order linear differential equation $(r(t)x')' + c(t)x = 0$ which is the special case $p = 2$ in (3). Instead of the terminology *half-linear equation* (which reflects the fact that the solution space of (3) is homogeneous but not generally additive, i.e., it has one half of the properties characterizing linearity), the terminology *equation with the scalar p-Laplacian* is sometimes used. The reason is that if $r(t) \equiv 1$, then the differential term in (3) is the special case $N = 1$ in the p-Laplacian operator

$$\Delta_p u(x) = \operatorname{div}\left(\|\nabla u(x)\|^{p-2}\nabla u(x)\right), \quad \nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right).$$

Elbert and Schneider [8] showed that (2) is also conditionally oscillatory with respect to the parameter μ with the oscillation constant $\mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$. Of course, if $p = 2$, we have $\mu_2 = \tilde{\mu} = \frac{1}{4}$. Equation (2) is viewed in [8] as a perturbation of the so-called *critical Euler half-linear differential equation*

$$\big(\Phi(x')\big)' + \frac{\gamma_p}{t^p}\Phi(x) = 0 \tag{4}$$

which is nonoscillatory and has a solution $x(t) = t^{\frac{p-1}{p}}$. Moreover, every linearly independent solution is asymptotically equivalent to the function $x(t) = Ct^{\frac{p-1}{p}} \log^{\frac{2}{p}} t$, $C \in \mathbb{R}$.

In our paper we are motivated by these results and also by the paper [5]. More precisely, we consider equation (3) which is supposed to be nonoscillatory with a solution h satisfying $h'(t) \neq 0$ for large t, say $t \geq T$. A prototype is the Euler equation (4) and its solution $h(t) = t^{\frac{p-1}{p}}$. Together with (3) we consider its perturbation

$$\big(r(t)\Phi(x')\big)' + \left[c(t) + \frac{\mu}{h^p(t)\big(\int^t R^{-1}(s)\,ds\big)^2 R(t)}\right]\Phi(x) = 0\,, \tag{5}$$

where

$$R(t) = r(t)h^2(t)|h'(t)|^{p-2}$$

and μ is a real parameter. By a direct computation one can verify that (5) reduces to (2) when $r(t) = 1$, $c(t) = \frac{\gamma_p}{t^p}$, and $h(t) = t^{\frac{p-1}{p}}$ (after a relabeling of the constant μ). It was shown in [5, Theorem 1] that if

$$\int^\infty R^{-1}(t)\,dt = \infty \tag{6}$$

and

$$\liminf_{t\to\infty} |G(t)| > 0\,, \quad G(t) = r(t)h(t)\Phi\big(h'(t)\big)\,, \tag{7}$$

then (5) is conditionally oscillatory with respect to μ with the oscillation constant $\tilde{\mu} = \frac{1}{2q}$, where q is the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, it was shown in [5, Theorem 2] that in the critical case $\mu = \frac{1}{2q}$ in (5) this equation is nonoscillatory and possesses a solution given by the asymptotic formula

$$x(t) = h(t)\Big(\int^t R^{-1}(s)\,ds\Big)^{\frac{1}{p}}\Big[1 + O\Big(\Big(\int^t R^{-1}(s)\,ds\Big)^{-1}\Big)\Big] \quad \text{as } t\to\infty. \tag{8}$$

Also, under the additional assumptions that

$$\int^\infty r^{1-q}(t)\,dt = \infty \tag{9}$$

and that the integral

$$\int_t^\infty \left[c(s) + \frac{1}{2qh^p(s)\big(\int^s R^{-1}(\tau)\,d\tau\big)^2 R(s)}\right] ds$$

is convergent and positive for large t, the solution given in (8) is the so-called principal solution of (5) with $\mu = \frac{1}{2q}$.

In our paper we complement these results. We deal with (5) in the subcritical case $\mu < \frac{1}{2q}$. We establish asymptotic formula for two linearly independent solutions of this equation and under additional assumptions we show which of these solutions is principal. We also deal with the critical case $\mu = \frac{1}{2q}$ and we prove an asymptotic formula for a solution which is linearly independent of (8).

