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OSTROWSKI'S TYPE INEQUALITIES FOR COMPLEX FUNCTIONS DEFINED ON UNIT CIRCLE WITH APPLICATIONS FOR UNITARY OPERATORS IN HILBERT SPACES

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Abstract. Some Ostrowski's type inequalities for the Riemann-Stieltjes integral $\int_a^b f\left(e^{it}\right) du\left(t\right)$ of continuous complex valued integrands $f\colon \mathcal{C}\left(0,1\right) \to \mathbb{C}$ defined on the complex unit circle $\mathcal{C}\left(0,1\right)$ and various subclasses of integrators $u\colon\ [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ of bounded variation are given. Natural applications for functions of unitary operators in Hilbert spaces are provided as well.

1. Introduction

The problem of approximating the Stieltjes integral $\int_a^b f\left(t\right) du\left(t\right)$ by the quantity $f\left(x\right)\left[u\left(b\right)-u\left(a\right)\right]$, which is a natural generalization of the Ostrowski problem analyzed in 1937 (see [6]), was apparently first considered in the literature by the author in 2000 (see [1]) where we obtained the following result:

$$\begin{aligned}\Big|\left[u\left(b\right)-u\left(a\right)\right]f\left(x\right)-\int_a^b f\left(t\right)\,du\left(t\right)\Big| &\le H\Big[\left(x-a\right)^r\bigvee_a^x\left(f\right)+\left(b-x\right)^r\bigvee_x^b\left(f\right)\Big]\\ &\le H\times\begin{cases}\left[\left(x-a\right)^r+\left(b-x\right)^r\right]\left[\frac{1}{2}\bigvee_a^b\left(f\right)+\frac{1}{2}\left|\bigvee_a^x\left(f\right)-\bigvee_x^b\left(f\right)\right|\right];\\ \left[\left(x-a\right)^{qr}+\left(b-x\right)^{qr}\right]^{\frac{1}{q}}\left[\left(\bigvee_a^x\left(f\right)\right)^p+\left(\bigvee_x^b\left(f\right)\right)^p\right]^{\frac{1}{p}}\\ \qquad\text{if}\quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left[\frac{1}{2}\left(b-a\right)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b\left(f\right)\,.\end{cases}\end{aligned}\tag{1.1}$$

for each $x\in[a,b]$, provided f is of bounded variation on $[a,b]$, $\bigvee_a^b\left(f\right)$ is its total variation on $[a,b]$, while $u\colon\ [a,b]\to\mathbb{R}$ is $r-H$-Hölder continuous, i.e., we recall that:

$$\left|u\left(x\right)-u\left(y\right)\right|\le H\left|x-y\right|^r\quad\text{for each}\quad x,y\in\left[a,b\right]\,,\tag{1.2}$$

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where $H > 0$ and $r \in (0, 1]$.

The dual case, i.e., when the *integrand* f is $q - K$-Hölder continuous and the *integrator* u is of bounded variation was obtained by the author in 2001 and can be stated as [2]

$$\left| [u(b) - u(a)] f(x) - \int_a^b f(t)\, du(t) \right| \leq K \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^q \bigvee_a^b (u) \tag{1.3}$$

for each $x \in [a, b]$.

The above inequalities provide, as important consequences, the following midpoint inequalities:

$$\left| [u(b) - u(a)] f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\, du(t) \right| \leq \begin{cases} \frac{1}{2^r}(b-a)^r H \bigvee_a^b (f) \\ \frac{1}{2^q}(b-a)^q K \bigvee_a^b (u) \end{cases} \tag{1.4}$$

which can be numerically implemented and provide a quadrature rule for approximating the Stieltjes integral $\int_a^b f(t)\, du(t)$.

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $\operatorname{Sp}(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f\colon [m, M] \to \mathbb{R}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieltjes integral:

$$\langle f(U) x, y \rangle = \int_{m-0}^M f(\lambda)\, d(\langle E_\lambda x, y \rangle), \tag{1.5}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \quad \text{and} \quad g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

On utilizing the spectral representation (1.5) and the Ostrowski's type inequality (1.3) we obtained the following result for continuous functions of selfadjoint operators:

Theorem 1 ([4])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $\operatorname{Sp}(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f\colon [m, M] \to \mathbb{R}$ is $r - H$-Hölder continuous on $[m, M]$, then we have the inequality*

$$\begin{aligned} |f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| &\leq H \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle\right) \left[\frac{1}{2}(M-m) + \left| s - \frac{m+M}{2} \right|\right]^r \\ &\leq H \|x\| \|y\| \left[\frac{1}{2}(M-m) + \left| s - \frac{m+M}{2} \right|\right]^r \end{aligned} \tag{1.6}$$

for any $x, y \in H$ and $s \in [m, M]$.

