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ASYMPTOTIC INTEGRATION OF DIFFERENTIAL EQUATIONS WITH SINGULAR p-LAPLACIAN

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Dedicated to professor Miroslav Bartušek on the occasion of his 70th birthday

Abstract. In this paper we deal with the problem of asymptotic integration of nonlinear differential equations with $p-$Laplacian, where $1 < p < 2$. We prove sufficient conditions under which all solutions of an equation from this class are converging to a linear function as $t \to \infty$.

1. Introduction

In the asymptotic theory of n-th order nonlinear ordinary differential equations

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}) \tag{1}$$

the classical problem is to establish conditions for the existence of a solution which asymptotically behaves as a polynomial of degree $1 \le m \le n-1$ as $t \to \infty$. The first paper concerning this problem was published by D. Caligo [5] in 1941 (see also [1]). He proved a result for that type of a linear second order differential equation. Since then many results concerning this problem for nonlinear differential equations have been proved, e.g. in the papers by D.S. Cohen [6], A. Constantin [7], [9] and [8], F.M. Dannan [10], T. Kusano and W.F. Trench [11] and [12], O. Lipovan [13], O.G. Mustafa, Y.V. Rogovchenko [17], Ch.G. Philos, I.K. Purnaras and P.Ch. Tsamatos [20], Y.V. Rogovchenko [22], S.P. Rogovchenko [21], J. Tong [23], F. Trench [24]. The paper by R.P. Agarwal, S.D. Djebali, T. Moussaoui and O.G. Mustafa [1] surveys the literature concerning the topic in asymptotic integration theory of ordinary differential equations. Several conditions under which all solutions of the one dimensional p-Laplacian equation

$$(|y'|^{p-1}y')' = f(t, y, y'), \quad p > 1 \tag{2}$$

behave asymptotically as $a + bt$ as $t \to \infty$ for some real numbers a, b are proved in [16] and some sufficient conditions for the existence of such solutions of the

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equation

$$\left(\Phi(y^{(n)})\right)' = f(t,y)\,, \quad n \geq 1\,, \tag{3}$$

where $\Phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with a locally Lipschitz inverse satisfying $\Phi(0) = 0$ are given in the paper [14]. We remark that in the papers [2], [3], [15] and [19] problems of the global existence, extendability and non-extendability of solutions of systems of equations with p-Laplacian are studied.

In this paper we prove sufficient conditions under which all solutions of a p-Laplace equation behave asymptotically as a linear function for $t \to \infty$. In its proof we apply the Bihari inequality. This technique was applied also in the paper [16] concerning a p-Laplace equation. In some of the above mentioned papers, also in the paper [14] concerning a p-Laplace equation, some results on the existence of solutions behaving like linear functions near the infinity are proved by using the Schauder fixed point theorem.

2. Asymptotic properties of one-dimensional singular p-Laplace equations

Consider the initial problem

$$(Q(t)\Phi_p(u'))' + f(t,u,u') = 0\,, \tag{4}$$

$$u(t_0) = u_0, u'(t_0) = u_1\,, \quad t_0 \geq 1\,, \tag{5}$$

where $\Phi_p(v) = |v|^{p-2}v$, $Q(t)$ is a continuous positive function. If $p > 1$ and $q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then $\Phi_q(v) = \Phi_p^{-1}(v)$. We need to assume $q > 2$. However in this case $1 < p < 2$ and this means that the p-Laplacian $\Phi_p(v)$ is singular.

Theorem 1. *Let the following conditions be satisfied:*

(C1) $1 < p < 2$*;*

(C2) *There exists a continuous nonnegative function* $h\colon \mathbb{R}_+ = [0,\infty) \to \mathbb{R}$*, continuous positive nondecreasing functions* $g_i\colon \mathbb{R}_+ \to \mathbb{R}$*,* $i = 1,2$ *and a positive number* k *such that*

$$|f(t,u,v)| \leq H(t)\Big[g_1\Big(\Big[\frac{|u|}{t}\Big]^k\Big) + g_2(|v|^k)\Big]$$

for all $(t,u,v) \in (0,\infty) \times \mathbb{R} \times \mathbb{R}$*;*

(C3)

$$\int_0^\infty H(s)^{\frac{1}{p-1}}\,ds < \infty\,;$$

(C4)

$$\int_{v_0}^\infty \frac{d\sigma}{g_1(\sigma^k)^{\frac{1}{p-1}} + g_2(\sigma^k)^{\frac{1}{p-1}}} = \frac{1}{k}\int_{v_0^k}^\infty \frac{\tau^{\frac{1}{k}-1}d\tau}{g_1(\tau)^{\frac{1}{p-1}} + g_2(\tau)^{\frac{1}{p-1}}} = \infty\,, \quad v_0 \geq 0\,;$$