2. Preliminaries

Suppose that (3) is nonoscillatory and x is its nontrivial solution for which $x(t) \neq 0$ for large t and $w = r\Phi(x'/x)$. Then w is a solution of the Riccati type differential equation

$$R[w] := w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0\,. \tag{10}$$

We will also use the so-called *modified Riccati equation* which is introduced as follows. Let h be a differentiable function, we put $v = h^p w - G$, where G is given by (7). Then we have the identity (see [3])

$$h^p R[w] = v' + hL[h] + (p-1)r^{1-q}h^{-q}H(v,G)\,,$$

where

$$H(v,G) = |v+G|^q - q\Phi^{-1}(G)v - |G|^q\,,$$

$\Phi^{-1}(s) = |s|^{q-2}s$ being the inverse function of Φ. In particular, if w is a solution of (10), v is a solution of the modified Riccati equation

$$v' + h(t)L[h(t)] + (p-1)r^{1-q}(t)h^{-q}(t)H\big(v, G(t)\big) = 0\,. \tag{11}$$

By a direct computation we have $H(0,G)=0=H_v(0,G)$ and if we replace the term $H(v,G)$ by its second degree Taylor polynomial $\frac{1}{2}H_{vv}(0,G)v^2$, we obtain the so-called *approximate Riccati equation*

$$u' + h(t)L[h(t)] + \frac{q}{2R(t)}u^2 = 0, \tag{12}$$

because $H_{vv}(0,G)=q(q-1)|G|^{q-2}$ and substituting for G we have

$$\frac{(p-1)}{2}r^{1-q}h^{-q}H_{vv}(0,G)v^2 = \frac{q}{2R}v^2\,.$$

Observe that equation (12) is the classical Riccati equation associated with the second order linear differential equation

$$\Big(\frac{2R(t)}{q}y'\Big)' + h(t)L[h(t)]y = 0\,.$$

In approximating (11) by (12) we will need the following estimate from [5].

Lemma 1. *If* $\varepsilon \in (0,1)$, (7) *holds, and*

$$b(\varepsilon) = \begin{cases} \left|\frac{q(q-2)}{6}\right|(1+\varepsilon)^{q-3}\,, & q \ge 3\,,\\ \left|\frac{q(q-2)}{6}\right|(1-\varepsilon)^{q-3}\,, & q < 3\,, \end{cases}$$

then

$$\Big|(p-1)r^{1-q}h^{-q}H(v,G) - \frac{q}{2R}v^2\Big| \le \frac{Lb(\varepsilon)}{R}|v|^3$$

for $|v/G| < \varepsilon$, *where* $L = \sup_{t>T}\left|\frac{1}{G(t)}\right|$ *for so large* T *that* $G(t)\neq 0$, $t \ge T$.

We also show that one of the established asymptotic formulas determines the so-called *principal solution* of half-linear equations which can be introduced as follows. Suppose that (3) is nonoscillatory. Then among all solutions of the associated Riccati equation (10) there exists the minimal solution $\tilde w$, minimal in the sense that any other solution w of (10) satisfies $w(t) > \tilde w(t)$ for large t. The principal solution $\tilde x$ of (3) is then defined as a solution which determines the minimal solution $\tilde w$ by the Riccati substitution $\tilde w = r\Phi(\tilde x'/\tilde x)$. The next statement gives a sufficient condition for a solution to be principal, see [2].

Proposition 1. *Suppose that* (3) *is nonoscillatory and* x *is its solution such that* $x'(t)\neq 0$ *for large* t. *Further, let* $\int^\infty r^{1-q}(t)\,dt = \infty$, $\int^\infty c(t)\,dt$ *be convergent and* $\int_t^\infty c(s)\,ds > 0$ *for large* t. *If*

$$\int^\infty \frac{dt}{r(t)x^2(t)|x'(t)|^{p-2}} = \infty\,,$$

then x *is the principal solution of* (3).

3. Asymptotic formulas

Our first main result reads as follows.

Theorem 1. *Consider equation* (5) *with* $0 < \mu < \frac{1}{2q}$ *and suppose that* (3) *is nonoscillatory and has a solution* h *such that* $h'(t) \neq 0$ *for* $t \geq T$*. If* (6) *and* (7) *hold, then equation* (5) *possesses a pair of linearly independent solutions given by the asymptotic formulas*

$$x_1(t) = h(t)\Big(\int_T^t R^{-1}(s)\,ds\Big)^{\frac{2\lambda_1}{p}}\left[1 + O\Big(\Big(\int_T^t R^{-1}(s)\,ds\Big)^{-1}\Big)\right], \tag{13}$$