The following dual result also holds:

Theorem 2 ([3]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f\colon [m, M] \to \mathbb{R}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality*

$$
\begin{aligned}
|f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| &\le \langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} \bigvee_m^s (f) \\
&\quad + \langle (1_H - E_s)x, x\rangle^{1/2} \langle (1_H - E_s)y, y\rangle^{1/2} \bigvee_s^M (f) \\
&\le \|x\| \|y\| \left(\frac{1}{2} \bigvee_m^M (f) + \frac{1}{2} \left| \bigvee_m^s (f) - \bigvee_s^M (f) \right| \right) \le \|x\| \|y\| \bigvee_m^M (f)
\end{aligned} \tag{1.7}
$$

for any $x, y \in H$ and for any $s \in [m, M]$, where 1_H is the identity operator on H.

Motivated by the above results, we investigate in the current paper the magnitude of the difference

$$
f\left(e^{is}\right)[u(b) - u(a)] - \int_a^b f\left(e^{it}\right) du(t) \quad \text{with} \quad s \in [a, b] \subseteq [0, 2\pi]
$$

for continuous complex valued function $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ defined on the complex unit circle $\mathcal{C}(0,1)$ and various subclasses of functions $u\colon [a, b] \subseteq [0, 2\pi] \to \mathbb{C}$ of bounded variation. Natural applications for functions of unitary operators in Hilbert spaces are provided as well.

2. Scalar Ostrowski's type inequalities

Theorem 3. *Assume that $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the following Hölder's type condition*

$$
|f(z) - f(w)| \le H |z - w|^r \tag{2.1}
$$

for any $w, z \in \mathcal{C}(0,1)$, where $H > 0$ and $r \in (0, 1]$ are given.

If $[a, b] \subseteq [0, 2\pi]$ and the function $u\colon [a, b] \to \mathbb{C}$ is of bounded variation on $[a, b]$, then

$$
\begin{aligned}
&\left| f(e^{is})[u(b) - u(a)] - \int_a^b f(e^{it})\, du(t) \right| \\
&\quad \le 2^r H \max_{t \in [a,b]} \left| \sin\left(\frac{s-t}{2}\right) \right|^r \bigvee_a^b (u)
\end{aligned} \tag{2.2}
$$

for any $s \in [a, b]$, where $\bigvee_a^b (u)$ denotes the total variation of u on the interval $[a, b]$.

Proof. Observe that

$$
f\left(e^{is}\right)[u(b) - u(a)] - \int_a^b f\left(e^{it}\right) du(t) = \int_a^b \left[f\left(e^{is}\right) - f\left(e^{it}\right)\right] du(t) \tag{2.3}
$$

for any $s \in [a, b]$.

It is known that if $p\colon\ [c,d]\to\mathbb{C}$ is a continuous function and $v\colon\ [c,d]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_c^d p(t)\,dv(t)\right|\le\max_{t\in[c,d]}|p(t)|\bigvee_c^d(v) \tag{2.4}$$

where $\bigvee_c^d(v)$ denotes the total variation of v on $[c,d]$.

Applying the property (2.4) to the identity (2.3) and utilizing the Hölder's type condition (2.1) we have successively

$$\begin{aligned}\left|f(e^{is})[u(b)-u(a)]-\int_a^b f(e^{it})\,du(t)\right| &= \max_{t\in[a,b]}\left|f\left(e^{is}\right)-f\left(e^{it}\right)\right|\bigvee_a^b(u)\\ &\le H\max_{t\in[a,b]}\left|e^{is}-e^{it}\right|^r\bigvee_a^b(u)\ .\end{aligned}\tag{2.5}$$

Since

$$\left|e^{is}-e^{it}\right|^2=\left|e^{is}\right|^2-2\operatorname{Re}\left(e^{i(s-t)}\right)+\left|e^{it}\right|^2=2-2\cos(s-t)=4\sin^2\left(\frac{s-t}{2}\right)$$

for any $t,\ s\in\mathbb{R}$, then

$$\left|e^{is}-e^{it}\right|^r=2^r\left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{2.6}$$

for any $t,\ s\in\mathbb{R}$.