(C5) *There exists a constant* $K > 0$ *such that*

$$Q(t) \geq Kt\,, \quad t \geq t_0 \geq 1\,.$$

Then for any solution $u(t)$ of the initial value problem (4), (5) *there exist* $a, b \in \mathbb{R}$ *such that*

$$\lim_{t\to\infty} |u(t) - (a+bt)| = 0\,.$$

Proof. First let us write the equation (4) in the form

$$\big(\Phi_p(h(t)u')\big)' + f(t,u,u') = 0\,, \tag{6}$$

where $h(t) = Q(t)^r = Q(t)^{q-1} = Q(t)^{\frac{1}{p-1}}$ ($r = q-1 = \frac{1}{p-1}$). From condition (C5) it follows that

$$h(t) \geq K^r t^r\,, \quad t \geq t_0 \geq 1\,. \tag{7}$$

If $u(t)$ is a solution of equation (4) satisfying the initial value condition (5), then

$$u'(t) = \frac{1}{h(t)}\Big\{\Phi_q\Big(\Phi_p(h(t_0)u_1) - \int_{t_0}^t f(s,u(s),u'(s))ds\Big)\Big\}\,, \tag{8}$$

$$u(t) = u_0 + \int_{t_0}^t \frac{1}{h(\tau)}\Big\{\Phi_q\Big(\Phi_p(h(t_0)u_1) - \int_{t_0}^\tau f(s,u(s),u'(s))ds\Big)\Big\}d\tau\,. \tag{9}$$

Using condition (C5) we obtain

$$\frac{1}{h(t)} = \frac{1}{Q(t)^r} \leq L\frac{1}{t^r}\,, \quad L = \frac{1}{K^r}$$

and

$$|u(t)| \leq |u_0|t + L\int_{t_0}^t \frac{1}{\tau^r}\Big(|\Phi_p(h(t_0)u_1)| + \int_{t_0}^\tau |f(s,u(s),u'(s))|\,ds\Big)^r\,d\tau\,.$$

Using the Hölder inequality (with r and $\frac{r}{r-1}$) and the inequality $(a_1+a_2+\cdots+a_m)^n \leq m^{n-1}(a_1^n+a_2^n+\cdots+a_m^n)$, $a_1,a_2,\ldots,a_m \geq 0$, $n \in \mathbb{N}$, and condition (C2) we obtain for $t \geq t_0 \geq 1$:

$$\begin{aligned}
|u(t)| &\leq |u_0|t + L\int_{t_0}^t \frac{1}{\tau^r}\Big(2^{r-1}|\Phi_p(h(t_0)u_1)|^r + 2^{r-1}\tau^{r-1}\int_0^\tau |f(s,u(s),u'(s))|^r\,ds\Big)\,d\tau\\
&\leq |u(t_0)|t + Lt2^{r-1}|\Phi_p(h(t_0)u_1)|^r + L2^{r-1}\int_0^t\int_{t_0}^s |f(\tau,u(\tau),u'(\tau))|^r d\tau\,ds\\
&\leq |u(t_0)|t + Lt2^{r-1}|\Phi_p(h(t_0)u_1)|^r\\
&\quad + L2^{r-1}t\int_{t_0}^t H(s)^r\Big(g_1\Big(\Big[\frac{|u(s)|}{s}\Big]^k\Big) + g_2(|u'(s)|^k)\Big)^r\,ds\\
&\leq |u(t_0)|t + Lt2^{r-1}|\Phi_p(h(t_0)u_1)|^r\\
&\quad + L4^{r-1}t\int_{t_0}^t H(s)^r\Big(g_1\Big(\Big[\frac{|u(s)|}{s}\Big]^k\Big)^r + g_2(|u'(s)|^k)^r\Big)\,ds\,.
\end{aligned}$$

