ARCHIVUM MATHEMATICUM (BRNO)
Tomus 52 (2016), 35–48

SINGULAR ϕ-LAPLACIAN THIRD-ORDER BVPS WITH DERIVATIVE DEPENDANCE

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Abstract. This work is devoted to the existence of solutions for a class of singular third-order boundary value problem associated with a ϕ-Laplacian operator and posed on the positive half-line; the nonlinearity also depends on the first derivative. The upper and lower solution method combined with the fixed point theory guarantee the existence of positive solutions when the nonlinearity is monotonic with respect to its arguments and may have a space singularity; however no Nagumo type condition is assumed. An example of application illustrates the applicability of the existence result.

1. Introduction

In this paper, we are concerned with the existence of positive solutions to the following third-order boundary value problem for a ϕ-Laplacian operator:

$$\begin{cases} (\phi(-x''))'(t) + f\big(t, x(t), x'(t)\big) = 0\,, \quad t > 0\,, \\ x(0) = \mu x'(0),\ x'(+\infty) = x''(+\infty) = 0\,, \end{cases} \tag{1.1}$$

where $\mu \geq 0$ is a constant and the function $f = f(t,x,y)\colon \mathbb{R}^+ \times (0,+\infty) \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous with possible space singularity at $x = 0$. Here $\mathbb{R}^+ = [0,+\infty)$. The operator of derivation $\phi\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous increasing homeomorphism such that $\phi(0) = 0$, extending the p-Laplacian $\varphi_p(s) = |s|^{p-1}s$, for $p > 1$.

Many applied problems modeling various phenomena in physics, epidemiology, combustion theory, mechanics (see, e.g., [2] and the references therein) are governed by boundary value problems (bvps for short) posed on the half-axis $[0,+\infty)$; we quote for instance the propagation of a flame in a long tube. A large amount of research papers have been devoted to these problems, in particular for the second-order boundary value problems; we refer the reader to [4], [5], [6], [7], [8], [9], [12], [15], [16], [17], and the references therein. However problems with higher-order differential equations on $[0,+\infty)$ have not been so extensively investigated; we can only cite [13], [14], [18], and [19]. When f does not depend on the first derivative, problem (1.1) in investigated in [11].

2010 *Mathematics Subject Classification*: primary 34B15; secondary 34B18, 34B40, 47H10.

Key words and phrases: third order, half-line, ϕ-Laplacian, singular problem, positive solution, derivative dependance, upper and lower solution.

Received December 1, 2014, revised November 2015. Editor O. Došlý.

DOI: 10.5817/AM2016-1-35

These recent papers have motivated the present work.

To study problem (1.1), we will employ an upper and lower solution technique adapted to this problem combined with the Schauder fixed point theorem. We only suppose a monotonic condition on f but no Nagumo-type restriction is assumed. This paper contains three sections. In Section 2, we present some preliminaries and basic notions needed in this paper. Problem (1.1) is rewritten as a nonlinear integral equation. In Section 3, we prove the main existence result when the nonlinearity f is monotonic with respect to x and y but may be singular at $x = 0$. The case where f is not singular at $x = 0$ is also considered with less hypotheses. An example of application is included to illustrate the existence theorem. We say that x is a solution of problem (1.1) if it belongs to the space

$$X = \{x \mid x \in C^2((0,\infty),\mathbb{R}) \text{ and } \phi(-x'') \in C^1((0,\infty),\mathbb{R})\} \tag{1.2}$$

and satisfies (1.1). x is called a positive solution if further $x(t) > 0$, for every $t \in (0, +\infty)$.

2. Preliminaries

The basic space to study problem (1.1) is

$$E = \left\{x \in C^1([0,\infty),\mathbb{R}) \mid \lim_{t\to+\infty} \frac{x(t)}{1+t} = \lim_{t\to+\infty} x'(t) = 0\right\}.$$

The motivation of the space E comes from the fact that the positivity of f and $x''(+\infty) = 0$ imply the concavity of x which in turn guarantees that x is nondecreasing for $x'(+\infty) = 0$. As a consequence, x has a possibly infinite limit at positive infinity. L'Hopital's rule then yields $\lim_{t\to+\infty} \frac{x(t)}{1+t} = \lim_{t\to+\infty} x'(t) = 0$. Notice that $(E, \|\cdot\|)$ is a Banach space with norm $\|x\| = \max\{\|x\|_1, \|x\|_2\}$, where $\|x\|_1 = \sup_{t\in\mathbb{R}^+} \frac{|x(t)|}{1+t}$ and $\|x\|_2 = \sup_{t\in\mathbb{R}^+} |x'(t)|$. However since for physical considerations, we are interested in positive solutions, the natural set for solutions is the positive cone:

$$S = \{x \in E \mid x(t) \geq 0, \text{ concave on } [0,+\infty),\ x(0) = \mu x'(0)\}. \tag{2.1}$$

This nonempty subset enjoys the following properties:

Lemma 2.1. *For every $x \in S$, there exists a positive constant $M_x > 0$ such that*

$$0 \leq x'(t) \leq M_x, \quad \forall\, t \geq 0.$$

Proof. Since x' is nonincreasing, then

$$0 = x'(+\infty) \leq x'(t) \leq x'(0) = M_x, \quad \forall\, t \geq 0.$$

□

The proof of the following lemma can be found in [11, Lemma 2.5].