Now, by (2.5) and (2.6) we deduce the desired result (2.2). □

Remark 1. If $a=0$ and $b=2\pi$, then for any $s\in[0,2\pi]$ there exists a unique $t\in[0,2\pi]$ such that $\frac{1}{2}|t-s|=\frac{\pi}{2}$, therefore $\max_{t\in[0,2\pi]}\left|\sin\left(\frac{s-t}{2}\right)\right|=1$ for all $s\in[0,2\pi]$ and we deduce from (2.2) the following inequality of interest

$$\left|f\left(e^{is}\right)[u(2\pi)-u(0)]-\int_0^{2\pi}f\left(e^{it}\right)\,du(t)\right|\le 2^rH\bigvee_0^{2\pi}(u) \tag{2.7}$$

that holds for each $s\in[0,2\pi]$.

Remark 2. If $[a,b]\subset[0,2\pi]$ and $0<b-a\le\pi$ then for all $t,s\in[a,b]$ we have $\frac{1}{2}|t-s|\le\frac{1}{2}(b-a)\le\frac{\pi}{2}$. Since the function sin is increasing on $\left[0,\frac{\pi}{2}\right]$, then we have successively that

$$\begin{aligned}\max_{t\in[a,b]}\left|\sin\left(\frac{s-t}{2}\right)\right| &= \sin\left(\max_{t\in[a,b]}\frac{1}{2}|t-s|\right)=\sin\left(\frac{1}{2}\max\{b-s,s-a\}\right)\\ &=\sin\left(\frac{1}{4}(b-a)+\frac{1}{2}\left|s-\frac{a+b}{2}\right|\right)\end{aligned}\tag{2.8}$$

for any $s\in[a,b]$.

Therefore, under the assumptions of Theorem 3 and if $[a,b] \subset [0,2\pi]$ with $0 < b-a \leq \pi$, then

$$\left|f(e^{is})[u(b)-u(a)] - \int_a^b f\left(e^{it}\right) du\left(t\right)\right|$$

$$\leq 2^r H \sin^r \left[\frac{1}{4}\left(b-a\right)+\frac{1}{2}\left|s-\frac{a+b}{2}\right|\right] \bigvee_a^b \left(u\right) \leq 2^r H \sin^r \left[\frac{1}{2}\left(b-a\right)\right] \bigvee_a^b \left(u\right) \tag{2.9}$$

for all $s \in [a,b]$.

In particular, the best inequality we can get from (2.9) is incorporated in (2.10)

$$\left|f\left(e^{\frac{a+b}{2}i}\right)\left[u\left(b\right)-u\left(a\right)\right] - \int_a^b f\left(e^{it}\right)\, du\left(t\right)\right| \leq 2^r H \sin^r \left[\frac{1}{4}\left(b-a\right)\right] \bigvee_a^b \left(u\right) .$$

The case when $f\colon \mathcal{C}\left(0,1\right) \to \mathbb{C}$ satisfies the Lipschitz condition $\left|f\left(z\right)-f\left(w\right)\right| \leq L\left|z-w\right|$ for any w, $z \in \mathcal{C}\left(0,1\right)$, where $L > 0$ is given, is of interest due to various examples one can consider. Also in this case we can show that the corresponding version of the inequality (2.11) is sharp.

Corollary 1. *Assume that $f\colon \mathcal{C}\left(0,1\right) \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on the circle $\mathcal{C}\left(0,1\right)$. If $[a,b] \subset [0,2\pi]$ with $0 < b-a \leq \pi$ and the function $u\colon [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then we have*

$$\left|f\left(e^{\frac{a+b}{2}i}\right)\left[u\left(b\right)-u\left(a\right)\right] - \int_a^b f\left(e^{it}\right) du\left(t\right)\right| \leq 2L \sin \left[\frac{1}{4}\left(b-a\right)\right] \bigvee_a^b \left(u\right) . \tag{2.11}$$

The constant 2 *cannot be replaced by a smaller quantity.*

Proof. We need to prove only the sharpness of the constant.

If we consider the function $f\colon \mathbb{C} \to \mathbb{C}$, $f\left(z\right) = z$, then obviously f is Lipschitzian with the constant $L = 1$. Also, consider in (2.11) $a = 0$ and $b = \pi$ to get

$$\left|i\left[u\left(\pi\right)-u\left(0\right)\right] - \int_0^\pi e^{it} du\left(t\right)\right| \leq \sqrt{2} \bigvee_0^\pi \left(u\right) . \tag{2.12}$$

Utilising the integration by parts formula for the Riemann-Stieltjes integral we have

$$\int_0^\pi e^{it} du\left(t\right) = e^{it} u\left(t\right)\big|_0^\pi - i\int_0^\pi e^{it} u\left(t\right)\, dt = -u\left(\pi\right)-u\left(0\right) - i\int_0^\pi e^{it} u\left(t\right)\, dt$$

and replacing into the inequality (2.12) we deduce

$$\left|i\left[u\left(\pi\right)-u\left(0\right)\right]+u\left(\pi\right)+u\left(0\right)+i\int_0^\pi e^{it} u\left(t\right)\, dt\right| \leq \sqrt{2} \bigvee_0^\pi \left(u\right)$$

which is equivalent with

$$\left|\left(i-1\right)u\left(\pi\right)+\left(i+1\right)u\left(0\right) - \int_0^\pi e^{it} u\left(t\right) dt\right| \leq \sqrt{2} \bigvee_0^\pi \left(u\right) \tag{2.13}$$

that holds for any functions of bounded variation $u\colon [0,\pi]\to\mathbb{C}$ and is of interest in itself.