$$x_2(t) = h(t)\Big(\int_T^t R^{-1}(s)\,ds\Big)^{\frac{2\lambda_2}{p}}\left[1 + O\Big(\Big(\int_T^t R^{-1}(s)\,ds\Big)^{-1}\Big)\right] \tag{14}$$

as $t \to \infty$*, where* $\lambda_{1,2}$ *are roots of the quadratic equation* $\lambda^2 - \lambda + \frac{q\mu}{2} = 0$*. Moreover, if* $\int^\infty r^{1-q}(t)\,dt = \infty$ *and the integral*

$$\int_t^\infty \left[c(s) + \frac{\mu}{h^p(t)\big(\int_T^s R^{-1}(\tau)\,d\tau\big)^2 R(s)}\right] ds$$

is convergent and positive for large t*, then the solution* x_1 *corresponding to the smaller root* λ_1 *is principal.*

Proof. First we consider the approximate Riccati equation corresponding to (5). Since h is a solution of (3), this equation takes the form

$$u' + \frac{\mu}{\big(\int_T^t R^{-1}(s)\,ds\big)^2 R(t)} + \frac{q}{2R(t)}u^2 = 0\,. \tag{15}$$

This is the classical Riccati equation associated with the equation

$$\left(\frac{2R(t)}{q}y'\right)' + \frac{\mu}{\big(\int_T^t R^{-1}(s)\,ds\big)^2 R(t)}y = 0$$

which is related to (15) by the substitution $u = \frac{2Ry'}{qy}$ and which is the same as the equation

$$\big(R(t)y'\big)' + \frac{q\mu}{2\big(\int_T^t R^{-1}(s)\,ds\big)^2 R(t)}y = 0. \tag{16}$$

The change of independent variable $s = \int_T^t R^{-1}(\tau)\,d\tau$, i.e., $\frac{d}{dt} = \frac{1}{R(t)}\frac{d}{ds}$, transforms (16) into the equation

$$\frac{d^2y}{ds^2} + \frac{q\mu}{2s^2}y = 0$$

which is the classical Euler second order differential equation whose pair of linearly independent solutions are $y_{1,2} = s^{\lambda_{1,2}}$, $\lambda_{1,2}$ being the roots of the quadratic equation $\lambda^2 - \lambda + \frac{q\mu}{2} = 0$. Hence solutions of (16) are $y_{1,2}(t) = \big(\int_T^t R^{-1}(s)\,ds\big)^{\lambda_{1,2}}$ and then

$$u(t) = \frac{2R(t)}{q}\frac{y'}{y} = \frac{2\lambda_{1,2}}{q\int_T^t R^{-1}(s)\,ds}\,.$$

We prove the asymptotic formula for x_1. The proof of the formula for x_2 is quite analogical, only λ_1 is replaced by λ_2. Let

$$\varphi(t) = \frac{4\lambda_1^3}{q^3\big(\int_T^t R^{-1}(s)\,ds\big)^2}$$

and consider the function space

$$\mathcal{V} = \big\{v \in C[T_1,\infty), |v(t)-u(t)| \le K\varphi(t)\big\}\,,$$

where the constants T_1 and K will be specified later. At this moment, T_1 is so large that $u(t) - K\varphi(t) > 0$ for $t > T_1$, i.e., $\mathcal{V}$ consists of positive functions. Such T_1 exists, because $\varphi(t) = o(u(t))$ as $t \to \infty$.

To prove (13) we establish first the asymptotic formula for a solution of the modified Riccati equation associated with (5) which in our particular case is the equation

$$v' + \frac{\mu}{\big(\int_T^t R^{-1}(s)\,ds\big)^2 R(t)} + (p-1)r^{1-q}(t)h^{-q}(t)H\big(v, G(t)\big) = 0\,. \tag{17}$$

We show that we can find this solution of (17) as a fixed point of the operator

$$\mathcal{F}(v)(t) = \frac{\mu}{\int_T^t R^{-1}(s)\,ds} + (p-1)\int_t^\infty r^{1-q}(s)h^{-q}(s)H\big(v(s), G(s)\big)\,ds\,. \tag{18}$$

Obviously, if $v = \mathcal{F}(v)$, differentiating this equality we see that v is a solution of (17), then $w = h^{-p}(v+G)$ is a solution of the Riccati equation associated with (5)

$$w' + c(t) + \frac{\mu}{h^p(t)\big(\int_T^t R^{-1}(s)\,ds\big)^2 R(t)} + (p-1)r^{1-q}(t)|w|^q = 0$$

and then $x(t) = \exp\big\{\int^t r^{1-q}(s)\Phi^{-1}(w(s))\,ds\big\}$ is a solution of (5).