Lemma 2.2. *Let $x \in S \setminus \{0\}$. Then there exists a positive constant λ_x such that*

(a) *for all $\theta > 1$,* $\quad x(t) \geq \frac{\lambda_x}{\theta}, \quad \forall\, t \in [1/\theta, \theta]$,

(b) *Let*

$$\rho(t) = \begin{cases} t\,, & t \in [0,1] \\ 1\,, & t \geq 1\,. \end{cases} \tag{2.2}$$

Then

$$x(t) \geq \lambda_x \rho(t)\,, \quad \forall\, t \geq 0\,.$$

Lemma 2.3. *Let x, $y \in S$ be such that $x'(t) \geq y'(t)$, for all $t \geq 0$. Then*

$$x(t) \geq y(t)\,, \quad \forall\, t \geq 0\,.$$

Proof. Since $x'(0) - y'(0) \geq 0$, then $x(0) - y(0) = \mu(x'(0) - y'(0)) \geq 0$. Now $x - y$ is nondecreasing implies that $(x-y)(t) \geq x(0) - y(0) \geq 0\,, \quad \forall\, t \geq 0$. □

Define the functional space

$$C_l([0,\infty),\mathbb{R}) = \left\{x \in C([0,\infty),\mathbb{R}) \mid \lim_{t\to+\infty} x(t) \text{ exists}\right\}.$$

Endowed with the norm $\|x\|_l = \sup_{t\in\mathbb{R}^+} |x(t)|$, this is a Banach space. A mapping defined on a Banach space is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets. We recall a classical compactness criterion:

Lemma 2.4 ([3])**.** *Let $M \subseteq C_l(\mathbb{R}^+,\mathbb{R})$. Then M is relatively compact in $C_l(\mathbb{R}^+,\mathbb{R})$ if the following three conditions hold:*

(a) *M is uniformly bounded in $C_l(\mathbb{R}^+,\mathbb{R})$.*

(b) *The functions belonging to M are almost equicontinuous on $\mathbb{R}^+$, i.e. equicontinuous on every compact interval of $\mathbb{R}^+$.*

(c) *The functions from M are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(t) - x(+\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $x \in M$.*

We can then deduce:

Lemma 2.5. *Let $M \subseteq E$. Then M is relatively compact in E if the following conditions hold:*

(a) *M is bounded in E,*

(b) *the functions belonging to $\{u \mid u(t) = \frac{x(t)}{1+t}, x \in M\}$ and to $\{z \mid z(t) = x'(t), x \in M\}$ are almost equicontinuous on $[0,+\infty)$,*

(c) *the functions belonging to $\{u \mid u(t) = \frac{x(t)}{1+t}, x \in M\}$ and to $\{z \mid z(t) = x'(t), x \in M\}$ are equiconvergent at $+\infty$.*

The Green's function of the linear problem $-x'' = x(0) - \mu x'(0) = x'(+\infty) = 0$ is

$$G(t,s) = \mu + \min(s,t)\,, \quad s,\, t \geq 0\,.$$

We have

Lemma 2.6. *Let $\delta \in C(\mathbb{R}^+, \mathbb{R}^+)$ be such that $\int_0^{+\infty} \delta(s)\, ds < +\infty$ and put $x(t) = \int_0^{+\infty} G(t,s)\delta(s)\, ds$. Then*

$$\begin{cases} x''(t) + \delta(t) = 0\,, & t > 0\,,\\ x(0) = \mu x'(0)\,, & \lim\limits_{t\to+\infty} x'(t) = 0\,. \end{cases} \tag{2.3}$$

Lemma 2.7. *Let $\delta \in C(\mathbb{R}^+, \mathbb{R}^+) \cap L^1(r, +\infty)$ for all $r > 0$ and*

$$\int_0^{+\infty} \phi^{-1}\Big(\int_s^{+\infty} \delta(\tau)\, d\tau\Big)\, ds < +\infty\,.$$

If $x(t) = \int_0^{+\infty} G(t,s)\phi^{-1}\big(\int_s^{+\infty} \delta(\tau)d\tau\big)\, ds$, then $x \in X$ and

$$\begin{cases} (\phi(-x''(t)))' + \delta(t) = 0\,, & t > 0\,,\\ x(0) = \mu x'(0)\,, & x'(+\infty) = x''(+\infty) = 0\,. \end{cases} \tag{2.4}$$

The proofs of the lemmas are immediate and are omitted.

3. Main results

We start with

Definition 3.1. A function $\alpha \in X$ is called lower solution of (1.1) if

$$\begin{cases} (\phi(-\alpha''(t)))' + f(t, \alpha(t), \alpha'(t)) \geq 0\,, \quad t > 0\\ \alpha(0) \leq \mu\alpha'(0), \quad \lim\limits_{t\to+\infty} \alpha'(t) \leq 0, \ \lim\limits_{t\to+\infty} \alpha''(t) \geq 0\,. \end{cases}$$

An upper solution of (1.1) is defined when the above inequalities are reversed. Assume that

($\mathcal{H}_1$) $f \in C(\mathbb{R}^+ \times (0, +\infty) \times \mathbb{R}^+, \mathbb{R}^+)$ and $f(t, x, y)$ is nonincreasing with respect to the second and third arguments.