Now, assume that there exists a constant $C>0$ such that

$$\left|(i-1)\,u(\pi)+(i+1)\,u(0)-\int_0^\pi e^{it}u(t)\,dt\right|\le C\bigvee_0^\pi(u) \tag{2.14}$$

for any functions of bounded variation $u\colon [0,\pi]\to\mathbb{C}$.

Consider the function $u\colon [0,\pi]\to\mathbb{R}$ with

$$u(t)=\begin{cases}0 & \text{if} \quad 0\le t<\pi\\ 1 & \text{if} \quad t=\pi\,.\end{cases}$$

Then u is of bounded variation, $\int_0^\pi e^{it}u(t)\,dt=0$, $\bigvee_0^\pi(u)=1$ and from (2.14) we get $C\ge\sqrt{2}$ showing that (2.14) is sharp and therefore (2.11) is sharp. □

Remark 3. The case of Riemann integral, namely when $u(t)=t$, $t\in[a,b]\subseteq[0,2\pi]$, is as follows

$$\left|f\left(e^{is}\right)-\frac{1}{b-a}\int_a^b f\left(e^{it}\right)\,du(t)\right|\le 2^rH\max_{t\in[a,b]}\left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{2.15}$$

for any $s\in[a,b]$ provided that $f\colon\mathcal{C}(0,1)\to\mathbb{C}$ satisfies the Hölder's type condition (2.1).

When u is an integral, then the following weighted integral inequality also holds.

Remark 4. If $w\colon [a,b]\subseteq[0,2\pi]\to\mathbb{C}$ is Lebesgue integrable on $[a,b]$ and $f\colon\mathcal{C}(0,1)\to\mathbb{C}$ satisfies the Hölder's type condition (2.1), then

$$\left|f\left(e^{is}\right)\int_a^b w(t)\,dt-\int_a^b f\left(e^{it}\right)w(t)\,dt\right|$$

$$\le 2^rH\max_{t\in[a,b]}\left|\sin\left(\frac{s-t}{2}\right)\right|^r\int_a^b|w(t)|\,dt \tag{2.16}$$

for any $s\in[a,b]$.

In particular, if $w(t)\ge0$ for $t\in[a,b]$ and $\int_a^b w(t)\,dt>0$ then

$$\left|f\left(e^{is}\right)-\frac{1}{\int_a^b w(t)\,dt}\int_a^b f\left(e^{it}\right)w(t)\,dt\right|\le 2^rH\max_{t\in[a,b]}\left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{2.17}$$

for any $s\in[a,b]$.

Theorem 4. *Assume that $f\colon\mathcal{C}(0,1)\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on the circle $\mathcal{C}(0,1)$. If $[a,b]\subseteq[0,2\pi]$ and the function $u\colon [a,b]\to\mathbb{C}$ is Lipschitzian with the constant $K>0$ on $[a,b]$, then*

$$\left|f\left(e^{is}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)\,du(t)\right|\le 4LK\left[\sin^2\left(\frac{s-a}{4}\right)+\sin^2\left(\frac{b-s}{4}\right)\right]$$

$$\le 8LK\sin^2\left(\frac{b-a}{4}\right) \tag{2.18}$$

for any $s \in [a, b]$.

Proof. It is well known that if $p\colon\ [a,b] \to \mathbb{C}$ is a Riemann integrable function and $v\colon\ [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $M > 0$, i.e.,

$$|f(s) - f(t)| \le M\,|s-t| \text{ for any } t, s \in [a,b]\ ,$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\ dv(t)\right| \le M \int_a^b |p(t)|\ dt\,. \tag{2.19}$$

Utilising the property (2.19), we have from (2.3) that

$$\left|f\left(e^{is}\right)[u(b)-u(a)] - \int_a^b f\left(e^{it}\right)\ du(t)\right| = \left|\int_a^b \left[f\left(e^{is}\right) - f\left(e^{it}\right)\right]\ du(t)\right|$$

$$\le K\int_a^b \left|f\left(e^{is}\right) - f\left(e^{it}\right)\right|\ dt \le KL\int_a^b \left|e^{is} - e^{it}\right|\ dt \tag{2.20}$$

for any $s \in [a,b]$.