The integral

$$\int^\infty r^{1-q}(t)h^{-q}(t)H\big(v(t), G(t)\big)\,dt < \infty$$

for $v \in \mathcal{V}$. Indeed, since the function H is increasing with respect to v for $v > 0$ and we have $0 < v(t) < u(t) + K\varphi(t)$ for $v \in \mathcal{V}$, hence (suppressing the integration argument)

$$\begin{aligned}
(p-1)&\int^\infty r^{1-q}h^{-q}H(v,G)\,dt \\
&\le \int^\infty \Big|(p-1)r^{1-q}h^{-q}H(v,G) - \frac{q}{2R}v^2\Big|\,dt + \int^\infty \frac{q}{2R}v^2\,dt \\
&\le Lb(\varepsilon)\int^\infty \frac{v^3}{R}\,dt + \frac{q}{2}\int^\infty \frac{v^2}{R}\,dt \\
&\le Lb(\varepsilon)\int^\infty \frac{(u+K\varphi)^3}{R}\,dt + \frac{q}{2}\int^\infty \frac{(u+K\varphi)^2}{R}\,dt < \infty\,,
\end{aligned}$$

because $\varphi(t) = o(u(t))$ as $t \to \infty$ by (6) and $\int^\infty \frac{u^2}{R}\,dt < \infty$.

The mapping $\mathcal{F}$ maps $\mathcal{V}$ into itself for suitably chosen constants K and T_1. Substituting for $\mathcal{F}(v)$ and u we have for $v \in \mathcal{V}$

$$\begin{aligned}
|\mathcal{F}(v)(t) - u(t)| &= \Big|(p-1)\int_t^\infty r^{1-q}h^{-q}H(v,G)\,ds - \frac{q}{2}\int_t^\infty \frac{u^2}{R}\,ds\Big| \\
&\le \int_t^\infty \Big|(p-1)r^{1-q}h^{-q}H(v,G) - \frac{q}{2}\frac{v^2}{R}\Big|\,ds + \frac{q}{2}\int_t^\infty \frac{|v^2-u^2|}{R}\,ds \\
&\le Lb(\varepsilon)\int_t^\infty \frac{v^3}{R}\,ds + \frac{q}{2}\int_t^\infty \frac{|u-v|(u+v)}{R}\,ds \\
&\le Lb(\varepsilon)\int_t^\infty \frac{(u+K\varphi)^3}{R}\,ds + \frac{q}{2}\int_t^\infty \frac{(2u+K\varphi)K\varphi}{R}\,ds\,.
\end{aligned}$$

Concerning the first integral on the previous line

$$\begin{aligned}
\int_t^\infty \frac{(u+K\varphi)^3}{R}\,ds &= \varphi(t) + \int_t^\infty \frac{1}{R}(3Ku^2\varphi + 3K^2u\varphi^2 + K^3\varphi^3)\,ds \\
&= \varphi(t) + o\big(\varphi(t)\big) = \varphi(t)\big(1+o(1)\big) \quad \text{as } \ t\to\infty\,.
\end{aligned}$$

As for the second integral, by a direct computation

$$\int_t^\infty \frac{\varphi^2(s)}{R(s)}\,ds = o\big(\varphi(t)\big) \quad \text{as } \ t\to\infty$$

and

$$qK\int_t^\infty \frac{u\varphi}{R}\,ds = \frac{8K\lambda_1^4}{q^3}\int_t^\infty \frac{1}{R\big(\int_T^s R^{-1}\big)^3}\,ds = \frac{4K\lambda_1^4}{q^3}\frac{1}{\big(\int_T^t R^{-1}\big)^2} = K\lambda_1\varphi(t)\,.$$

Observe that for $0<\mu<\frac{1}{2q}$ both roots $\lambda_{1,2}$ of the quadratic equation $\lambda^2-\lambda+\frac{q\mu}{2}=0$ are in the interval $(0,1)$.

