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PROLOGATION OF POISSON 2-FORM ON WEIL BUNDLES

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Abstract. In this paper, M denotes a smooth manifold of dimension n, A a Weil algebra and M^A the associated Weil bundle. When (M,ω_M) is a Poisson manifold with 2-form ω_M, we construct the 2-Poisson form $\omega^A_{M^A}$, prolongation on M^A of the 2-Poisson form ω_M. We give a necessary and sufficient condition for that M^A be an A-Poisson manifold.

1. Introduction

1.1. **Weil algebra and Weil bundle.** In what follows, all structures are assumed to be of class C^∞. We denote by M a smooth differential manifold, $C^\infty(M)$ the algebra of differentiable functions on M and by $\mathfrak{X}(M)$, the $C^\infty(M)$-module of vectors field on M.

A Weil algebra is a real, unitary, commutative algebra of finite dimension with a unique maximal ideal of codimension 1 on $\mathbb{R}$, [15].
Let A be a Weil algebra and $\mathfrak{m}$ be its maximal ideal. We have $A=\mathbb{R}\oplus\mathfrak{m}$ and the first projection

$$A=\mathbb{R}\oplus\mathfrak{m}\to\mathbb{R}$$

is a homomorphism of algebras which is surjective, called augmentation and the unique non zero integer $h\in\mathbb{N}$ such that $\mathfrak{m}^h\neq(0)$ and $\mathfrak{m}^{h+1}=(0)$ is the height of A, [15].

If M is a smooth manifold, and A a Weil algebra of maximal ideal $\mathfrak{m}$, an infinitely near point to $x\in M$ of kind A is a homomorphism of algebras

$$\xi\colon C^\infty(M)\to A$$

such that $[\xi(f)-f(x)]\in\mathfrak{m}$ for any $f\in C^\infty(M)$. i.e. the real part of $\xi(f)$ is exactly $f(x)$, [15].

We denote by M^A_x the set of all infinitely near points to $x\in M$ of kind A and $M^A=\bigcup\limits_{x\in M}M^A_x$ the manifold of infinitely near points of kind A. We have $\dim M^A=\dim M\times\dim A$, [6].

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When both M and N are smooth manifolds and when $h\colon M \to N$ is a differentiable application, then the map

$$h^A\colon M^A \to N^A\,, \quad \xi \mapsto h^A(\xi)\,,$$

such that, for any $g \in C^\infty(N)$,

$$\left[h^A(\xi)\right](g) = \xi(g \circ h)$$

is also differentiable. When h is a diffeomorphism, it is the same for h^A, [1].

Moreover, if $\varphi\colon A \to B$ is a homomorphism of Weil algebras, for any smooth manifold M, the map

$$\varphi_M\colon M^A \to M^B\,, \quad \xi \mapsto \varphi \circ \xi$$

is differentiable. In particular, the augmentation

$$A \to \mathbb{R}$$

defines for any smooth manifold M, the projection

$$\pi_M\colon M^A \to M\,,$$

which assigns every infinitely near point to $x \in M$ to its origin x. Thus (M^A, π_M, M) defines the bundle of infinitely near points or simply Weil bundle (see [3], [6], [15], [9]).

If (U, φ) is a local chart of M with coordinate functions $(x_1, x_2, \dots, x_n)$, the application

$$U^A \to A^n\,, \quad \xi \mapsto \big(\xi(x_1), \xi(x_2), \dots, \xi(x_n)\big)\,,$$

is a bijection from U^A into an open of A^n. The manifold M^A is a smooth manifold modeled over A^n, that is to say an A-manifold of dimension n, [2], [13].

The set, $C^\infty(M^A, A)$ of differentiable functions on M^A with values in A is a commutative, unitary algebra over A. When one identitifies $\mathbb{R}^A$ with A, for $f \in C^\infty(M)$, the map

$$f^A\colon M^A \to A\,, \quad \xi \mapsto \xi(f)$$

is differentiable and the map

$$C^\infty(M) \to C^\infty(M^A, A)\,, \quad f \mapsto f^A\,,$$

is an injective homomorphism of algebras and we have:

$$(f+g)^A = f^A + g^A\,; \quad (\lambda \cdot f)^A = \lambda \cdot f^A\,; \quad (f \cdot g)^A = f^A \cdot g^A$$

for $\lambda \in \mathbb{R}$, f, $g \in C^\infty(M)$.

We denote $\mathfrak{X}(M^A)$, the set of all vector fields on M^A. According to [2], [8] we have the following equivalent assertions:

Theorem 1. *The following assertions are equivalent:*

(1) *A vector field on M^A is a differentiable section of the tangent bundle (TM^A, π_{M^A}, M^A).*

(2) *A vector field on M^A is a derivation of $C^\infty(M^A)$.*

(3) *A vector field on M^A is a derivation of $C^\infty(M^A, A)$ which is A-linear.*

(4) *A vector field on M^A is a linear map $X\colon C^\infty(M) \to C^\infty(M^A, A)$ such that*

$$X(f \cdot g) = X(f) \cdot g^A + f^A \cdot X(g)\,, \quad \textit{for any} \quad f, g \in C^\infty(M)\,.$$

Consequently [8],

Theorem 2. *The map*

$$\mathfrak{X}(M^A) \times \mathfrak{X}(M^A) \longrightarrow \mathfrak{X}(M^A), (X, Y) \longmapsto [X, Y] = X \circ Y - Y \circ X$$

is skew-symmetric A-bilinear and defines a structure of A-Lie algebra over $\mathfrak{X}(M^A)$.

Thus, if $\mathrm{Der}_A[C^\infty(M^A, A)]$ denotes the $C^\infty(M^A, A)$-module of derivations of $C^\infty(M^A, A)$ which are A-linear, a vector field on M^A is a derivation of $C^\infty(M^A, A)$ which is A-linear i.e. a A-linear map

$$X\colon C^\infty(M^A, A) \to C^\infty(M^A, A)$$

such that

$$X(\varphi \cdot \psi) = X(\varphi) \cdot \psi + \varphi \cdot X(\psi), \quad \text{for any} \quad \varphi, \psi \in C^\infty(M^A, A)\,.$$

Thus, we have

$$\mathfrak{X}(M^A) = \mathrm{Der}_A \left[C^\infty(M^A, A)\right].$$

Proposition 1 ([2], [8])**.** *If $\theta\colon C^\infty(M) \to C^\infty(M)$ is a vector field on M, then there exists one and only one A-linear derivation*

$$\theta^A\colon C^\infty(M^A, A) \to C^\infty(M^A, A)$$

such that

$$\theta^A(f^A) = [\theta(f)]^A$$

for any $f \in C^\infty(M)$.

Proposition 2 ([2], [8])**.** *If θ, θ_1 and θ_2 are vector fields on M and if $f \in C^\infty(M)$, then we have:*

(1) $\quad (\theta_1 + \theta_2)^A = \theta_1^A + \theta_2^A$;

(2) $\quad (f \cdot \theta)^A = f^A \cdot \theta^A$;

(3) $\quad [\theta_1, \theta_2]^A = [\theta_1^A, \theta_2^A]$.

Corollary 1. *The map*

$$\mathfrak{X}(M) \to \mathrm{Der}_A \left[C^\infty(M^A, A)\right], \theta \mapsto \theta^A$$

is an injective homomorphism of $\mathbb{R}$-Lie algebras. If $\mu\colon A \to A$, is a $\mathbb{R}$-endomorphism, and $\theta\colon C^\infty(M) \to C^\infty(M)$ a vector field on M, then

$$\theta^A(\mu \circ f^A) = \mu \circ \left[\theta(f)\right]^A \quad \textit{for any} \quad f \in C^\infty(M)\,.$$

1.2. **Poisson manifold.** We recall that a Poisson structure on a smooth manifold M is due to the existence of a bracket $\{\,,\}_M$ on $C^\infty(M)$ such that the pair $(C^\infty(M), \{\,,\}_M)$ is a real Lie algebra such that, for any $f \in C^\infty(M)$ the map

$$\mathrm{ad}(f)\colon C^\infty(M) \to C^\infty(M)\,, \quad g \mapsto \{f, g\}_M$$

is a derivation of commutative algebra i.e.

$$\{f, g \cdot h\}_M = \{f, g\}_M \cdot h + g \cdot \{f, h\}_M$$

for f, g, $h \in C^\infty(M)$. In this case we say that $C^\infty(M)$ is a Poisson algebra and M is a Poisson manifold ([5], [14], [11]).