This yields

$$\frac{|u(t)|}{t} \leq A_1 + B\int_{t_0}^t H(s)^r\Big(g_1\Big(\Big[\frac{|u(s)|}{s}\Big]^k\Big)^r + g_2(|u'(s)|^k)^r\Big)\,ds\,,$$

where $A_1 = |u(t_0)| + L2^{r-1}|\Phi_p(h(t_0)u_1)|^r$, $B = 4^{r-1}L$. One can show that

$$\frac{|u(t)|}{t} \le z(t)\,, \quad |u'(t)| \le z(t)\,, \tag{10}$$

where

$$z(t) = A + B\int_{t_0}^{t} H(s)^r \Big(g_1\Big(\Big[\frac{|u(s)|}{s}\Big]^k\Big)^r + g_2(|u'(s)|^k)^r\Big)\,ds\,,$$

$A = A_1 + |u_1|$. Since the functions g_1, g_2 are nondecreasing, the inequalities (10) yield

$$z(t) \le A + B\int_{t_0}^{t} H(s)^r \big(g_1(z(s)^k)^r + g_2(z(s)^k)^r\big)\,ds$$

and from the Bihari inequality it follows

$$\Omega(z(t)) \le K_1 := \Omega(A) + B\int_{t_0}^{\infty} H(s)^r\,ds < \infty\,,$$

where

$$\Omega(v) = \int_{v_0}^{v} \frac{d\sigma}{g_1(\sigma^k)^r + g_2(\sigma^k)^r}\,, \quad r = q-1\,.$$

From inequalities (10) we have

$$\frac{|u(t)|}{t} \le K := \Omega^{-1}(K_1) < \infty\,, \quad |u'(t)| \le K, \quad t \ge t_0\,. \tag{11}$$

Since

$$\begin{aligned}\int_{t_0}^{t} |f(s,u(s),u'(s))|\,ds &\le \int_{t_0}^{t} H(s)\Big(g_1\Big(\Big[\frac{|u(s)|}{s}\Big]^k\Big) + g_2(|u'(s)|^k)\Big)\,ds\\ &\le z(t) \le K\,, \quad t \ge t_0\,,\end{aligned}$$

the integral $\int_{t_0}^{\infty} |f(s,u(s),u'(s))|\,ds$ exists.

From (11) it follows that there exists $a \in \mathbb{R}$ such that

$$\lim_{t\to\infty} u'(t) = a$$

and by using the L'Hospital rule we obtain

$$\lim_{t\to\infty} \frac{|u(t)|}{t} = \lim_{t\to\infty} u'(t) = a\,.$$

Therefore there exist a, $b \in \mathbb{R}$ such that $u(t) = at + b + o(t)$ as $t \to \infty$. □

Example. Let $t_0 = 1$, $1 < p < 2$, $0 < k \le 1$, $H(t)$ be a nonnegative, continuous function on $[0,\infty)$ with $\int_1^{\infty} H(s)^{\frac{1}{p-1}}\,ds < \infty$ and

$$f(t,u,v) = H(t)\Big(u^{\frac{(p-1)(1-k)}{k}} \ln^{p-1} u + v^{\frac{(p-1)(1-k)}{k}}\Big)\,, \quad u,v > 0,\ t \in [0,\infty)\,.$$

If $g_1(u) := u^{\frac{(p-1)(1-k)}{k}} \ln^{p-1} u$, $g_2(v) := v^{\frac{(p-1)(1-k)}{k}}$, $Q(t) := t$, $t \ge 1$, then

$$\int_{v_0^k}^{\infty} \frac{\tau^{\frac{1}{k}-1}d\tau}{g_1(\tau)^{p-1} + g_2(\tau)^{p-1}} = \int_{v_0^k}^{\infty} \frac{d\tau}{\ln\tau + \tau} = \infty$$

(see [7]) and thus all conditions of Theorem 1 are satisfied.