Altogether,

$$\big|\mathcal{F}(v)(t) - u(t)\big| \le \big(Lb(\varepsilon) + \lambda_1 K + o(1)\big)\varphi(t)$$

as $t\to\infty$. Now, let $T_1 > T$ be so large that the term $o(1)$ in the previous computation is $o(1)\le 1$ for $t > T_1$. Then for $K \ge \frac{Lb(\varepsilon)+1}{1-\lambda_1}$ we have

$$\big|\mathcal{F}(v)(t) - u(t)\big| \le K\varphi(t) \quad \text{for } \ t\ge T_1\,,$$

i.e., $\mathcal{F}$ maps $\mathcal{V}$ into itself. Moreover, $\mathcal{F}(\mathcal{V})$ is bounded and since the derivatives

$$[\mathcal{F}(v)(t)]' = -\frac{\mu}{\big(\int_T^t R^{-1}(s)\,ds\big)^2 R(t)} - (p-1)r^{1-q}(t)h^{-q}(t)H\big(v,G(t)\big)$$

are bounded on compact subintervals of $[T_1,\infty)$, $\mathcal{F}(\mathcal{V})$ is also equicontinuous. Now, by the Schauder–Tichonoff fixed point theorem $\mathcal{F}$ has a fixed point in $\mathcal{V}$, i.e., a solution of (17) satisfying

$$v(t) = u(t) + O\big(\varphi(t)\big) \quad \text{as } \ t\to\infty\,.$$

Substituting for v in the formula $w = h^{-p}(v + G)$ we have

$$w = r\Phi(x'/x) = h^{-p}\big(u + O(\varphi) + G\big) = h^{-p}G\Big(1 + \frac{2\lambda_1}{qG\int_T^t R^{-1}} + O(\varphi/G)\Big)$$
$$= r\Phi(h'/h)\left(1 + \frac{2\lambda_1}{qG\int_T^t R^{-1}} + O\Big(G^{-1}\Big(\int_T^t R^{-1}\Big)^{-2}\Big)\right)$$

as $t \to \infty$. Hence, by the binomial expansion (since $\frac{q-1}{q} = \frac{1}{p}$ and (7) holds)

$$\frac{x'}{x} = \frac{h'}{h}\left[1 + \frac{2\lambda_1}{qG\int_T^t R^{-1}} + O\left(G^{-1}\Big(\int_T^t R^{-1}\Big)^{-2}\right)\right]^{q-1}$$
$$= \frac{h'}{h}\left[1 + \frac{2\lambda_1}{pG\int_T^t R^{-1}} + O\left(G^{-1}\Big(\int_T^t R^{-1}\Big)^{-2}\right)\right]$$
$$= \frac{h'}{h} + \frac{2\lambda_1}{pR\int_T^t R^{-1}} + O\left(R^{-1}\Big(\int_T^t R^{-1}\Big)^{-2}\right)$$

as $t \to \infty$. Therefore, integrating the last formula we have the asymptotic formula for the solution

$$x_1(t) = h(t)\exp\left\{\frac{2\lambda_1}{p}\log\int_T^t R^{-1}\right\}\cdot\exp\left\{O\left(\Big(\int_T^t R^{-1}\Big)^{-1}\right)\right\}$$
$$= h(t)\Big(\int_T^t R^{-1}\Big)^{\frac{2\lambda_1}{p}}\left[1 + O\left(\Big(\int_T^t R^{-1}\Big)^{-1}\right)\right]$$

as $t \to \infty$. The same asymptotic formula, with λ_2 instead of λ_1, we have for the solution x_2.

Finally, concerning the solution x_1, since the term $O\Big(\big(\int_T^t R^{-1}\big)^{-1}\Big)$ can be differentiated and its derivative tends to 0 as $t \to \infty$, we have

$$x_1' = h'\Big(\int_T^t R^{-1}\Big)^{\frac{2\lambda_1}{p}}\left[1 + \frac{2\lambda_1 h}{ph'\big(\int_T^t R^{-1}\big)R} + O\left(\Big(\int_T^t R^{-1}\Big)^{-1}\right)\right]$$
$$= h'\Big(\int_T^t R^{-1}\Big)^{\frac{2\lambda_1}{p}}\big[1 + o(1)\big] \quad \text{as} \quad t \to \infty\,,$$

which implies that

$$rx_1^2|x_1'|^{p-2} \sim rh^2|h'|^{p-2}\Big(\int_T^t R^{-1}\Big)^{\frac{4\lambda_1}{p}}\Big(\int_T^t R^{-1}\Big)^{\frac{2\lambda_1(p-2)}{p}} = R\Big(\int_T^t R^{-1}\Big)^{2\lambda_1}.$$