Let $\Omega_{\mathbb{R}}[C^\infty(M)]$ be the $C^\infty(M)$-module of Kähler differentials of $C^\infty(M)$ and

$$\delta_M\colon C^\infty(M) \to \Omega_{\mathbb{R}}[C^\infty(M)]\,, \quad f \mapsto \overline{f \otimes 1_{C^\infty(M)} - 1_{C^\infty(M)} \otimes f}$$

the canonical derivation which the image of δ_M generates the $C^\infty(M)$-module $\Omega_{\mathbb{R}}[C^\infty(M)]$ i.e. for $x \in \Omega_{\mathbb{R}}[C^\infty(M)]$,

$$x = \sum_{i \in I\colon \text{finite}} f_i \cdot \delta_M(g_i)\,,$$

with f_i, $g_i \in C^\infty(M)$ for any $i \in I$ ([4], [11], [10]).

The manifold M is a Poisson manifold if and only if there exists a skew-symmetric 2-form

$$\omega_M\colon \Omega_{\mathbb{R}}\big[C^\infty(M)\big] \times \Omega_{\mathbb{R}}\big[C^\infty(M)\big] \to C^\infty(M)$$

such that for any f and g in $C^\infty(M)$,

$$\{f, g\}_M = -\omega_M\big[\delta_M(f), \delta_M(g)\big]$$

defines a structure of Lie algebra over $C^\infty(M)$, [11], [10]. In this case, we say that ω_M is the Poisson 2-form of the Poisson manifold M and we denote (M, ω_M) the Poisson manifold of Poisson 2-form ω_M.

The main goal of this paper is to study the prolongation of the Poisson 2-form ω_M of Poisson manifold on Weil bundles.

2. The algebra of Kähler forms on $C^\infty(M^A, A)$

Definition 1. The $C^\infty(M^A)$-module of Kähler differentials of $C^\infty(M^A)$ is the set

$$\Omega_{\mathbb{R}}\big[C^\infty(M^A)\big] = \frac{J}{J^2}$$

where J is the $C^\infty(M^A)$-submodule of $C^\infty(M^A) \bigotimes_{\mathbb{R}} C^\infty(M^A)$ generated by the elements of the form $F \otimes 1_{C^\infty(M^A)} - 1_{C^\infty(M^A)} \otimes F$ with $F \in C^\infty(M^A)$. Thus, the map

$$d_{M^A}\colon C^\infty(M^A) \to \Omega_{\mathbb{R}}[C^\infty(M^A)], F \mapsto \overline{F \otimes 1_{C^\infty(M^A)} - 1_{C^\infty(M^A)} \otimes F}$$

is a derivation and the image of d_{M^A} generates $\Omega_{\mathbb{R}}[C^\infty(M^A)]$.

The A-algebra $C^\infty(M^A, A) \bigotimes\limits_A C^\infty(M^A, A)$ admits a structure of $C^\infty(M^A, A)$--module defined by the homomorphism of A-algebras

$$C^\infty(M^A, A) \to C^\infty(M^A, A) \underset{A}{\otimes} C^\infty(M^A, A), \varphi \mapsto \varphi \otimes 1_{C^\infty(M^A,A)}.$$

In this case, we say that $C^\infty(M^A, A) \bigotimes\limits_A C^\infty(M^A, A)$ admits a structure of $C^\infty(M^A, A)$-module defined by the first factor. The second factor is defined by

$$C^\infty(M^A, A) \to C^\infty(M^A, A) \underset{A}{\otimes} C^\infty(M^A, A)\,, \quad \varphi \mapsto 1_{C^\infty(M^A,A)} \otimes \varphi\,.$$

The map

$$C^\infty(M^A, A) \times C^\infty(M^A, A) \to C^\infty(M^A, A)\,, \quad (\varphi, \psi) \mapsto \varphi \cdot \psi$$

being A-bilinear, then there exists a unique A-linear map

$$m\colon C^\infty(M^A, A) \underset{A}{\otimes} C^\infty(M^A, A) \to C^\infty(M^A, A)$$

such that

$$m(\varphi \otimes \psi) = \varphi \cdot \psi\,.$$

The kernel of m is the $C^\infty(M^A, A)$-submodule of $C^\infty(M^A, A) \otimes C^\infty(M^A, A)$ generated by the elements of the form $\varphi \otimes 1_{C^\infty(M^A,A)} - 1_{C^\infty(M^A,A)} \otimes \varphi$ with $\varphi \in C^\infty(M^A, A)$.

We denote $\Omega_A[C^\infty(M^A, A)]$, the $C^\infty(M^A, A)$-module of Kähler differentials of $C^\infty(M^A, A)$ which are A-linears. In this case, for $\varphi \in C^\infty(M^A, A)$, we denote $\overline{\varphi \otimes 1_{C^\infty(M^A,A)} - 1_{C^\infty(M^A,A)} \otimes \varphi}$, the class of $\varphi \otimes 1_{C^\infty(M^A,A)} - 1_{C^\infty(M^A,A)} \otimes \varphi$ in $C^\infty(M^A, A)$.

The map

$$C^\infty(M) \to \Omega_{\mathbb{A}}\big[C^\infty(M^A, A)\big]\,, \quad f \mapsto \overline{f^A \otimes 1_{C^\infty(M^A,A)} - 1_{C^\infty(M^A,A)} \otimes f^A}$$

is a derivation.
Thus,

Proposition 3. *There exists a unique A-linear derivation*

$$\delta^A_{M^A} : C^\infty(M^A, A) \to \Omega_A\big[C^\infty(M^A, A)\big]$$

such that

$$\delta^A_{M^A}(f^A) = \big[\delta_M(f)\big]^A$$

for any $f \in C^\infty(M)$.

Proof. Let

$$\delta^A_{M^A} : C^\infty(M^A, A) \xrightarrow{\sigma^{-1}} A \otimes C^\infty(M^A) \xrightarrow{\mathrm{id}_A \otimes d_{M^A}} A \otimes \Omega_{\mathbb{R}}\big[C^\infty(M^A)\big] \xrightarrow{\varpi} \Omega_A\big[C^\infty(M^A, A)\big]$$

be that map, where

$$\sigma^{-1} : \varphi = \sum_{\alpha=1}^{\dim A} (a^*_\alpha \circ \varphi) \cdot a_\alpha \longmapsto \sum_{\alpha=1}^{\dim A} a_\alpha \otimes (a^*_\alpha \circ \varphi)$$

with $(a_\alpha)_{\alpha=1,\dots,\dim A}$ a basis of A and $(a^*_\alpha)_{\alpha=1,\dots,\dim A}$ the dual basis of the basis $(a_\alpha)_{\alpha=1,\dots,\dim A}$,

$$\begin{aligned}\mathrm{id}_A \otimes d_{M^A} \colon \sum_{\alpha=1}^{\dim A} a_\alpha \otimes (a^*_\alpha \circ \varphi) &\mapsto \sum_{\alpha=1}^{\dim A} a_\alpha \otimes d_{M^A}(a^*_\alpha \circ \varphi)\\ &= \sum_{\alpha=1}^{\dim A} a_\alpha \otimes \left[\overline{(a^*_\alpha \circ \varphi) \otimes 1_{C^\infty(M^A)} - 1_{C^\infty(M^A)} \otimes (a^*_\alpha \circ \varphi)}\right],\end{aligned}$$

$$\begin{aligned}\varpi \colon &\sum_{\alpha=1}^{\dim A} a_\alpha \otimes d_{M^A}(a^*_\alpha \circ \varphi)\\ &\longmapsto \sum_{\alpha=1}^{\dim A} \left[\overline{(a^*_\alpha \circ \varphi)a_\alpha \otimes 1_{C^\infty(M^A,A)} - 1_{C^\infty(M^A,A)} \otimes (a^*_\alpha \circ \varphi)a_\alpha}\right].\end{aligned}$$

Thus,

$$\begin{aligned}\delta^A_{M^A}(\varphi) &= [\varpi \circ (\mathrm{id}_A \otimes d_{M^A}) \circ \sigma^{-1}](\varphi)\\ &= \sum_{\alpha=1}^{\dim A} \overline{\left[(a^*_\alpha \circ \varphi)a_\alpha \otimes 1_{C^\infty(M^A,A)} - 1_{C^\infty(M^A,A)} \otimes (a^*_\alpha \circ \varphi)a_\alpha\right]}.\end{aligned}$$