($\mathcal{H}_2$) For every $\lambda > 0$,

$$\int_0^{+\infty} f(\tau, \lambda\rho(\tau), 0)\, d\tau < +\infty$$

and

$$\int_0^{+\infty} \phi^{-1}\Big(\int_s^{+\infty} f\big(\tau, \lambda\rho(\tau), 0\big)\, d\tau\Big)\, ds < +\infty\,.$$

($\mathcal{H}_3$) There exists a function $a \in S \setminus \{0\}$ such that the function b defined by

$$b(t) = \int_0^{+\infty} G(t,s)\phi^{-1}\Big(\int_s^{+\infty} f\big(\tau, a(\tau), a'(\tau)\big)\, d\tau\Big)\, ds$$

satisfies

$$b'(t) \geq \int_t^{+\infty} \phi^{-1}\Big(\int_s^{+\infty} f\big(\tau, b(\tau), b'(\tau)\big)\, d\tau\Big)\, ds \geq a'(t)\,, \quad \forall\, t \geq 0\,.$$

Define the fixed point operator T on E by

$$Tx(t) = \int_0^{+\infty} G(t,s)\phi^{-1}\Big(\int_s^{+\infty} f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau\Big)\,ds\,.$$

Remark 3.1. (a) Since

$$(Tx)'(t) = \int_t^{+\infty} \phi^{-1}\Big(\int_s^{+\infty} f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau\Big)\,ds,$$

the inequalities in $(\mathcal{H}_3)$ read

$$b'(t) \geq (Tb)'(t) \geq a'(t)\,, \quad \forall\, t \geq 0 \tag{3.1}$$

or equivalently

$$(Ta)'(t) \geq (T^2a)'(t) \geq a'(t)\,, \quad \forall\, t \geq 0\,.$$

(b) Since $a \in S$ then $b \in S$ and so $Tb \in S$. Using part (a) and Lemma 2.3, we get the estimates

$$Ta(t) \geq T^2a(t) \geq a(t)\,, \quad \forall\, t \geq 0\,. \tag{3.2}$$

Lemma 3.1. *Let $(\mathcal{H}_1)$–$(\mathcal{H}_2)$ hold. Then the operator T maps $S \setminus \{0\}$ into $X \cap S$. Moreover*

$$\begin{cases} \big(\phi(-(Tx)'')\big)'(t) + f\big(t, x(t), x'(t)\big) = 0\,, & t > 0\,, \\ (Tx)(0) = \mu (Tx)'(0)\,, \quad (Tx)'(+\infty) = (Tx)''(+\infty) = 0\,. \end{cases} \tag{3.3}$$

Proof. (a) For $\lambda > 0$, let

$$F_\lambda(t) = \int_0^{+\infty} G(t,s)\phi^{-1}\Big(\int_s^{+\infty} f\big(\tau, \lambda\rho(\tau), 0\big)\,d\tau\Big)\,ds\,.$$

Then

$$\lim_{t\to+\infty} \frac{F_\lambda(t)}{1+t} = 0\,.$$

By the convergence of the second integral in $(\mathcal{H}_2)$, we get

$$\lim_{t\to+\infty} F'_\lambda(t) = \lim_{t\to+\infty} \int_t^{+\infty} \phi^{-1}\Big(\int_s^{+\infty} f\big(\tau, \lambda\rho(\tau), 0\big)\,d\tau\Big)\,ds = 0\,.$$

Then F_λ is monotone nondecreasing and

$$\lim_{t\to+\infty} \frac{F_\lambda(t)}{1+t} = \begin{cases} 0\,, & \text{if} \quad \lim\limits_{t\to+\infty} F_\lambda(t) < \infty\,, \\ \lim\limits_{t\to+\infty} F'_\lambda(t) = 0\,, & \text{if} \quad \lim\limits_{t\to+\infty} F_\lambda(t) = \infty\,. \end{cases}$$

(b) For $x \in S \setminus \{0\}$, Lemmas 2.1 and 2.2 guarantee the existence of $\lambda_x > 0$ such that for all positive t, $x(t) \geq \lambda_x \rho(t)$ and $x'(t) \geq 0$. $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ with Part (a) imply

$$\begin{aligned} \frac{Tx(t)}{1+t} &= \frac{\int_0^{+\infty} G(t,s)\phi^{-1}\big(\int_s^{+\infty} f(\tau, x(\tau), x'(\tau))d\tau\big)\,ds}{1+t} \\ &\leq \frac{\int_0^{+\infty} G(t,s)\phi^{-1}\big(\int_s^{+\infty} f(\tau, \lambda_x\rho(\tau), 0)d\tau\big)\,ds}{1+t} = \frac{F_{\lambda_x}(t)}{1+t} \end{aligned}$$

and

$$\begin{aligned}(Tx)'(t) &= \int_t^{+\infty} \phi^{-1}\Big(\int_s^{+\infty} f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau\Big)\,ds\\ &\leq \int_t^{+\infty} \phi^{-1}\Big(\int_s^{+\infty} f\big(\tau, \lambda_x\rho(\tau), 0\big)\,d\tau\Big)\,ds\,.\end{aligned}$$