Since λ_1 is the smaller root of the equation $\lambda^2 - \lambda + \frac{q\mu}{2} = 0$, it is $\lambda_1 < \frac{1}{2}$ and hence

$$\int^\infty \frac{dt}{r(t)x_1^2(t)|x_1'(t)|^{p-2}}\,dt \sim \int^\infty \frac{dt}{R(t)\big(\int_T^t R^{-1}(s)\,ds\big)^{2\lambda_1}} = \infty$$

which means, by Proposition 1, that x_1 is the principal solution. □

In the next statement we turn our attention to (5) in the critical case $\mu = \frac{1}{2q}$, i.e., we consider the equation

$$\big(r(t)\Phi(x')\big)' + \left[c(t) + \frac{1}{2qh^p(t)\big(\int_T^t R^{-1}(s)\,ds\big)^2 R(t)}\right]\Phi(x) = 0\,. \tag{19}$$

In [5] we proved that this equation has a solution given by the asymptotic formula (8). In the next theorem we establish an asymptotic formula for a solution of (19) which is linearly independent of (8).

Theorem 2. *Suppose that* (6) *and* (7) *hold. Equation* (19) *has the pair of linearly independent solutions*

$$x_1(t) = h(t)\Big(\int_T^t R^{-1}(s)\,ds\Big)^{\frac{1}{p}}\left[1 + O\left(\Big(\int_T^t R^{-1}(s)\,ds\Big)^{-1}\right)\right],$$

$$x_2(t) = h(t)\Big(\int_T^t R^{-1}(s)\,ds\Big)^{\frac{1}{p}}\log^{\frac{2}{p}}\Big(\int_T^t R^{-1}(s)\,ds\Big)\left[1 + O\left(\Big(\int_T^t R^{-1}(s)\,ds\Big)^{-1}\right)\right]$$

as $t \to \infty$.

Proof. As we have already mentioned, the formula for x_1 is proved in [5, Theorem 2], so we prove only the formula for x_2. Since the proof is similar to that in [5], we skip some details.

The approximate Riccati equation associated with (19) is

$$u' + \frac{1}{2q\big(\int_T^t R^{-1}(s)\,ds\big)^2 R(t)} + \frac{q}{2R(t)}u^2 = 0\,. \tag{20}$$

This equation is the classical Riccati equation associated with the linear equation (by the substitution $u = \frac{2Ry'}{qy}$)

$$\big(R(t)y'\big)' + \frac{1}{4\big(\int_T^t R^{-1}(s)\,ds\big)^2 R(t)}y = 0$$

which has the pair of linearly independent solutions (we again introduce the new independent variable s given by $\frac{d}{dt} = R^{-1}(t)\frac{d}{ds}$)

$$y_1(t) = \Big(\int_T^t R^{-1}(s)\,ds\Big)^{\frac{1}{2}}, \quad y_2(t) = \Big(\int_T^t R^{-1}(s)\,ds\Big)^{\frac{1}{2}}\log\Big(\int_T^t R^{-1}(s)\,ds\Big).$$

The solution y_1 leads to the asymptotic formula for x_1, while concerning y_2, the associated solution of (20) is

$$u = \frac{2Ry_2'}{qy_2} = \frac{1}{q\big(\int_T^t R^{-1}(s)\,ds\big)}\left(1 + \frac{2}{\log\big(\int_T^t R^{-1}(s)\,ds\big)}\right).$$

Now, we define

$$\varphi(t) = \int_t^\infty \frac{d\tau}{q^3 R(\tau)\big(\int_T^\tau R^{-1}(s)\,ds\big)^3} = \frac{1}{2q^3\big(\int_T^t R^{-1}(s)\,ds\big)^2}$$

and we consider the space

$$\mathcal{V} = \big\{v \in C[T_1, \infty) : |v(t) - u(t)| \le K\varphi(t)\,, \quad t \ge T_1\big\}$$

with suitably chosen T_1, K. In this space, we consider the operator

$$\mathcal{F}(v)(t) = \frac{1}{2q\big(\int_T^t R^{-1}(s)\,ds\big)} + (p-1)\int_t^\infty r^{1-q}(s)h^{-q}(s)H\big(v(s), G(s)\big)\,ds$$

which maps $\mathcal{V}$ into itself for suitable T_1, K. By the Schauder–Tichonoff theorem this operator has a fixed point