For any φ, $\psi \in C^\infty(M^A, A)$, we have

$$\begin{aligned}\delta^A_{M^A}(\varphi+\psi) &= \left[\varpi \circ (\mathrm{id}_A \otimes d_{M^A}) \circ \sigma^{-1}\right](\varphi+\psi)\\ &= \left[\varpi \circ (\mathrm{id}_A \otimes d_{M^A})\right]\left(\sigma^{-1}(\varphi)+\sigma^{-1}(\psi)\right)\\ &= \left[\varpi \circ (\mathrm{id}_A \otimes d_{M^A})\right]\left(\sigma^{-1}(\varphi)\right) + \left[\varpi \circ (\mathrm{id}_A \otimes d_{M^A})\right]\left(\sigma^{-1}(\psi)\right)\\ &= \left[\varpi \circ (\mathrm{id}_A \otimes d_{M^A}) \circ \sigma^{-1}\right](\varphi) + \left[\varpi \circ (\mathrm{id}_A \otimes d_{M^A}) \circ \sigma^{-1}\right](\psi)\\ &= \delta^A_{M^A}(\varphi) + \delta^A_{M^A}(\psi).\end{aligned}$$

For any $\varphi \in C^\infty(M^A, A)$ and $a \in A$, we have

$$\begin{aligned}\delta^A_{M^A}(a\cdot\varphi) &= \left[\varpi \circ (\mathrm{id}_A \otimes d_{M^A}) \circ \sigma^{-1}\right](a\cdot\varphi)\\ &= \left[\varpi \circ (\mathrm{id}_A \otimes d_{M^A})\right]\left(\sigma^{-1}(a\cdot\varphi)\right)\\ &= \left[\varpi \circ (\mathrm{id}_A \otimes d_{M^A})\right]\left(a\cdot\sigma^{-1}(\varphi)\right)\\ &= a\cdot\left[\varpi \circ (\mathrm{id}_A \otimes d_{M^A})\right]\left(\sigma^{-1}(\varphi)\right)\\ &= a\cdot\delta^A_{M^A}(\varphi).\end{aligned}$$

For any φ, $\psi \in C^{\infty}(M^A, A)$, we have

$$\begin{aligned}
\delta^A_{M^A}(\varphi\cdot\psi) &= \big[\varpi\circ(\mathrm{id}_A \otimes d_{M^A})\circ\sigma^{-1}\big](\varphi\cdot\psi)\\
&= \big[\varpi\circ(id_A\otimes d_{M^A})\big]\big(\sigma^{-1}(\varphi\cdot\psi)\big)\\
&= \big[\varpi\circ(\mathrm{id}_A \otimes d_{M^A})\big]\big(\sigma^{-1}(\varphi)\cdot\sigma^{-1}(\psi)\big)\\
&= \big[\varpi\circ(\mathrm{id}_A \otimes d_{M^A})\big]\big(\sigma^{-1}(\varphi)\big)\cdot\psi+\varphi\cdot\big[\varpi\circ(\mathrm{id}_A \otimes d_{M^A})\big]\big(\sigma^{-1}(\psi)\big)\\
&= \big[\varpi\circ(\mathrm{id}_A \otimes d_{M^A})\circ\sigma^{-1}\big](\varphi)\cdot\psi+\varphi\cdot\big[\varpi\circ(\mathrm{id}_A \otimes d_{M^A})\circ\sigma^{-1}\big](\psi)\\
&= [\varpi\circ(\mathrm{id}_A \otimes d_{M^A})\circ\sigma^{-1}](\varphi)+[\varpi\circ(\mathrm{id}_A \otimes d_{M^A})\circ\sigma^{-1}](\psi)\\
&= \delta^A_{M^A}(\varphi)\cdot\psi+\varphi\cdot\delta^A_{M^A}(\psi)\,.
\end{aligned}$$

As

$$\delta_M\colon C^{\infty}(M)\to\Omega_{\mathbb{R}}\big[C^{\infty}(M)\big]$$

is a derivation, then the map

$$C^{\infty}(M)\to\Omega_A\big[C^{\infty}(M^A,A)\big]\,,\quad f\mapsto\big[\delta_M(f)\big]^A$$

is a derivation. Thus, for any $f\in C^{\infty}(M)$

$$\begin{aligned}
\delta^A_{M^A}(f^A) &= \varpi\circ(\mathrm{id}_A \otimes d_{M^A})\circ\sigma^{-1}(f^A)\\
&= \sum_{\alpha=1}^{\dim A}\overline{\big[(a^*_\alpha\circ f^A)a_\alpha\otimes 1_{C^\infty(M^A,A)}-1_{C^\infty(M^A,A)}\otimes(a^*_\alpha\circ f^A)a_\alpha\big]}\\
&= \overline{\sum_{\alpha=1}^{\dim A}(a^*_\alpha\circ f^A)a_\alpha\otimes 1_{C^\infty(M^A,A)}-1_{C^\infty(M^A,A)}\otimes\sum_{\alpha=1}^{\dim A}(a^*_\alpha\circ f^A)a_\alpha}\\
&= \overline{f^A\otimes 1_{C^\infty(M^A,A)}-1_{C^\infty(M^A,A)}\otimes f^A}\\
&= \big[\overline{f\otimes 1_{C^\infty(M)}-1_{C^\infty(M)}\otimes f}\big]^A
\end{aligned}$$

i.e.

$$\delta^A_{M^A}(f^A)=\big[\delta_M(f)\big]^A\,.$$

□

Proposition 4. *The map*

$$\Omega_{\mathbb{R}}\big[C^{\infty}(M)\big]\to\Omega_A[C^{\infty}(M^A,A)]\,,\quad x\mapsto x^A$$

is an injective homomorphism of $\mathbb{R}$*-modules.*

Proof. Let

$$\Psi\colon\Omega_{\mathbb{R}}\big[C^{\infty}(M)\big]\to\Omega_A\big[C^{\infty}(M^A,A)\big]\,,\quad x\mapsto x^A$$

be that map. □

For any x, $y \in \Omega_{\mathbb{R}}[C^\infty(M)]$,

$$\begin{aligned}\Psi(x+y) &= (x+y)^A \\ &= \Big(\sum_{i\in I:\ \text{finite}} f_i \cdot \delta_M(f_i') + \sum_{j\in I:\ \text{finite}} g_j \cdot \delta_M(g_j')\Big)^A \\ &= \Big(\sum_{i\in I:\ \text{finite}} f_i \cdot \delta_M(f_i')\Big)^A + \Big(\sum_{j\in I:\ \text{finite}} g_j \cdot \delta_M(g_j')\Big)^A \\ &= x^A + y^A\,.\end{aligned}$$

For any $x \in \Omega_{\mathbb{R}}[C^\infty(M)]$ and for $\lambda \in \mathbb{R}$,

$$\begin{aligned}\Psi(\lambda \cdot x) &= (\lambda \cdot x)^A \\ &= \Big(\lambda \cdot \sum_{i\in I:\ \text{finite}} f_i \cdot \delta_M(f_i')\Big)^A \\ &= \lambda \cdot \Big(\sum_{i\in I:\ \text{finite}} f_i \cdot \delta_M(f_i')\Big)^A \\ &= \lambda \cdot x^A\,.\end{aligned}$$

The pair $(\Omega_A[C^\infty(M^A, A)], \delta^A_{M^A})$ satisfies the following universal property: for every $C^\infty(M^A, A)$-module E and every A-derivation

$$\Phi\colon C^\infty(M^A, A) \to E\,,$$

there exists a unique $C^\infty(M^A, A)$-linear map

$$\widetilde{\Phi}\colon \Omega_A[C^\infty(M^A, A)] \to E$$

such that

$$\widetilde{\Phi} \circ \delta^A_{M^A} = \Phi\,.$$

In other words, there exists a unique $\widetilde{\Phi}$ which makes the following diagram commutative

$$\begin{tikzcd} \Omega_A[C^\infty(M^A, A)] \arrow[dr, "\widetilde{\Phi}"] & \\ C^\infty(M^A, A) \arrow[u, "\delta^A_{M^A}"] \arrow[r, "\Phi"'] & E\,. \end{tikzcd}$$

This fact implies the existence of a natural isomorphism of $C^\infty(M^A, A)$-modules

$$\mathrm{Hom}_{C^\infty(M^A,A)}\big(\Omega_A[C^\infty(M^A, A)], E\big) \to \mathrm{Der}_A[C^\infty(M^A, A)], E),\ \ \psi \longmapsto \psi \circ \delta^A_{M^A}\,.$$

In particular, if $E = C^\infty(M^A, A)$, we have

$$\begin{aligned}\Omega_A[C^\infty(M^A, A)]^* &\simeq \mathrm{Der}_A[C^\infty(M^A, A)] \\ &= \mathfrak{X}(M^A)\,.\end{aligned}$$