Hence $\lim\limits_{t\to+\infty}\frac{Tx(t)}{1+t}=0$ and $\lim\limits_{t\to+\infty}(Tx)'(t)=0$. Then $Tx\in E$. In fact, we even have that $Tx\in X\cap S$ for $Tx(t)\geq 0$, $Tx(0)=\mu(Tx)'(0)$, and

$$(Tx)''(t) = -\phi^{-1}\Big(\int_t^{+\infty} f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau\Big) \leq 0\,.$$

Thus (3.3) is satisfied. □

We are now in position to prove our main existence result:

Theorem 3.1. *Assume that assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ hold. Then the boundary value problem* (1.1) *has at least one positive solution $x\in X$ such that $x(t)\geq\lambda_0\rho(t)$ and $0\leq x'(t)\leq M$, $\forall\, t\geq 0$, for some positive constants λ_0, M.*

Proof.
Step 1. Upper and lower solution method. Taking into account Remark 3.1, using (3.1), (3.2) and the monotonicity of f, we have for all $t>0$

$$\begin{cases}\big(\phi(-(Tb)'')\big)'(t)+f\big(t,Tb(t),(Tb)'(t)\big)\\ \qquad\geq \big(\phi(-(Tb)'')\big)'(t)+f\big(t,b(t),b'(t)\big)=0\\ (Tb)(0)=\mu(Tb)'(0)\,,\ (Tb)'(+\infty)=0\,,\ (Tb)''(+\infty)=0\end{cases}\tag{3.4}$$

and

$$\begin{cases}\big(\phi(-(Ta)'')\big)'(t)+f\big(t,Ta(t),(Ta)'(t)\big)\\ \qquad\leq \big(\phi(-(Ta)'')\big)'(t)+f\big(t,a(t),a'(t)\big)=0\,,\\ (Ta)(0)=\mu(Ta)'(0)\,,\ (Ta)'(+\infty)=0\,,\ (Ta)''(+\infty)=0\,.\end{cases}\tag{3.5}$$

Therefore the functions $\alpha=Tb$, $\beta=Ta$ are lower and upper solutions of problem (1.1), respectively with $\alpha\leq\beta$ and $\alpha'\leq\beta'$.

Step 2. Consider the truncated problem:

$$\begin{cases}\big(\phi(-x'')\big)'(t)+f^*\big(t,x(t),x'(t)\big)=0\,, \quad t>0\,,\\ x(0)=\mu x'(0)\,,\ x'(+\infty)=x''(+\infty)=0\,,\end{cases}\tag{3.6}$$

where

$$f^*(t,x,y)=\begin{cases}\widetilde{f}(t,\alpha,y)\,, & x<\alpha(t)\,,\\ \widetilde{f}(t,x,y)\,, & \alpha(t)\leq x\leq\beta(t)\,,\\ \widetilde{f}(t,\beta,y)\,, & x>\beta(t)\,,\end{cases}\tag{3.7}$$

with

$$\widetilde{f}(t,x,y)=\begin{cases} f(t,x,\alpha')\,, & y<\alpha'(t)\,,\\ f(t,x,y)\,, & \alpha'(t)\le y\le\beta'(t)\,,\\ f(t,x,\beta')\,, & y>\beta'(t)\,.\end{cases}$$

To prove that problem (3.6) has a positive solution, consider the operator $A\colon E\to E$ defined by

$$Ax(t)=\int_0^{+\infty}G(t,s)\phi^{-1}\Big(\int_s^{+\infty}f^*\big(\tau,x(\tau),x'(\tau)\big)\,d\tau\Big)\,ds\,.$$

Then a fixed point of the operator A is a solution of the boundary value problem (3.6). Since $\alpha\in S\setminus\{0\}$, by Lemmas 2.1 and 2.2, there exists a positive constant λ_α such that $\alpha(t)\ge\lambda_\alpha\rho(t)$, $\forall\,t\ge 0$ and $\alpha'(t)\ge 0,\forall\,t\ge 0$. Since $f(t,x,y)$ is nonincreasing in x and y, then

$$f^*(t,x,y)\le f\big(t,\alpha(t),\alpha'(t)\big)\le f\big(t,\lambda_\alpha\rho(t),0\big)\,, \tag{3.8}$$

for all positive t.

(a) $A(E)\subseteq E$. For $x\in E$ and $t\in\mathbb{R}^+$, we have

$$\begin{aligned}\frac{Ax(t)}{1+t}&=\frac{\int_0^{+\infty}G(t,s)\phi^{-1}(\int_s^{+\infty}f^*(\tau,x(\tau),x'(\tau))d\tau)\,ds}{1+t}\\ &\le\frac{\int_0^{+\infty}G(t,s)\phi^{-1}(\int_s^{+\infty}f(\tau,\lambda_\alpha\rho(\tau),0)d\tau)ds}{1+t}=\frac{F_{\lambda_\alpha}(t)}{1+t}\end{aligned}$$

and

$$\begin{aligned}(Ax)'(t)&=\int_t^{+\infty}\phi^{-1}\Big(\int_s^{+\infty}f^*\big(\tau,x(\tau),x'(\tau)\big)\,d\tau\Big)\,ds\\ &\le\int_t^{+\infty}\phi^{-1}\Big(\int_s^{+\infty}f\big(\tau,\lambda_\alpha\rho(\tau),0\big)\,d\tau\Big)\,ds\,.\end{aligned}$$

Then $\lim\limits_{t\to+\infty}\frac{Ax(t)}{1+t}=0$ and $\lim\limits_{t\to+\infty}(Tx)'=0$ which implies that $A(E)\subseteq E$.