Remark 1. Let us define the following classes of functions defined on the region $D \subset (0,\infty)\times\mathbb{R}\times\mathbb{R}$:

$$\mathcal{C}_i = \{f(t,u,v) : f \in C(D) \text{ and satisfies the condition } (Ki)\}\,, \quad i = 0,1,2\,,$$

where (K0) is given by the conditions (C2), (C3), (C4) from Theorem 1,

(K1)
$$|f(t,u,v)| \le h_1(t)\Big[g_1\Big(\Big[\frac{|u|}{t}\Big]^k\Big) + h_2(t)g_2(|v|^k) + h_3(t)\Big]$$
for all $(t,u,v)\in(0,\infty)\times\mathbb{R}\times\mathbb{R}$ with
$$\int_0^\infty h_j(s)^{\frac{1}{p-1}}ds < \infty\,, \quad j = 1,2,3$$
and
$$\int_{v_0}^\infty \frac{d\sigma}{g_1(\sigma^k)^{\frac{1}{p-1}} + g_2(\sigma^k)^{\frac{1}{p-1}}} = \frac{1}{k}\int_{v_0^k}^\infty \frac{\tau^{\frac{1}{k}-1}d\tau}{g_1(\tau)^{\frac{1}{p-1}} + g_2(\tau)^{\frac{1}{p-1}}} = \infty\,, \quad v_0 \ge 0\,;$$

(K2)
$$|f(t,u,v)| \le h_4(t)\Big[g_1\Big(\Big[\frac{|u|}{t}\Big]^k\Big)g_2(|v|^k) + h_5(t)\Big]$$
for all $(t,u,v)\in(0,\infty)\times\mathbb{R}\times\mathbb{R}$ with
$$\int_0^\infty h_j(s)^{\frac{1}{p-1}}ds < \infty\,, \quad j = 4,5$$
and
$$\int_{v_0}^\infty \frac{d\sigma}{g_1(\sigma^k)^{\frac{1}{p-1}}g_2(\sigma^k)^{\frac{1}{p-1}}} = \frac{1}{k}\int_{v_0^k}^\infty \frac{\tau^{\frac{1}{k}-1}d\tau}{g_1(\tau)^{\frac{1}{p-1}}g_2(\tau)^{\frac{1}{p-1}}} = \infty\,, \quad v_0 \ge 0\,.$$

Proposition 2. *It holds*
$$\mathcal{C}_1 \subset \mathcal{C}_0\,, \quad \mathcal{C}_2 \subset \mathcal{C}_0\,.$$

This proposition is a corollary of Proposition 2 from [18]. If we substitute conditions (K1) or (K2) instead of conditions (C1), (C2), (C3) in Theorem 1 we obtain results which are corollaries of Theorem 1. This type of results with these classes of nonlinearities are proved in [22], [21] and also in [16], separately.

Remark 2. Since we study equation (6) with $1 < p < 2$ we need condition (C5). This condition is not necessary in the case studied in [16].

Theorem 3. *Let conditions* (C1)–(C5) *of Theorem 1 be satisfied. Then any solution* $u\colon [0,T)\to\mathbb{R}$ *with* $0 < T < \infty$ *can be extended to the right beyond* T.

Proof. Let $u\colon [0,T)\to\mathbb{R}$ be a solution of equation (4) with $0 < T < \infty$ satisfying the initial value condition (5), which cannot be extended to the right beyond T. Then $\lim_{t\to T^-}|u(t)| = \infty$. However from inequality (10) we have

$$|u(t)| \le t|z(t)|\,, \quad t \ge 1\,, \tag{12}$$

where

$$z(t) \le A + B \int_{t_0}^{t} H(s)^r \left(g_1(z(s)^k)^r + g_2(z(s)^k)^r \right) ds\,, \tag{13}$$

and by applying the Bihari inequality we obtain that $|z(t)| \le K$ for all $t \in [1, \infty)$, where $K > 0$ is a constant. However from the inequality (12) we have $|u(t)| \le TK$ for all $t \in [1, \infty)$ and it is a contradiction. □

Theorem 4. *Let conditions* (C1)–(C4) *of Theorem 1 be satisfied and suppose that there exists a solution* $u\colon [1, T) \to \mathbb{R}$ *of equation* (4) *with* $0 < T < \infty$ *which cannot be extended to the right of* T. *Then* $G(+\infty) < \infty$, *where*

$$G(v) = \int_{v_0}^{v} \frac{d\sigma}{g_1(\sigma^k)^{\frac{1}{p-1}} + g_2(\sigma^k)^{\frac{1}{p-1}}}\,, \quad v \ge v_0 \ge 0\,.$$

This theorem can be proved by a modification of the procedure used in the proof of Lemma 3.6 from [18].

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References

[1] Agarwal, R.P., Djebali, S., Moussaoui, T., Mustafa, O.G., *On the asymptotic integration of nonlinear differential equations*, J. Comput. Appl. Math **202** (2007), 352–376.