For any $p \in \mathbb{N}$, $\Lambda^p(\Omega_A[C^\infty(M^A, A)]) = \mathfrak{L}^p_{sks}(\Omega_A[C^\infty(M^A, A)], C^\infty(M^A, A))$ denotes the $C^\infty(M^A, A)$-module of skew-symmetric multilinear forms of degree p from $\Omega_A[C^\infty(M^A, A)]$ into $C^\infty(M^A, A)$ and

$$\Lambda\big(\Omega_A[C^\infty(M^A, A)]\big) = \bigoplus_{p\in\mathbb{N}} \Lambda^p\big(\Omega_A[C^\infty(M^A, A)]\big)$$

the exterior $C^\infty(M^A, A)$-algebra of $\Omega_A[C^\infty(M^A, A)]$.

$$\Lambda^0\big(\Omega_A[C^\infty(M^A, A)]\big) = C^\infty(M^A, A)\,,$$

$$\Lambda^1\big(\Omega_A[C^\infty(M^A, A)]\big) = \Omega_A\big[C^\infty(M^A, A)\big]^*\,.$$

We denote,

$$\delta^A_{M^A} = \delta_{M^A} \colon \Lambda\big(\Omega_A[C^\infty(M^A, A)]\big) \to \Lambda\big(\Omega_A[C^\infty(M^A, A)]\big)$$

a unique derivation, of degree $+1$, which extends the canonical derivation

$$\delta^0_{M^A} \colon C^\infty(M^A, A) \to \Omega_A[C^\infty(M^A, A)]\,.$$

For any φ, ψ, ψ_1, $\psi_2, \dots, \psi_p \in C^\infty(M^A, A)$ and $\omega \in \Omega_A[C^\infty(M^A, A)]^*$, we get

1. $\delta_{M^A}(\varphi \cdot \delta_{M^A}(\psi_1) \wedge \cdots \wedge \delta_{M^A}(\psi_p) = \delta_{M^A}(\varphi) \wedge \delta_{M^A}(\psi_1) \wedge \cdots \wedge \delta_{M^A}(\psi_p)$.
2. $\delta^1_{M^A}[\psi \cdot \delta^0_{M^A}(\varphi)] = \delta^0_{M^A}(\psi) \wedge \delta^0_{M^A}(\varphi)$.
3. $\delta^1_{M^A}(\varphi \cdot \omega) = \delta^0_{M^A}(\varphi) \wedge \omega + \varphi \cdot \delta^1_{M^A}(\omega)$.

Proposition 5. *If $\eta \in \Lambda^p(\Omega_{\mathbb{R}}[C^\infty(M)])$, then $\eta^A \in \Lambda^p(\Omega_A[C^\infty(M^A, A)])$.*

Proof. Indeed, for any $\eta \in \Lambda^p(\Omega_{\mathbb{R}}[C^\infty(M)])$, η is of the form $\delta_M(f_1) \wedge \cdots \wedge \delta_M(f_p)$ with $f_1, f_2, \dots, f_p \in C^\infty(M)$.

$$\begin{aligned} \eta^A &= [\delta_M(f_1) \wedge \cdots \wedge \delta_M(f_p)]^A \\ &= [\delta_M(f_1)]^A \wedge \cdots \wedge [\delta_M(f_p)]^A \\ &= \delta^0_{M^A}(f_1^A) \wedge \cdots \wedge \delta^0_{M^A}(f_p^A)\,. \end{aligned}$$

Thus, the $C^\infty(M^A, A)$-module $\Lambda^p(\Omega_A[C^\infty(M^A, A)])$ is generated by elements of the form

$$\eta^A = \delta^0_{M^A}(\varphi_1) \wedge \cdots \wedge \delta^0_{M^A}(\varphi_p)$$

with $\varphi_1 = f_1^A, \dots, \varphi_p = f_p^A \in C^\infty(M^A, A)$. □

The algebra

$$\Lambda\big(\Omega_A[C^\infty(M^A, A)]\big) = \bigoplus_{p\in\mathbb{N}} \Lambda^p\big(\Omega_A[C^\infty(M^A, A)]\big)$$

is the algebra of Kähler forms on $C^\infty(M^A, A)$.

The pair $(\Lambda(\Omega_A[C^\infty(M^A, A)], \delta^A_{M^A}))$ is a differential complex and the map

$$A \times \Omega_{\mathbb{R}}[C^\infty(M)] \longmapsto \Omega_A[C^\infty(M^A, A)], (a, x) \longmapsto a \cdot x^A$$

induces the morphism of the differential complex $(A \bigotimes \Lambda(\Omega_{\mathbb{R}}[C^\infty(M)]), \mathrm{id}_A \otimes \delta_M))$ into the differential complex $(\Lambda(\Omega_A[C^\infty(M^A, A)]), \delta^A_{M^A})$.

3. Lie derivative with respect to a derivation on M^A

Let
$$\theta\colon C^\infty(M) \to C^\infty(M)$$
be a derivation and
$$\sigma_\theta\colon \left[\Omega_{\mathbb{R}}[C^\infty(M)]\right]^p \longrightarrow \Lambda^p\big(\Omega_{\mathbb{R}}[C^\infty(M)]\big)\,,$$
be the $C^\infty(M)$-skew-symmetric multilinear map such that for any $x_1, x_2, \ldots, x_p \in \Omega_{\mathbb{R}}[C^\infty(M)]$,
$$\sigma_\theta(x_1, x_2, \ldots, x_p) = \sum_{i=1}^{p} (-1)^{i-1} \widetilde{\theta}(x_i) \cdot x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_p\,,$$
where
$$\widetilde{\theta}\colon \Omega_{\mathbb{R}}[C^\infty(M)] \to C^\infty(M)$$
is a unique $C^\infty(M)$-linear map such that $\widetilde{\theta} \circ \delta_M = \theta$. Then,
$$\sigma^A_{\theta^A}\colon \left[\Omega_A[C^\infty(M^A, A)]\right]^p \to \Lambda^p\big(\Omega_A[C^\infty(M^A, A)]\big)$$
is a unique $C^\infty(M^A, A)$-skew-symmetric multilinear map such that
$$\sigma^A_{\theta^A}(x_1^A, x_2^A, \ldots, x_p^A) = [\sigma_\theta(x_1, x_2, \ldots, x_p)]^A\,.$$
We denote
$$\widetilde{\sigma^A_{\theta^A}}\colon \Lambda^p(\Omega_A[C^\infty(M^A, A)] \to \Lambda^{p-1}\big(\Omega_A[C^\infty(M^A, A)]\big)\,,$$
the unique $C^\infty(M^A, A)$-skew-symmetric multilinear map such that
$$\widetilde{\sigma^A_{\theta^A}}(x_1^A \wedge x_2^A \wedge \cdots \wedge x_p^A) = \sigma^A_{\theta^A}(x_1^A, x_2^A, \ldots, x_p^A)$$
i.e. $\sigma^A_{\theta^A}$ induces a derivation
$$i_{\theta^A} = \widetilde{\sigma^A_{\theta^A}}\colon \Lambda(\Omega_A[C^\infty(M^A, A)]) \to \Lambda\big(\Omega_A[C^\infty(M^A, A)]\big)$$
of degree -1.

Proposition 6. *For any $\theta \in \mathrm{Der}_{\mathbb{R}}[C^\infty(M)]$ and for any $\eta \in \Lambda^p(\Omega_{\mathbb{R}}[C^\infty(M)])$, we have*
$$i_{\theta^A}(\eta^A) = [i_\theta(\eta)]^A\,.$$

Proof. If $\eta \in \Lambda^p(\Omega_{\mathbb{R}}[C^\infty(M)])$, then there exists $f_1, f_2, \ldots, f_p \in C^\infty(M)$, such that $\eta = \delta_M(f_1) \wedge \cdots \wedge \delta_M(f_p)$. Thus,
$$\begin{aligned} i_{\theta^A}(\eta^A) &= i_{\theta^A}\big(\,[\delta_M(f_1) \wedge \cdots \wedge \delta_M(f_p)]^A\,\big) \\ &= i_{\theta^A}([\delta_M(f_1)]^A \wedge \cdots \wedge [\delta_M(f_p)]^A) \\ &= \sigma^A_{\theta^A}\big([\delta_M(f_1)]^A, \ldots, [\delta_M(f_p)]^A\big) \end{aligned}$$

$$
\begin{aligned}
&= \left[\sigma_\theta(\delta_M(f_1), \dots, \delta_M(f_p))\right]^A \\
&= \left[i_\theta(\delta_M(f_1) \wedge \dots \wedge \delta_M(f_p)\right]^A) \\
&= \left[i_\theta(\eta)\right]^A .
\end{aligned}
$$