(b) *A is continuous.*

Let $\{x_n\}_{n\ge1}\subseteq E$ be a sequence converging to some limit $x_0\in E$. Then

$$\begin{aligned}\|Ax_n-Ax_0\|_1&=\sup_{t\in\mathbb{R}^+}\frac{|Ax_n(t)-Ax_0(t)|}{1+t}\\ &=\sup_{t\in\mathbb{R}^+}\int_0^{+\infty}\frac{G(t,s)}{1+t}\Big|\phi^{-1}\Big(\int_s^{+\infty}f^*\big(\tau,x_n(\tau),x_n'(\tau)\big)\,d\tau\Big)\\ &\quad-\phi^{-1}\Big(\int_s^{+\infty}f^*\big(\tau,x_0(\tau),x_0'(\tau)\big)\,d\tau\Big)\Big|\,ds\end{aligned}$$

$$\le \max(1,\mu)\int_0^{+\infty}\phi^{-1}\Big(\int_s^{+\infty} f^*\big(\tau,x_n(\tau),x_n'(\tau)\big)\,d\tau\Big) - \phi^{-1}\Big(\int_s^{+\infty} f^*\big(\tau,x_0(\tau)x_0'(\tau)\big)\,d\tau\Big)\Big|\,ds$$

and

$$\begin{aligned}\|Ax_n - Ax_0\|_2 &= \sup_{t\in\mathbb{R}^+}\big|(Ax_n)'(t)-(Ax_0)'(t)\big|\\ &\le \sup_{t\in\mathbb{R}^+}\int_t^{+\infty}\Big|\phi^{-1}\Big(\int_s^{+\infty} f^*\big(\tau,x_n(\tau),x_n'(\tau)\big)\,d\tau\Big)\\ &\quad - \phi^{-1}\Big(\int_s^{+\infty} f^*\big(\tau,x_0(\tau),x_0'(\tau)\big)\,d\tau\Big)\Big|\,ds\\ &\le \int_t^{+\infty}\phi^{-1}\Big(\int_s^{+\infty} f^*\big(\tau,x_n(\tau),x_n'(\tau)\big)\,d\tau\Big)\\ &\quad - \phi^{-1}\Big(\int_s^{+\infty} f^*\big(\tau,x_0(\tau),x_0'(\tau)\big)\,d\tau\Big)\Big|\,ds\,.\end{aligned}$$

Since

$$\begin{aligned}\Big|\phi^{-1}\Big(\int_s^{+\infty} f^*\big(\tau,x_n(\tau),x_n'(\tau)\big)d\tau\Big) - \phi^{-1}\Big(\int_s^{+\infty} f^*\big(\tau,x_0(\tau)\big),x_0'(\tau)\,d\tau\Big)\Big|&\\ \le 2\phi^{-1}\Big(\int_0^{+\infty} f\big(\tau,\lambda_\alpha\rho(\tau)\big),0\Big)\,d\tau\,,&\end{aligned}$$

then the continuity of f^*, ϕ^{-1}, assumption $(\mathcal{H}_2)$, and the Lebesgue dominated convergence theorem guarantee that $\|Ax_n - Ax_0\| \longrightarrow 0$, as $n \longrightarrow +\infty$.

(c) $A(E)$ *is relatively compact.* We will make use of Lemma 2.5.

(i) $A(E)$ *is uniformly bounded.* For $x \in E$, we have

$$\begin{aligned}\|Ax\|_1 = \sup_{t\in\mathbb{R}^+}\frac{|Ax(t)|}{1+t} &\le \sup_{t\in\mathbb{R}^+}\int_0^{+\infty}\frac{G(t,s)}{1+t}\phi^{-1}\Big(\int_s^{+\infty} f^*\big(\tau,x(\tau),x'(\tau)\big)\,d\tau\Big)\,ds\\ &\le \max(1,\mu)\int_0^{+\infty}\phi^{-1}\Big(\int_s^{+\infty} f^*\big(\tau,x(\tau),x'(\tau)\big)d\tau\Big)\,ds\\ &\le \max(1,\mu)\int_0^{+\infty}\phi^{-1}\Big(\int_s^{+\infty} f\big(\tau,\lambda_\alpha\rho(\tau),0\big)\,d\tau\Big)\,ds < +\infty\end{aligned}$$

and

$$\|Ax\|_2 = \sup_{t\in\mathbb{R}^+}|(Ax)'(t)| \le \int_0^{+\infty}\phi^{-1}\Big(\int_s^{+\infty} f\big(\tau,\lambda_\alpha\rho(\tau),0\big)d\tau\Big)\,ds < +\infty.$$