$$v(t) = u(t) + O\big(\varphi(t)\big) \quad \text{as } \ t \to \infty\,.$$

Now

$$\begin{aligned} w &= r\Phi(x'/x) = h^{-p}(v+G) = h^{-p}\big(u + G + O(\varphi)\big) \\ &= h^{-p}\left[\frac{1}{q\big(\int_T^t R^{-1}(s)\,ds\big)}\left(1 + \frac{2}{\log\big(\int_T^t R^{-1}(s)\,ds\big)}\right) + rh\Phi(h')\right. \\ &\quad \left. + \, O\left(\left(\int_T^t R^{-1}(s)\,ds\right)^{-2}\right)\right] \\ &= r\Phi(h'/h)\left[1 + \frac{1}{qG\big(\int_T^t R^{-1}(s)\,ds\big)} + \frac{2}{qG\big(\int_T^t R^{-1}(s)\,ds\big)\log\big(\int_T^t R^{-1}(s)\,ds\big)}\right. \\ &\quad \left. + \, O\left(G^{-1}\left(\int_T^t R^{-1}(s)\,ds\right)^{-2}\right)\right] \quad \text{as } \ t \to \infty\,. \end{aligned}$$

Hence, again using the binomial formula

$$\begin{aligned} \frac{x'}{x} = \frac{h'}{h} &+ \frac{1}{pR(t)\big(\int_T^t R^{-1}(s)\,ds\big)} + \frac{2}{pR(t)\big(\int_T^t R^{-1}(s)\,ds\big)\log\big(\int_T^t R^{-1}(s)\,ds\big)} \\ &+ O\left(R^{-1}(t)\left(\int_T^t R^{-1}(s)\,ds\right)^{-2}\right) \quad \text{as } \ t \to \infty \end{aligned}$$

and integrating this equality we obtain the asymptotic formula for x_2. □

4. Open problems

(i) We have shown in [5] that if (6) and (7) hold, then (5) is conditionally oscillatory with respect to μ with the oscillation constant $\tilde{\mu} = \frac{1}{2q}$. Assumptions (6), (7) are used there in replacing the modified Riccati equation (11) by its approximation (12).

Consider the subcritical *linear* Euler equation

$$x'' + \frac{\gamma}{t^2}x = 0\,, \quad \gamma < \frac{1}{4}\,,$$

as the special case of (3) with $p = 2$, $r(t) = 1$, $c(t) = \gamma t^{-2}$. Solutions of this equation are $x_{1,2} = t^{\lambda_{1,2}}$, where $\lambda_{1,2}$ are the roots of the quadratic equation $\lambda^2 - \lambda + \gamma = 0$, i.e., $\lambda_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1-4\gamma}\right)$. Take as a solution h one of the solutions $h(t) = t^{\lambda_{1,2}}$. We have $R = rh^2 = t^{2\lambda_{1,2}}$ and

$$\frac{1}{2qh^2\big(\int^t R^{-1}\big)^2 rh^2} = \frac{1}{4} t^{-4\lambda_{1,2}} \left(\frac{t^{-2\lambda_{1,2}+1}}{1-2\lambda_{1,2}}\right)^{-2} = \frac{1}{4}\Big(1 - 1 \pm \sqrt{1-4\gamma}\Big)^2 t^{-2}$$
$$= \frac{1-4\gamma}{4t^2} = \Big(\frac{1}{4} - \gamma\Big) t^{-2}.$$

Hence

$$c + \frac{1}{2qh^2\big(\int^t R^{-1}\big)^2 rh^2} = \frac{\gamma}{t^2} + \frac{\frac{1}{4} - \gamma}{t^2} = \frac{1}{4t^2}$$

and (5) reduces to the critical linear Euler equation in this particular case.