For $p = 1$, we have

$$
\begin{aligned}
i_{\theta^A} = \widetilde{\sigma^A_{\theta^A}} &: \Lambda^1\big(\Omega_A[C^\infty(M^A, A)]\big) \\
&= \Omega_A[C^\infty(M^A, A)]^* \to \Lambda^0(\Omega_A[C^\infty(M^A, A)]) = C^\infty(M^A, A)\,,
\end{aligned}
$$

and for any $y \in \Omega_{\mathbb{R}}[C^\infty(M)]$,

$$i_{\theta^A}(y^A) = \widetilde{\theta^A}(y^A)\,.$$

For $p = 2$, we have

$$\sigma^A_{\theta^A} \colon \Omega_A\big[C^\infty(M^A, A)\big] \times \Omega_A\big[C^\infty(M^A, A)\big] \to C^\infty(M^A, A)$$

and for any x, $y \in \Omega_{\mathbb{R}}[C^\infty(M)]$,

$$\sigma^A_{\theta^A}(x^A, y^A) = \widetilde{\theta^A}(x^A) \cdot y^A - \widetilde{\theta^A}(y^A) \cdot x^A\,.$$

Thus, the map

$$i_{\theta^A} \colon \Lambda^2\big(\Omega_A[C^\infty(M^A, A)]\big) \to \Omega_A\big[C^\infty(M^A, A)\big]^*$$

is the unique $C^\infty(M^A, A)$-linear map such that

$$i_{\theta^A}(x^A \wedge y^A) = \sigma^A_{\theta^A}(x^A, y^A) = \widetilde{\theta^A}(x^A) \cdot y^A - \widetilde{\theta^A}(y^A) \cdot x^A\,.$$

□

Definition 2. The Lie derivative with respect to $D \in \mathrm{Der}_A[C^\infty(M^A, A)]$ is the derivation of degree 0

$$\mathfrak{L}_D = i_D \circ \delta^A_{M^A} + \delta^A_{M^A} \circ i_D \colon \Lambda\big(\Omega_A[C^\infty(M^A, A)]\big) \to \Lambda\big(\Omega_A[C^\infty(M^A, A)]\big)\,.$$

Proposition 7. *For any $\theta \in \mathfrak{X}(M)$, the map*

$$\mathfrak{L}_{\theta^A} \colon \Lambda\big(\Omega_A[C^\infty(M^A, A)]\big) \to \Lambda\big(\Omega_A[C^\infty(M^A, A)]\big)$$

is a unique A-linear derivation such that

$$\mathfrak{L}_{\theta^A}(\eta^A) = [\mathfrak{L}_\theta(\eta)]^A\,,$$

for any $\eta \in \Lambda(\Omega_{\mathbb{R}}[C^\infty(M)])$.

Proof. For any $\eta \in \Lambda(\Omega_{\mathbb{R}}[C^\infty(M)])$, we have

$$\begin{aligned}
\mathfrak{L}_{\theta^A}(\eta^A) &= i_{\theta^A}\left[\delta^A_{M^A}(\eta^A)\right] + \delta^A_{M^A}\left[i_{\theta^A}(\eta^A)\right] \\
&= i_{\theta^A}\left([\delta_M(\eta)]^A\right) + \delta^A_{M^A}\left([i_\theta(\eta)]^A\right) \\
&= \left(i_\theta[\delta_M(\eta)]\right)^A + \left(\delta_M[i_\theta(\eta)]\right)^A \\
&= \left(i_\theta[\delta_M(\eta)] + \delta_M[i_\theta(\eta)]\right)^A \\
&= \left[\mathfrak{L}_\theta(\eta)\right]^A .
\end{aligned}$$

□

Proposition 8. *For any $\theta \in \mathfrak{X}(M)$, for any $x \in \Omega_{\mathbb{R}}[C^\infty(M)]$ and for any $f \in C^\infty(M)$, we have*

$$\mathfrak{L}_{f^A.\theta^A}(x^A) = [\mathfrak{L}_{f\cdot\theta}(x)]^A. \tag{1}$$

$$\mathfrak{L}_{\theta^A}(f^A \cdot x^A) = [\mathfrak{L}_\theta(f \cdot x)]^A. \tag{2}$$

$$\mathfrak{L}_{\theta^A}\left[\delta^A_{M^A}(f^A)\right] = \left[\mathfrak{L}_\theta(\delta_M(f))\right]^A. \tag{3}$$

Proof. For any $\theta \in \mathfrak{X}(M)$, for any $x \in \Omega_{\mathbb{R}}[C^\infty(M)]$ and for any $f \in C^\infty(M)$, we have

1. $$\begin{aligned}
\mathfrak{L}_{f^A.\theta^A}(x^A) &= i_{f^A.\theta^A}\left[\delta^A_{M^A}(x^A)\right] + \delta^A_{M^A}\left[i_{f^A.\theta^A}(x^A)\right] \\
&= f^A \cdot i_{\theta^A}\left[\delta^A_{M^A}(x^A)\right] + \delta^A_{M^A}\left[f^A \cdot i_{\theta^A}(x^A)\right] \\
&= f^A \cdot i_{\theta^A}\left([\delta_M(x)]^A\right) + \delta^A_{M^A}(f^A \cdot [i_\theta(x)]^A) \\
&= f^A \cdot i_{\theta^A}([\delta_M(x)]^A) + i_{\theta^A}(x^A) \cdot \delta^A_{M^A}(f^A) + f^A \cdot \delta^A_{M^A}\left[i_{\theta^A}(x^A)\right] \\
&= f^A \cdot (i_\theta[\delta_M(x)])^A + [i_\theta(x)]^A \cdot [\delta_M(f)]^A + f^A \cdot (\delta_M[i_\theta(x)])^A \\
&= (f \cdot i_\theta[\delta_M(x)] + i_\theta(x) \cdot \delta_M(f) + f \cdot \delta_M[i_\theta(x)])^A \\
&= (f \cdot i_\theta[\delta_M(x)] + \delta_M(f \cdot [i_\theta(x)])^A \\
&= (i_{f\cdot\theta}[\delta_M(x)] + \delta_M[i_{f\cdot\theta}(x)])^A \\
&= [\mathfrak{L}_{f\cdot\theta}(x)]^A .
\end{aligned}$$

Thus, $$\mathfrak{L}_{f^A.\theta^A}(x^A) = [\mathfrak{L}_{f\cdot\theta}(x)]^A .$$

2. $$\begin{aligned}
\mathfrak{L}_{\theta^A}(f^A \cdot x^A) &= i_{\theta^A}\left[\delta^A_{M^A}(f^A \cdot x^A)\right] + \delta^A_{M^A}\left[i_{\theta^A}(f^A \cdot x^A)\right] \\
&= i_{\theta^A}\left[\delta^A_{M^A}(f^A)\Lambda x^A + f^A \cdot \delta^A_{M^A}(x^A)\right] + \delta^A_{M^A}\left[f^A \cdot i_{\theta^A}(x^A)\right] \\
&= \theta^A(f^A) \cdot x^A - \delta^A_{M^A}(f^A) \cdot \widetilde{\theta^A}(x^A) + f^A \cdot \theta^A(x^A) \\
&\quad + \delta^A_{M^A}(f^A) \cdot \widetilde{\theta^A}(x^A) + f^A \cdot \delta^A_{M^A}\left[i_{\theta^A}(x^A)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \big(\theta(f)\cdot x - \delta_M(f)\cdot\widetilde{\theta}(x) + f\cdot\theta(x)\\
&\quad + \delta_M(f)\cdot\widetilde{\theta}(x) + f\cdot\delta_M\big[i_\theta(x)\big]\big)^A\\
&= \big(i_\theta\big[\delta_M(f)\Lambda x + f\cdot\delta_M(x)\big] + \delta_M\big[f\cdot i_\theta(x)\big]\big)^A\\
&= \big(i_\theta\big[\delta_M(f\cdot x)\big] + \delta_M\big[i_\theta(f\cdot x)\big]\big)^A\\
&= \big[\mathfrak{L}_\theta(f\cdot x)\big]^A.
\end{aligned}
$$

3. $$
\begin{aligned}
\mathfrak{L}_{\theta^A}\left[\delta^A_{M^A}(f^A)\right] &= i_{\theta^A}\left[\delta^A_{M^A}(\delta^A_{M^A}(f^A))\right] + \delta^A_{M^A}\left[i_{\theta^A}(\delta^A_{M^A}(f^A))\right]\\
&= 0 + \delta^A_{M^A}\left[\theta^A(f^A)\right]\\
&= \delta^A_{M^A}([\theta(f)]^A)\\
&= \left(\delta_M\left[\theta(f)\right]\right)^A\\
&= (0 + \delta_M\left[\theta(f)\right])^A\\
&= \left(i_\theta\left[\delta_M(\delta_M(f))\right] + \delta_M\left[i_\theta(\delta_M(f))\right]\right)^A\\
&= \left[\mathfrak{L}_\theta(\delta_M(f))\right]^A.
\end{aligned}
$$

□

Proposition 9. *For any $D \in \mathrm{Der}_A[C^\infty(M^A, A)]$, $X \in \Omega_A[C^\infty(M^A, A)]$, and $\varphi \in C^\infty(M^A, A)$, we have*

(1) $\mathfrak{L}_{\varphi\cdot D}(X) = \varphi\cdot\mathfrak{L}_D(X) + \widetilde{D}(X)\cdot\delta^A_{M^A}(\varphi)$;

(2) $\mathfrak{L}_D(\varphi\cdot X) = D(\varphi)\cdot X + \varphi\cdot\mathfrak{L}_D(X)$;

(3) $\mathfrak{L}_D\left[\delta^A_{M^A}(\varphi)\right] = \delta^A_{M^A}\left[D(\varphi)\right]$.