(ii) *Almost equicontinuity.* For a given $T>0$, $x\in E$, and $t,t'\in[0,T]$ $(t>t')$, we have

$$\begin{aligned}\Big|\frac{Ax(t)}{1+t}-\frac{Ax(t')}{1+t'}\Big| &\le \int_0^{+\infty}\Big|\frac{G(t,s)}{1+t}-\frac{G(t',s)}{1+t'}\Big|\phi^{-1}\Big(\int_s^{+\infty} f^*\big(\tau,x(\tau),x'(\tau)\big)\,d\tau\Big)\,ds\\ &\le \int_0^{T}\Big|\frac{G(t,s)}{1+t}-\frac{G(t',s)}{1+t'}\Big|\phi^{-1}\Big(\int_s^{+\infty} f\big(\tau,\lambda_\alpha\rho(\tau),0\big)\,d\tau\Big)\,ds\\ &\quad+\Big|\frac{t+\mu}{1+t}-\frac{t'+\mu}{1+t'}\Big|\int_T^{+\infty}\phi^{-1}\Big(\int_s^{+\infty} f\big(\tau,\lambda_\alpha\rho(\tau),0\big)\,d\tau\Big)\,ds\,.\end{aligned}$$

Similarly

$$\begin{aligned}|((Ax)'(t))-((Ax)'(t'))| &= \Big|\int_t^{+\infty}\phi^{-1}\Big(\int_s^{+\infty} f^*\big(\tau,x(\tau),x'(\tau)\big)\,d\tau\Big)\,ds\\ &\quad-\int_{t'}^{+\infty}\phi^{-1}\Big(\int_s^{+\infty} f^*\big(\tau,x(\tau),x'(\tau)\big)\,d\tau\Big)\,ds\Big|\\ &\le \int_{t'}^{t}\phi^{-1}\Big(\int_s^{+\infty} f\big(\tau,\lambda_\alpha\rho(\tau),0\big)\,d\tau\Big)\,ds.\end{aligned}$$

Hence by $(\mathcal{H}_2)$, for any $\varepsilon>0$ and $T>0$, there exists $\delta>0$ such that $\big|\frac{Ax(t)}{1+t}-\frac{Ax(t')}{1+t'}\big|<\varepsilon$ and $|(Ax)'(t)-(Ax)'(t')|<\varepsilon$ for all $t,t'\in[0,T]$ with $|t-t'|<\delta$. As a consequence $\{\frac{A(E)}{1+t}\}$ and $\{A(E)\}'$ are almost equicontiuous.

(iii) *$\frac{A(E)}{1+t}$ and $(A(E))'$ are equiconvergent at $+\infty$.* Since $\lim\limits_{t\to+\infty}\frac{Ax(t)}{1+t}=0$ and $\lim\limits_{t\to+\infty}(Ax)'(t)=0$, then $(\mathcal{H}_2)$ yields

$$\begin{aligned}&\lim_{t\to+\infty}\sup_{x\in E}\Big|\frac{Ax(t)}{1+t}-\lim_{t\to+\infty}\frac{Ax(t)}{1+t}\Big|\\ &\qquad= \lim_{t\to+\infty}\sup_{x\in E}\frac{\int_0^{+\infty}G(t,s)\phi^{-1}(\int_s^{+\infty}f^*(\tau,x(\tau),x'(\tau))d\tau)\,ds}{1+t}\\ &\qquad\le \lim_{t\to+\infty}\frac{\int_0^{+\infty}G(t,s)\phi^{-1}(\int_s^{+\infty}f(\tau,\lambda_\alpha\rho(\tau),0)d\tau)\,ds}{1+t}\\ &\qquad= \lim_{t\to+\infty}\frac{F_{\lambda_\alpha}(t)}{1+t}=0\end{aligned}$$

and

$$\begin{aligned}&\lim_{t\to+\infty}\sup_{x\in E}\big|(Ax)'(t)-\lim_{t\to+\infty}(Ax)'(t)\big|\\ &\qquad= \lim_{t\to+\infty}\sup_{x\in E}\int_t^{+\infty}\phi^{-1}\Big(\int_s^{+\infty}f^*\big(\tau,x(\tau),x'(\tau)\big)\,d\tau\Big)\,ds\\ &\qquad\le \lim_{t\to+\infty}\int_t^{+\infty}\phi^{-1}\Big(\int_s^{+\infty}f\big(\tau,\lambda_\alpha\rho(\tau),0\big)\,d\tau\Big)\,ds=0\,.\end{aligned}$$

By Lemma 2.5, the range $A(E)$ is relatively compact so the Schauder fixed point theorem (see, e.g., [1]), guarantees that the operator A has at least one fixed point

$x \in E$ which in fact lies in X by Lemma 3.1; of course x is solution of problem (3.6).

Step 3. Problem (1.1) *has at least one positive solution.*
We only check that $\alpha(t) \leq x(t) \leq \beta(t)$ and $\alpha'(t) \leq x'(t) \leq \beta'(t)$, $\forall\, t \in \mathbb{R}^+$. Since x is a solution of (3.6), we have

$$x(0) = \mu x'(0)\,, \quad \lim_{t\to+\infty} x'(t) = \lim_{t\to+\infty} x''(t) = 0\,. \tag{3.9}$$

The function $f(t,x,y)$ being nonincreasing in x and y, we have

$$f\big(t,\beta(t),\beta'(t)\big) \leq f^*(t,x,x') \leq f\big(t,\alpha(t),\alpha'(t)\big)\,,\ \forall\, t \in \mathbb{R}^+\,. \tag{3.10}$$

Then (3.1) and (3.2) yield

$$f\big(t,b(t),b'(t)\big) \leq f^*(t,x,x') \leq f\big(t,a(t),a'(t)\big)\,,\ \forall\, t \in \mathbb{R}^+\,. \tag{3.11}$$