If we try to modify this computation to the perturbed subcritical Euler half-linear equation

$$\big(\Phi(x')\big)' + \frac{\gamma}{t^p}\Phi(x) = 0, \quad \gamma < \gamma_p = \Big(\frac{p-1}{p}\Big)^p,$$

we take $h(t) = t^\lambda$, where λ is a root of the equation

$$(p-1)\big(|\lambda|^p - \Phi(\lambda)\big) + \gamma = 0\,. \tag{21}$$

In this case

$$R = rh^2|h'|^{p-2} = |\lambda|^{p-2} t^{2\lambda} t^{(\lambda-1)(p-2)} = |\lambda|^{p-2} t^{\lambda p - p + 2}$$

and

$$2qh^p\Big(\int^t R^{-1}\Big)^2 R = 2qt^{p\lambda}|\lambda|^{p-2} t^{\lambda p - p + 2}\Big(\frac{t^{-p\lambda+p-1}}{-p\lambda+p-1}\Big)^2$$
$$= \frac{2qt^p}{|\lambda|^{p-2}(-p\lambda + p - 1)^2}\,.$$

Then

$$c + \frac{1}{2qh^p\big(\int^t R^{-1}\big)^2 R} = t^{-p}\Big(\gamma + \frac{1}{2q}|\lambda|^{p-2}(-p\lambda+p-1)^2\Big)\,. \tag{22}$$

If we take $\lambda = \lambda_1$, where $\lambda_1 < \frac{p-1}{p}$ is the smaller root of (21), we have

$$G = rh\Phi(h') = \Phi(\lambda_1) t^{\lambda_1} t^{(p-1)(\lambda_1 - 1)} = \Phi(\lambda_1) t^{p\lambda_1 - p + 1} \to 0$$

as $t \to \infty$ because $p\lambda_1 - p + 1 < 0$. If we take $\lambda = \lambda_2$, where λ_2 is the bigger root of (21), then $G(t) \to \infty$ as $t \to \infty$, but

$$\int^\infty R^{-1}(t)\,dt = |\lambda_2|^{2-p}\int^\infty t^{-2+p-p\lambda_2}\,dt < \infty$$

because $-2 + p - p\lambda < -1$. Hence, in both cases assumptions (6), (7) are not satisfied so we cannot apply the method used in the previous proofs where the modified Riccati equation (11) was replaced by its approximation (12). Nevertheless, we conjecture, based on the linear case and also on the fact that the power t^{-p} appeared in (22), that (5) is an equation with the critical oscillation constant also in this particular case. This leads to the conjecture that we have the identity for the roots of (21)

$$\gamma + \frac{1}{2q}|\lambda|^{p-2}\big(-p\lambda + p - 1\big)^2 = \gamma_p = \Big(\frac{p-1}{p}\Big)^p .$$

(ii) Consider now equation (5) in the supercritical case $\mu > \frac{1}{2q}$. Then this equation is oscillatory. Approximate Riccati equation (12) is the classical Riccati equation corresponding to the linear Sturm–Liouville equation

$$\big(R(t)y'\big)' + \frac{q\mu}{2\big(\int^t R^{-1}\big)^2 R} y = 0 \,, \quad \mu > \frac{1}{2q}$$

(related to (12) by the substitution $u = \frac{2Ry'}{qy}$). The linearly independent solutions of this equation are

$$y_{1,2}(t) = \Big(\int^t R^{-1}(s)\, ds\Big)^{\frac{1}{2}} \times \begin{cases} \cos\Big(\beta \log\big(\int^t R^{-1}(s)\, ds\big)\Big) \,, \\ \sin\Big(\beta \log\big(\int^t R^{-1}(s)\, ds\big)\Big) \,, \end{cases}$$

where $\beta = \frac{1}{2}\sqrt{2q\mu - 1}$ as can be verified by a direct computation. Then

$$u_{1,2} = \frac{2Ry'_{1,2}}{y_{1,2}} = \frac{1}{q\int^t R^{-1}} \times \begin{cases} 1 - 2\beta \tan\big(\beta \log \int^t R^{-1}\big) \,, \\ 1 + 2\beta \cot\big(\beta \log \int^t R^{-1}\big) \,. \end{cases}$$

Now, if we accept the assumption that the approximate Riccati equation well approximates the modified Riccati equation also in the oscillatory case, this leads, using the same idea as in the nonoscillatory case, to the conjecture that the solutions of (5) in the supercritical case are

$$x_1(t) \sim h(t)\Big(\int^t R^{-1}(s)\, ds\Big)^{\frac{1}{p}} \cos^{\frac{2}{p}}\Big(\beta \log\Big(\int^t R^{-1}(s)\, ds\Big)\Big) \,,$$

$$x_2(t) \sim h(t)\Big(\int^t R^{-1}(s)\, ds\Big)^{\frac{1}{p}} \sin^{\frac{2}{p}}\Big(\beta \log\Big(\int^t R^{-1}(s)\, ds\Big)\Big) \,.$$

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