Proof. For any $D \in \mathrm{Der}_A[C^\infty(M^A, A)]$, $X \in \Omega_A[C^\infty(M^A, A)]$, and $\varphi \in C^\infty(M^A, A)$, we have

1. $$
\begin{aligned}
\mathfrak{L}_{\varphi\cdot D}(X) &= i_{\varphi\cdot D}[\delta^A_{M^A}(X)] + \delta^A_{M^A}[i_{\varphi\cdot D}(X)]\\
&= \varphi\cdot i_D\big[\delta^A_{M^A}(X)\big] + \delta^A_{M^A}\big[\varphi\cdot i_D(X)\big]\\
&= \varphi\cdot i_D\big[\delta^A_{M^A}(X)\big] + i_D(X)\cdot\delta^A_{M^A}(\varphi) + \varphi\cdot\delta^A_{M^A}\big[i_D(X)\big]\\
&= \varphi\cdot\mathfrak{L}_D(X) + \widetilde{D}(X)\cdot\delta^A_{M^A}(\varphi).
\end{aligned}
$$

2.
$$\begin{aligned}\mathfrak{L}_D(\varphi\cdot X) &= i_D\left[\delta^A_{M^A}(\varphi\cdot X)\right]+\delta^A_{M^A}\left[i_D(\varphi\cdot X)\right]\\ &= i_D\left[\delta^A_{M^A}(\varphi)\Lambda X+\varphi\cdot\delta^A_{M^A}(X)\right]+\delta^A_{M^A}\left[\varphi\cdot i_D(X)\right]\\ &= \widetilde{D}[\delta^A_{M^A}(\varphi)]\cdot X-\delta^A_{M^A}(\varphi)\cdot\widetilde{D}(X)+\varphi\cdot i_D[\delta^A_{M^A}(X)]\\ &\quad+\delta^A_{M^A}(\varphi)\cdot\widetilde{D}(X)+\varphi\cdot\delta^A_{M^A}\left[i_D(X)\right]\\ &= D(\varphi)\cdot X+\varphi\cdot i_D[\delta^A_{M^A}(X)]+\varphi\cdot\delta^A_{M^A}\left[i_D(X)\right]\\ &= D(\varphi)\cdot X+\varphi\cdot(i_D[\delta^A_{M^A}(X)]+\delta^A_{M^A}\left[i_D(X)\right])\\ &= D(\varphi)\cdot X+\varphi\cdot\mathfrak{L}_D(X)\,.\end{aligned}$$

3.
$$\begin{aligned}\mathfrak{L}_D[\delta^A_{M^A}(\varphi)] &= i_D\left[\delta^A_{M^A}\big(\delta^A_{M^A}(\varphi)\big)\right]+\delta^A_{M^A}\left[i_D\big(\delta^A_{M^A}(\varphi)\big)\right]\\ &= 0+\delta^A_{M^A}\left[\widetilde{D}\circ\delta^A_{M^A}(\varphi)\right]\\ &= \delta^A_{M^A}\left[D(\varphi)\right].\end{aligned}$$

□

4. The Poisson 2-form on Weil bundles

We recall that, when M is a smooth manifold, A a Weil algebra and M^A the associated Weil bundle, the A-algebra $C^\infty(M^A,A)$ is a Poisson algebra over A if there exists a bracket $\{,\}$ on $C^\infty(M^A,A)$ such that the pair $(C^\infty(M^A,A),\{,\})$ is a Lie algebra over A satisfying

$$\{\varphi,\psi_1\cdot\psi_2\}=\{\varphi,\psi_1\}\cdot\psi_2+\psi_1\cdot\{\varphi,\psi_2\}$$

for any φ, ψ_1, $\psi_2\in C^\infty(M^A,A)$. When $C^\infty(M^A,A)$ is a Poisson A-algebra, we will say that the manifold M^A is a A-Poisson manifold [1],[7].

When $(M,\{,\})$ is a Poisson manifold, the map

$$\mathrm{ad}\colon C^\infty(M)\to \mathrm{Der}_{\mathbb{R}}[C^\infty(M)]\,,\quad f\to \mathrm{ad}(f)$$

such that $[\mathrm{ad}(f)]\,(g)=\{f,g\}$ for any $g\in C^\infty(M)$, is a derivation. Thus:

Proposition 10. *There exists a derivation*

$$\mathrm{ad}^A\colon C^\infty(M^A,A)\to \mathrm{Der}_A[C^\infty(M^A,A)]$$

such that

$$\mathrm{ad}^A(f^A)=[\mathrm{ad}(f)]^A\,.$$

Let's consider the following diagram commutative:

$$\begin{tikzcd} C^\infty(M^A,A) \arrow[r, "\widetilde{\tau}"] & \mathrm{Der}_A[C^\infty(M^A,A)] \\ C^\infty(M) \arrow[u, "\gamma_M"] \arrow[r, "\mathrm{ad}"] & \mathrm{Der}_{\mathbb{R}}[C^\infty(M)] \arrow[u, "\Phi"'] \end{tikzcd}$$

i.e.

$$\widetilde{\tau}\circ\gamma_M=\Phi\circ\mathrm{ad}\,,$$

where

$$\gamma_M\colon C^\infty(M)\to C^\infty(M^A,A),\ \ f\longmapsto f^A$$

and

$$\Phi\colon\ \mathrm{Der}_{\mathbb{R}}[C^\infty(M)]\to \mathrm{Der}_A[C^\infty(M^A,A)],\ \ \theta\to\theta^A\,.$$

For any $f\in C^\infty(M)$, we have

$$\widetilde{\tau}\circ\gamma_M(f)=\widetilde{\tau}(f^A)$$

and

$$\Phi\circ\mathrm{ad}(f)=\Phi[\mathrm{ad}(f)]=[\mathrm{ad}(f)]^A\,.$$

Thus, there exists $\mathrm{ad}^A=\widetilde{\tau}$ such that

$$\mathrm{ad}^A(f^A)=[\mathrm{ad}(f)]^A\,.$$

As

$$\mathrm{ad}^A:C^\infty(M^A,A)\to \mathrm{Der}_A[C^\infty(M^A,A)]$$

is a derivation, then there exists a unique $C^\infty(M^A,A)$-linear map

$$\widetilde{\mathrm{ad}^A}\colon\Omega_A[C^\infty(M^A,A)]\to \mathrm{Der}_A[C^\infty(M^A,A)]$$

such that

$$\widetilde{\mathrm{ad}^A}\circ\delta^A_{M^A}=\mathrm{ad}^A\ .$$

Let's consider the canonical isomorphism

$$\sigma_{M^A}\colon\Omega_A[C^\infty(M^A,A)]^*\longrightarrow \mathrm{Der}_A[C^\infty(M^A,A)],\Psi\longmapsto\Psi\circ\delta^A_{M^A}$$

and let

$$\sigma^{-1}_{M^A}\circ\widetilde{\mathrm{ad}^A}\colon\Omega_A[C^\infty(M^A,A)]\xrightarrow{\widetilde{\mathrm{ad}^A}}\mathrm{Der}_A[C^\infty(M^A,A)]\xrightarrow{\sigma^{-1}_{M^A}}\Omega_A[C^\infty(M^A,A)]^*$$

be the map.