Noting that $a, b \in S \setminus \{0\}$, we get by Lemma 3.1

$$\big(\phi(-\beta''(t))\big)' = \big(\phi(-Ta)''(t))\big)' = -f\big(t,a(t),a'(t)\big)\,,\ \forall\, t \in \mathbb{R}^+\,,$$

$$\big(\phi(-\alpha''(t))\big)' = \big(\phi(-Tb)''(t))\big)' = -f\big(t,b(t),b'(t)\big)\,,\ \forall\, t \in \mathbb{R}^+\,.$$

Combining this with (3.1), (3.2), (3.9)–(3.11), and Lemma 3.1, we obtain that for all positive t

$$\big(\phi(-\beta''(t))\big)' - \big(\phi(-x''(t))\big)' = -f\big(t,a(t),a'(t)\big) + f^*\big(t,x(t),x'(t)\big) \leq 0\,.$$

Then the function z defined by $z(t) = (\phi(-\beta''(t))) - (\phi(-x''(t)))$ is nonincreasing in $\mathbb{R}^+$. Moreover $z(+\infty) = 0$ implies $z(t) \geq 0,\ \forall\, t \geq 0$. Hence $(\beta - x)''(t) \leq 0,\ \forall\, t \in \mathbb{R}^+$ which implies that $(\beta - x)'$ is nonincreasing in $\mathbb{R}^+$. In addition $(\beta - x)'(+\infty) = 0$, then $(\beta - x)'(t) \geq 0,\ \ \forall\, t \in \mathbb{R}^+$ and $x'(t) \leq \beta'(t),\ \ \forall\, t \in \mathbb{R}^+$. Finally, Lemma 2.3 implies that $x(t) \leq \beta(t)$, for all $t \in \mathbb{R}^+$. The estimates $x'(t) \geq \alpha'(t)$ and $x(t) \geq \alpha(t)$, for all $t \in \mathbb{R}^+$ are proved similarly; we omit the details. Therefore, x is a solution of (1.1). Finally, since $\alpha, \beta \in S \setminus \{0\}$, by Lemmas 2.1 and 2.2, there exist two positive constants $\lambda_0 = \lambda_\alpha$ and $M = M_\beta$ such that $x(t) \geq \alpha(t) \geq \lambda_0 \rho(t)$ and $0 \leq x'(t) \leq \beta'(t) \leq M,\ \ \forall\, t \in \mathbb{R}^+$, as claimed. ☐

When $f(t,x,y)$ has no singularity at $x = 0$, i.e. $f\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous function, then for all $x, y \geq 0$, $f(t,x,y) \leq f(t,0,0)$. We have obtained the following

Theorem 3.2. *Assume that assumption* $(\mathcal{H}_1)$ *holds with*
$(\mathcal{H}_2)'$

$$0 < \int_0^{+\infty} f(\tau,0,0)d\tau < +\infty \quad \textit{and} \quad \int_0^{+\infty} \phi^{-1}\Big(\int_s^{+\infty} f(\tau,0,0)d\tau\Big)\,ds < +\infty\,.$$

Then problem (1.1) *has at least one positive solution* $x \in E$ *and, by Lemma 3.1, we even have that* $x \in X$*. In addition,* $x(t) \geq \lambda_0 \rho(t)$ *and* $0 \leq x'(t) \leq M$*, for some* $M, \lambda_0 > 0$.

The proof follows the same line as that of Theorem 3.1. We only have to check that $T(S) \subset S \cap X$ and if $a(t) = 0,\ \ t \geq 0$, then the condition $(\mathcal{H}_3)$ holds. Also the condition $(\mathcal{H}_2)'$ implies that the functions $\beta = Ta = b$ and $\alpha = Tb$ belong to $S \setminus \{0\}$.

Example 3.1. Consider the singular bvp

$$\begin{cases} \big(\phi(-x''(t))\big)' + f\big(t, x(t), x'(t)\big) = 0\,, \\ x(0) = \mu x'(0)\,, \quad \lim\limits_{t\to+\infty} x'(t) = \lim\limits_{t\to+\infty} x''(t) = 0\,, \end{cases} \tag{3.12}$$

where $0 \leq \mu \leq \frac{8}{3}$, $\phi(x) = x^{\frac{1}{3}}$, $f(t,x,y) = e^{-t} m(t) g(x) \psi(y)$,

$$m(t) = \begin{cases} t^3\,, & t \in [0,1]\,, \\ \frac{1}{t^2}\,, & t \geq 1\,, \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{x}\,, & x \in (0,1]\,, \\ 1\,, & x \geq 1\,, \end{cases}$$

$$\psi(y) = \begin{cases} \frac{1}{2} + \frac{1}{y+1}\,, & y \in [0,1]\,, \\ 1\,, & y \geq 1\,. \end{cases}$$

Next, we verify the assumptions in Theorem 3.1.

$(\mathcal{H}_1)$ $f \in C(\mathbb{R}^+ \times (0,+\infty) \times \mathbb{R}^+, \mathbb{R}^+)$ and $f(t,x,y)$ is nonincreasing with respect to x and y, for every positive t.