[2] Bartušek, M., Medveď, M., *Existence of global solutions for systems of second-order functional-differential equations with p-Laplacian*, EJDE **40** (2008), 1–8.

[3] Bartušek, M., Pekárková, E., *On the existence of proper solutions of quasilinear second order differential equations*, EJQTDE **1** (2007), 1–14.

[4] Bihari, I., *A generalization of a lemma of Bellman and its applications to uniqueness problems of differential equations*, Acta Math. Hungar. **7** (1956), 81–94.

[5] Caligo, D., *Comportamento asintotico degli integrali dell'equazione* $y''(x) + A(x)y(x) = 0$, *nell'ipotesi* $\lim_{x\to+\infty} A(x) = 0$, Boll. Un. Mat. Ital. (2) **3** (1941), 286–295.

[6] Cohen, D.S., *The asymptotic behavior of a class of nonlinear differntial equations*, Proc. Amer. Math. Soc. **18** (1967), 607–609.

[7] Constantin, A., *On the asymptotic behavior of second order nonlinear differential equations*, Rend. Mat. Appl. (7) **13** (4) (1993), 627–634.

[8] Constantin, A., *Solutions globales d'équations différentielles perturbées*, C. R. Acad. Sci. Paris Sér. I Math. **320** (11) (1995), 1319–1322.

[9] Constantin, A., *On the existence of positive solutions of second order differential equations*, Ann. Mat. Pura Appl. (4) **184** (2) (2005), 131–138.

[10] Dannan, F.M., *Integral inequalities of Gronwall-Bellman-Bihari type and asymptotic behavior of certain second order nonlinear differential equations*, J. Math. anal. Appl. **108** (1) (1985), 151–164.

[11] Kusano, T., Trench, W.F., *Existence of global solutions with prescribed asymptotic behavior for nonlinear ordinary differential equations*, J. London Math. Soc.(2) **31** (3) (1985), 478–486.

[12] Kusano, T., Trench, W.F., *Existence of global solutions with prescribed asymptotic behavior for nonlinear ordinary differential equations*, Ann. Mat. Pura Appl. (4) **142** (1985), 381–392.

[13] Lipovan, O., *On the asymptotic behaviour of the solutions to a class of second order nonlinear differential equations*, Glasgow Math. J. **45** (1) (2003), 179–187.

[14] Medveď, M., Moussaoui, T., *Asymptotic integration of nonlinear $\Phi-$Laplacian differential equations*, Nonlinear Anal. **72** (2010), 1–8.

[15] Medveď, M., Pekárková, E., *Existence of global solutions for systems of second-order differential equations with p-Laplacian*, EJDE **2007** (136) (2007), 1–9.

[16] Medveď, M., Pekárková, E., *Long time behavior of second order differential equations with p-Laplacian*, EJDE **2008** (108) (2008), 1–12.

[17] Mustafa, O.G., Rogovchenko, Y.V., *Global existence of solutions with prescribed asymptotic behavior for second-order nonlinear differential equations*, Nonlinear Anal. **51** (2002), 339–368.

[18] Mustafa, O.G., Rogovchenko, Y.V., *Asymptotic behavior of nonoscillatory solutions of second-order nonlinear differential equations*, Dynamic Systems and Applications **4** (2004), 312–319.

[19] Pekárková, E., *Estimations of noncontinuable solutions of second order differential equations with p-Laplacian*, Arch. Math.(Brno) **46** (2010), 135–144.

[20] Philos, Ch.G., Purnaras, I.K., Tsamatos, P.Ch., *Large time asymptotic to polynomials solutions for nonlinear differential equations*, Nonlinear Anal. **59** (2004), 1157–1179.

[21] Rogovchenko, S.P., Rogovchenko, Y.V., *Asymptotics of solutions for a class of second order nonlinear differential equations*, Portugal. Math. **57** (1) (2000), 17–32.

[22] Rogovchenko, Y.V., *On asymptotic behavior of solutions for a class of second order nonlinear differential equations*, Collect. Math. **49** (1) (1998), 113–120.

[23] Tong, J., *The asymptotic behavior of a class of nonlinear differential equations of second order*, Proc. Amer. Math. Soc. **54** (1982), 235–236.

[24] Trench, W.F., *On the asymptotic behavior of solutions of second order linear differential equations*, Proc. Amer. Math. Soc. **54** (1963), 12–14.

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