Proposition 11. *If (M,ω_M) is a Poisson manifold, then the map,*

$$\omega^A_{M^A}\colon\Omega_A[C^\infty(M^A,A)]\times\Omega_A\big[C^\infty(M^A,A)\big]\to C^\infty(M^A,A)$$

such that for any X, $Y\in\Omega_A[C^\infty(M^A,A)]$

$$\omega^A_{M^A}(X,Y)=-\big[\sigma^{-1}_{M^A}\circ\widetilde{\mathrm{ad}^A}(X)\big](Y)$$

is a skew-symmetric 2-form on $\Omega_A[C^\infty(M^A,A)]$ such that

$$\omega^A_{M^A}(x^A,y^A)=\big[\omega_M(x,y)\big]^A\,,$$

for any x and y in $\Omega_{\mathbb{R}}[C^\infty(M)]$.

Proof. For any $X \in \Omega_A[C^\infty(M^A, A)]$, we have $X = \sum\limits_{i \in I:\ \text{fini}} \varphi_i \cdot \delta^A_{M^A}(\psi_i)$, with $\varphi_i \in C^\infty(M^A, A)$, $\psi_i \in C^\infty(M^A, A)$.

$$\begin{aligned}
\omega_{M^A}(X, X) &= -\big[\sigma^{-1}_{M^A} \circ \widetilde{\mathrm{ad}^A}(X)\big](X) \\
&= -\sum_{j \in I:\ \text{finite}} \varphi_j \cdot \big[\sigma^{-1}_{M^A} \circ \widetilde{\mathrm{ad}^A}(X)\big]\delta^A_{M^A}(\psi_j) \\
&= -\sum_{j \in I:\ \text{finite}} \varphi_j \cdot \big[\widetilde{\mathrm{ad}^A}(X)\big](\psi_j) \\
&= -\sum_{j,k \in I:\ \text{finite}} \varphi_j \cdot \varphi_k \cdot \big[\widetilde{\mathrm{ad}^A}(\delta^A_{M^A}(\psi_k))\big](\psi_j) \\
&= -\sum_{j,k \in I:\ \text{finite}} \varphi_j \cdot \varphi_k \cdot \big[\,\mathrm{ad}^A(\psi_k)\big](\psi_j) \\
&= -\sum_{j,k \in I:\ \text{finite}} \varphi_j \cdot \varphi_k \cdot \{\psi_k, \psi_j\} \\
&= 0\,.
\end{aligned}$$

For any X_1, X_2 and $Y \in \Omega_A[C^\infty(M^A, A)]$ and for any $\varphi \in C^\infty(M^A, A)$, we have

$$\begin{aligned}
\omega_{M^A}[(\varphi \cdot X_1 + X_2), Y] &= -\big[\sigma^{-1}_{M^A} \circ \widetilde{\mathrm{ad}^A}(\varphi \cdot X_1 + X_2)\big](Y) \\
&= -\big(\sigma^{-1}_{M^A}[\varphi \cdot \widetilde{\mathrm{ad}^A}(X_1) + \widetilde{\mathrm{ad}^A}(X_2)]\big)(Y) \\
&= -\varphi \cdot (\sigma^{-1}_{M^A}[\widetilde{\mathrm{ad}^A}(X_1)](Y) + \big(\sigma^{-1}_{M^A}[\widetilde{\mathrm{ad}^A}(X_2)]\big)(Y) \\
&= \varphi \cdot \omega_{M^A}(X_1, Y) + \omega_{M^A}(X_2, Y)\,.
\end{aligned}$$

For any x and y in $\Omega_{\mathbb{R}}[C^\infty(M)]$,

$$x^A = \sum_{i \in I:\ \text{fini}} f^A_i \cdot \delta^A_{M^A}(f'^A_i) \quad \text{and} \quad y^A = \sum_{j \in I:\ \text{fini}} g^A_j \cdot \delta^A_{M^A}(g'^A_j)\,.$$

Thus,

$$\begin{aligned}
\omega^A_{M^A}(x^A, y^A) &= -[\sigma^{-1}_{M^A} \circ \widetilde{\mathrm{ad}^A}(x^A)](y^A) \\
&= -\Big[\sigma^{-1}_{M^A} \circ \widetilde{\mathrm{ad}^A}\Big(\sum_{i \in I:\ \text{finite}} f^A_i \cdot \delta^A_{M^A}(f'^A_i)\Big)\Big]\Big(\sum_{j \in I:\ \text{finite}} g^A_j \cdot \delta^A_{M^A}(g'^A_j)\Big) \\
&= -\sum_{i,j \in I:\ \text{finite}} f^A_i \cdot g^A_j \cdot \big[\sigma^{-1}_{M^A} \circ \widetilde{\mathrm{ad}^A}(\delta^A_{M^A}(f'^A_i))\big]\big(\delta^A_{M^A}(g'^A_j)\big)
\end{aligned}$$

$$
\begin{aligned}
&= - \sum_{i,j\in I:\ \text{finite}} f_i^A \cdot g_j^A \cdot \left[\sigma_{M^A}^{-1} \circ \widetilde{\mathrm{ad}^A}(\delta_{M^A}^A(f_i'^A))\right]\left(\delta_{M^A}^A(g_j'^A)\right) \\
&= - \sum_{i,j\in I:\ \text{finite}} f_i^A \cdot g_j^A \cdot \left[\sigma_{M^A}^{-1}(\widetilde{\mathrm{ad}^A} \circ \delta_{M^A}^A(f_i'^A))\right]\left(\delta_{M^A}^A(g_j'^A)\right) \\
&= - \sum_{i,j\in I:\ \text{finite}} f_i^A \cdot g_j^A \cdot \left[\sigma_{M^A}^{-1}(\mathrm{ad}^A(f_i'^A))\right]\left(\delta_{M^A}^A(g_j'^A)\right) \\
&= - \sum_{i,j\in I:\ \text{finite}} f_i^A \cdot g_j^A \cdot \left[\sigma_{M^A}^{-1}(\mathrm{ad}(f_i')^A)\right]\left(\delta_M(g_j')\right)^A \\
&= - \Big[\sum_{i,j\in I:\ \text{finite}} f_i \cdot g_j \cdot \left[\sigma_M^{-1}(\mathrm{ad}(f_i'))\right]\left(\delta_M(g_j')\right) \Big]^A \\
&= - \Big[\sum_{i,j\in I:\ \text{finite}} f_i \cdot g_j \cdot \left[\sigma_M^{-1}(\widetilde{\mathrm{ad}} \circ \delta_M(f_i'))\right]\left(\delta_M(g_j')\right) \Big]^A \\
&= - \Big[\sigma_M^{-1} \circ \widetilde{\mathrm{ad}}\Big(\sum_{i\in I:\ \text{finite}} f_i \cdot \delta_M(f_i')\Big)\Big(\sum_{j\in I:\ \text{finite}} g_j \cdot \delta_M(g_j')\Big)\Big]^A \\
&= [\omega_M(x,y)]^A .
\end{aligned}
$$

□

Proposition 12. *When (M, ω_M) is a Poisson manifold of Poisson 2-form ω_M, then $(M^A, \omega_{M^A}^A)$ is a Poisson manifold.*

Proof. For any f and g in $C^\infty(M)$,

$$
\begin{aligned}
\omega_{M^A}^A\left(\delta_{M^A}^A(f^A), \delta_{M^A}^A(g^A)\right) &= \omega_{M^A}^A\left([\delta_M(f)]^A, [\delta_M(g)]^A\right) \\
&= \left[\omega_M(\delta_{M^A}(f), \delta_M(g))\right]^A \\
&= -\{f,g\}_M^A ,
\end{aligned}
$$

and

$$
\omega_{M^A}^A(x^A, y^A) = \left[\omega_M(x,y)\right]^A ,
$$

for any x, $y \in \Omega_{\mathbb{R}}[C^\infty(M)]$. We deduce that $(M^A, \omega_{M^A}^A)$ is a Poisson manifold. □

Theorem 3. *The manifold M^A is a Poisson manifold if and only if there exists a skew-symmetric 2-form*

$$
\omega_{M^A}^A : \Omega_A\left[C^\infty(M^A, A)\right] \times \Omega_A\left[C^\infty(M^A, A)\right] \to C^\infty(M^A, A)
$$

such that for any φ and ψ in $C^\infty(M^A, A)$,

$$
\{\varphi, \psi\}_{M^A} = -\omega_{M^A}^A\left(\delta_{M^A}^A(\varphi), \delta_{M^A}^A(\psi)\right)
$$

defines a structure of A-Lie algebra over $C^\infty(M^A, A)$. Moreover, for any f and g in $C^\infty(M)$,

$$\{f^A, g^A\}_{M^A} = \{f, g\}_M^A .$$

Proof. Indeed, according to the previous proposition, the bracket

$$\{\varphi, \psi\}_{M^A} = -\omega_{M^A}^A\big(\delta_{M^A}^A(\varphi), \delta_{M^A}^A(\psi)\big)$$

defines a structure of A-Lie algebra over $C^\infty(M^A, A)$. For any f and g in $C^\infty(M)$,

$$\{f^A, g^A\}_{M^A} = -\omega_{M^A}^A\big(\delta_{M^A}^A(f^A), \delta_{M^A}^A(g^A)\big) = \{f, g\}_M^A .$$