$(\mathcal{H}_2)$ For all $\lambda > 0$,

$$\int_0^{+\infty} f(\tau, \lambda\rho(\tau), 0)\, d\tau = \frac{2}{\lambda}$$

and

$$\int_0^{+\infty} \phi^{-1}\Big(\int_s^{+\infty} f(\tau, \lambda\rho(\tau), 0)\, d\tau\Big)\, ds < +\infty\,.$$

$(\mathcal{H}_3)$ We set $a_0(t) = t + 1$ and $a = Ta_0$, i.e.

$$a(t) = \int_0^{+\infty} G(t,s)\phi^{-1}\Big(\int_s^{+\infty} e^{-\tau} m(\tau)\, d\tau\Big)\, ds\,.$$

Then for all positive t

$$\begin{aligned} a(t) &\leq \int_0^{+\infty} G(t,s)\phi^{-1}\Big(\int_s^{+\infty} e^{-\tau}\, d\tau\Big) \\ &\leq \int_0^{+\infty} (s + 8/3)\phi^{-1}(e^{-s})\, ds \leq 1 \leq a_0(t) \end{aligned}$$

and

$$\begin{aligned}
a'(t) &= \int_t^{+\infty} \phi^{-1}\Big(\int_s^{+\infty} e^{-\tau} m(\tau)\, d\tau\Big) \\
&\le \int_t^{+\infty} \phi^{-1}\Big(\int_s^{+\infty} e^{-\tau} d\tau\Big) \\
&\le \int_t^{+\infty} \phi^{-1}(e^{-s})\, ds \le 1 = a_0'(t)\,.
\end{aligned}$$

Hence

$$\begin{aligned}
b(t) &= \int_0^{+\infty} G(t,s)\phi^{-1}\Big(\int_s^{+\infty} e^{-\tau} m(\tau) g\big(a(\tau)\big)\psi\big(a'(\tau)\big)\, d\tau\Big)\, ds \\
&\ge \int_0^{+\infty} G(t,s)\phi^{-1}\Big(\int_s^{+\infty} e^{-\tau} m(\tau) g\big(a_0(\tau)\big)\psi\big(a_0'(\tau)\big)\, d\tau\Big)\, ds = a(t)
\end{aligned}$$

and $b'(t) \ge a'(t)$. As a consequence

$$\begin{aligned}
\int_t^{+\infty} & \phi^{-1}\Big(\int_s^{+\infty} f\big(\tau, b(\tau), b'(\tau)\big)\, d\tau\Big)\, ds \\
&= \int_t^{+\infty} \phi^{-1}\Big(\int_s^{+\infty} e^{-\tau} m(\tau) g\big(b(\tau)\big)\psi\big(b'(\tau)\big)\, d\tau\Big)\, ds \\
&\le \int_t^{+\infty} \phi^{-1}\Big(\int_s^{+\infty} e^{-\tau} m(\tau) g\big(a(\tau)\big)\psi\big(a'(\tau)\big)\, d\tau\Big)\, ds = b'(t)\,.
\end{aligned}$$

Then the first inequality of $(\mathcal{H}_3)$ holds. Finally, since $g \ge 1$ and $\psi \ge 1$, we get

$$\begin{aligned}
\int_t^{+\infty} & \phi^{-1}\Big(\int_s^{+\infty} f\big(\tau, b(\tau), b'(\tau)\big)\, d\tau\Big)\, ds \\
&\ge \int_t^{+\infty} \phi^{-1}\Big(\int_s^{+\infty} e^{-\tau} m(\tau)\, d\tau\Big)\, ds = a'(t)\,,
\end{aligned}$$

then the second inequality of $(\mathcal{H}_3)$ holds too.

By Theorem 3.1, problem (3.12) has at least one positive solution x lying between T^2a and Ta.

4. Concluding remarks

(a) Owing to Lemma 2.3, S is partially ordered by the relation $x' \le y'$. Then the monotonic assumption on the nonlinearity f, namely Assumption $(\mathcal{H}_1)$, implies that the fixed point integral operator T is monotonic nonincreasing. From (3.2) in Remark 3.1, it is easily seen that

$$a \le T^2a \le T^4a \le \cdots \le T^{(2m)}(a) \le \cdots \le T^{(2m-1)}(a) \le \cdots \le T^3a \le Ta\,.$$

Hence the subsequences $T^{(2m)}(a)$ and $T^{(2m+1)}(a)$ are monotonic nondecreasing and nonincreasing sequences respectively; since they are further uniformly bounded in the interval $[a, Ta]$, then they are convergent to some limits y and z, respectively, with $a \le y \le z \le Ta$. In fact, we even have that, for all integer m, $y \geq T^{(2m)}(a)$ and $z \leq T^{(2m+1)}(a)$. Therefore, we deduce that the solution set for problem (1.1) lies in the smaller interval $[y, z]$.

(b) Problem (1.1) is considered in [11] when f does not depend on the first derivative. When f depends as well on the first derivative, it has also been studied in [10] via a topological method; the hypotheses on the nonlinearity rather involve the growth of the function $f(t, (1+t)x, y)$. In [14], the authors investigated problem (1.1) with a nonlinearity satisfying some local growth conditions.

(c) In this work, problem (1.1) was treated with the method of upper and lower solutions but with no Nagumo-type growth condition on the nonlinearity f, as generally assumed. We point out that the first derivative x' is not at all involved in $(\mathcal{H}_2)$ which is rather related to the half-line problem setting.

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