In this case, we will say that $\omega_{M^A}^A$ is the Poisson 2-form of the A-Poisson manifold M^A and we denote $(M^A, \omega_{M^A}^A)$ the A-Poisson manifold of Poisson 2-form $\omega_{M^A}^A$. $\square$

Proposition 13. *When (M, ω_M) is a Poisson manifold of Poisson 2-form ω_M, then for any x, $y \in \Omega_{\mathbb{R}}[C^\infty(M)]$ and for any f, $g \in C^\infty(M)$, we get*

(1) $$[\widetilde{\mathrm{ad}^A}(x^A)](f^A) = ([\widetilde{\mathrm{ad}}(x)](f))^A.$$

(2) $$[\widetilde{\widetilde{\mathrm{ad}^A}(x^A)}](y^A) = ([\widetilde{\widetilde{\mathrm{ad}}(x)}](y))^A.$$

(3) $$\mathfrak{L}_{\widetilde{\mathrm{ad}^A}[\delta_{M^A}^A(f^A)]}(g^A) = (\mathfrak{L}_{\widetilde{\mathrm{ad}}[\delta_M(f)]}(g))^A.$$

Proof. 1. For any $x \in \Omega_{\mathbb{R}}[C^\infty(M)], x^A = g^A \cdot \delta_{M^A}^A(h^A)$ with g and h in $C^\infty(M)$, and for any f, $g \in C^\infty(M)$, we have

$$\begin{aligned}
[\widetilde{\mathrm{ad}^A}(x^A)](f^A) &= [\widetilde{\mathrm{ad}^A}(g^A \cdot \delta_{M^A}^A(h^A))](f^A) \\
&= [g^A \cdot \widetilde{\mathrm{ad}^A} \circ \delta_{M^A}^A(h^A))](f^A) \\
&= [g^A \cdot \mathrm{ad}^A(h^A)](f^A) \\
&= g^A \cdot [\mathrm{ad}(h)]^A(f^A) \\
&= (g \cdot [\mathrm{ad}(h)](f))^A \\
&= (g \cdot \widetilde{\mathrm{ad}}[\delta_M(h)](f))^A \\
&= (\widetilde{\mathrm{ad}}[g \cdot \delta_M(h)](f))^A \\
&= (\widetilde{\mathrm{ad}}[g \cdot \delta_M(h)](f))^A \\
&= ([\widetilde{\mathrm{ad}}(x)](f))^A .
\end{aligned}$$

2. When $y \in \Omega_{\mathbb{R}}[C^\infty(M)], y^A = g^A \cdot \delta^A_{M^A}(h^A)$ with g and h in $C^\infty(M)$

$$
\begin{aligned}
\widetilde{[\mathrm{ad}^A(x^A)]}(y^A) &= \widetilde{[\mathrm{ad}^A(x^A)]}(g^A \cdot \delta^A_{M^A}(h^A)) \\
&= g^A \cdot (\widetilde{[\mathrm{ad}^A(x^A)]} \circ \delta^A_{M^A})(h^A) \\
&= g^A \cdot [\widetilde{\mathrm{ad}^A}(x^A)](h^A) \\
&= g^A \cdot [\widetilde{\mathrm{ad}^A}(x^A)](h^A) \\
&= (g \cdot [\widetilde{\mathrm{ad}}(x)](h))^A \\
&= (g \cdot \widetilde{[\mathrm{ad}(x)]} \circ \delta_M)(h))^A \\
&= (\widetilde{[\mathrm{ad}(x)]}(g \cdot \delta_M(h)))^A \\
&= (\widetilde{[\mathrm{ad}(x)]}(y))^A \,.
\end{aligned}
$$

$\square$

Proposition 14. *When (M, ω_M) is a Poisson manifold of Poisson 2-form ω_M, then for any X, $Y \in \Omega_A[C^\infty(M^A, A)]$ and for any φ, $\psi \in C^\infty(M^A, A)$, we get*

(1) $\quad [\widetilde{ad^A}(X)](\varphi) = -\omega^A_{M^A}(X, \delta^A_{M^A}(\varphi))$;

(2) $\quad \widetilde{[ad^A(X)]}(Y) = -\omega^A_{M^A}(X, Y)$;

(3) $\quad \mathfrak{L}_{\widetilde{ad^A}[\delta^A_{M^A}(\varphi)]}\delta^A_{M^A}(\psi) = \delta^A_{M^A}(\{\varphi, \psi\}_{M^A})$.

Proof. When X and $Y \in \Omega_A[C^\infty(M^A, A)]$, $X = \sum_{i \in I:\ \text{fini}} \varphi_i \cdot \delta^A_{M^A}(\varphi'_i)$, $Y = \sum_{j \in J:\ \text{fini}} \psi_j \cdot \delta^A_{M^A}(\psi'_i)$ with $\varphi_i, \varphi'_i, \psi_j, \psi'_j \in C^\infty(M^A, A)$

1.
$$
\begin{aligned}
[\widetilde{\mathrm{ad}^A}(X)](\varphi) &= \Big[\widetilde{\mathrm{ad}^A}\Big(\sum_{i \in I:\ \text{fini}} \varphi_i \cdot \delta^A_{M^A}(\varphi'_i)\Big)\Big](\varphi) \\
&= \sum_{i \in I:\ \text{fini}} \varphi_i \cdot (\widetilde{\mathrm{ad}^A}[\delta^A_{M^A}(\varphi'_i)])(\varphi) \\
&= \sum_{i \in I:\ \text{fini}} \varphi_i \cdot [ad^A(\varphi'_i)](\varphi) \\
&= \sum_{i \in I:\ \text{fini}} \varphi_i \cdot \{\varphi'_i, \varphi\}_{M^A} \\
&= -\sum_{i \in I:\ \text{fini}} \varphi_i \cdot \omega^A_{M^A}(\delta^A_{M^A}(\varphi'_i), \delta^A_{M^A}(\varphi)) \\
&= -\omega^A_{M^A}\Big(\sum_{i \in I:\ \text{fini}} \varphi_i \cdot \delta^A_{M^A}(\varphi'_i), \delta^A_{M^A}(\varphi)\Big) \\
&= -\omega^A_{M^A}(X, \delta^A_{M^A}(\varphi)) \,.
\end{aligned}
$$

2. $$\begin{aligned}\widetilde{[\widetilde{\mathrm{ad}^A}(X)]}(Y) &= \widetilde{[\widetilde{\mathrm{ad}^A}(X)]}\Big(\sum_{j\in J:\ \text{fini}} \psi_j \cdot \delta^A_{M^A}(\psi'_j)\Big)\\ &= \sum_{j\in J:\ \text{fini}} \psi_j \cdot \widetilde{[\widetilde{\mathrm{ad}^A}(X)]}(\delta^A_{M^A}(\psi'_j))\\ &= \sum_{j\in J:\ \text{fini}} \psi_j \cdot (\widetilde{[\widetilde{\mathrm{ad}^A}(X)]} \circ \delta^A_{M^A})(\psi'_j))\\ &= \sum_{j\in J:\ \text{fini}} \psi_j \cdot [\widetilde{\mathrm{ad}^A}(X)](\psi'_j)\\ &= -\sum_{j\in J:\ \text{fini}} \psi_j \cdot \omega^A_{M^A}(X, \delta^A_{M^A}(\psi'_j))\\ &= -\omega^A_{M^A}\Big(X, \sum_{j\in J:\ \text{fini}} \psi_j \cdot \delta^A_{M^A}(\psi'_j)\Big)\\ &= -\omega^A_{M^A}(X,Y)\,;\end{aligned}$$

3. $$\begin{aligned}\mathfrak{L}_{\widetilde{\mathrm{ad}^A}[\delta^A_{M^A}(\varphi)]}\delta^A_{M^A}(\psi) &= \mathfrak{L}_{\mathrm{ad}^A(\varphi)}\delta^A_{M^A}(\psi)\\ &= \delta^A_{M^A}[\mathrm{ad}^A(\varphi)(\psi)]\\ &= \delta^A_{M^A}(\{\varphi,\psi\}_{M^A})\,.\end{aligned}$$

□

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