Since, by (2.6), $\left|e^{is} - e^{it}\right| = 2\left|\sin\left(\frac{s-t}{2}\right)\right|$ for any $t,\ s \in \mathbb{R}$, then

$$\begin{aligned}\int_a^b \left|e^{is} - e^{it}\right|\ dt &= 2\int_a^b \left|\sin\left(\frac{s-t}{2}\right)\right| dt\\ &= 2\Big[\int_a^s \sin\left(\frac{s-t}{2}\right)dt + \int_s^b \sin\left(\frac{t-s}{2}\right)dt\Big]\\ &= 2\Big[1-\cos\left(\frac{s-a}{2}\right)\Big] + 2\Big[1-\cos\left(\frac{b-s}{2}\right)\Big]\\ &= 4\Big[\sin^2\left(\frac{s-a}{4}\right) + \sin^2\left(\frac{b-s}{4}\right)\Big] \le 8\sin^2\left(\frac{b-a}{4}\right)\end{aligned} \tag{2.21}$$

for any $s \in [a,b] \subseteq [0, 2\pi]$, and the inequality (2.18) is proved. □

The best inequality we can get from (2.18) is incorporated in

Corollary 2. *With the assumptions in Theorem 4 we have the inequality*

$$\left|f\left(e^{\frac{a+b}{2}i}\right)[u(b)-u(a)] - \int_a^b f\left(e^{it}\right)\ du(t)\right| \le 8LK\sin^2\left(\frac{b-a}{8}\right). \tag{2.22}$$

The multiplicative constant 8 *cannot be replaced by a smaller quantity.*

Proof. We need to prove only the sharpness of the constant.

If we consider the function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z$, then obviously f is Lipschitzian with the constant $L = 1$. Also, consider in (2.22) $a = 0$ and $b = 2\pi$ to get

$$\left| -[u(2\pi) - u(0)] - \int_0^{2\pi} e^{it} du(t)\right| \le 4K\,. \tag{2.23}$$

Utilising the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\int_0^{2\pi} e^{it} du(t) = e^{it}u(t)\Big|_0^{2\pi} - i\int_0^{2\pi} e^{it}u(t)\ dt = u(2\pi) - u(0) - i\int_0^{2\pi} e^{it}u(t)\ dt\,,$$

which inserted in (2.23) produces the inequality

$$\Big| -2\left[u(2\pi)-u(0)\right]+i\int_0^{2\pi} e^{it}u(t)\,dt\Big| \le 4K$$

which is equivalent with

$$\Big|\int_0^{2\pi} e^{it}u(t)\,dt-\frac{2}{i}\left[u(2\pi)-u(0)\right]\Big| \le 4K \tag{2.24}$$

that holds for any K-Lipschitzian function $u\colon [0,2\pi]\to\mathbb{C}$ and is of interest in itself.

Now, assume that the inequality (2.24) holds with a constant $D>0$, namely

$$\Big|\int_0^{2\pi} e^{it}u(t)\,dt-\frac{2}{i}\left[u(2\pi)-u(0)\right]\Big| \le DK \tag{2.25}$$

for any K-Lipschitzian function $u\colon [0,2\pi]\to\mathbb{C}$.

Consider $u\colon [0,2\pi]\to\mathbb{R}$, $u(t)=|t-\pi|$. Then, by the continuity property of the modulus we have that u is Lipschitzian with the constant $K=1$.

We also have that

$$\begin{aligned}\int_0^{2\pi} e^{it}u(t)\,dt &= \int_0^{2\pi} e^{it}|t-\pi|\,dt=\int_0^{2\pi}|t-\pi|\left(\cos t+i\sin t\right)dt\\ &=\int_0^{2\pi}|t-\pi|\cos t\,dt+i\int_0^{2\pi}|t-\pi|\sin t\,dt\,.\end{aligned}$$

Observe that by symmetry reasons $\int_0^{2\pi}|t-\pi|\sin t\,dt=0$ and

$$\int_0^{2\pi}|t-\pi|\cos t\,dt=2\int_0^{\pi}(\pi-t)\cos t\,dt=2\Big[(\pi-t)\sin t\big|_0^{\pi}+\int_0^{\pi}\sin t\,dt\Big]=4$$

and by (2.25) we get $D\ge 4$ which proves the desired sharpness of the constant 8 in (2.22). □

Remark 5. If $u(t)=t$, $t\in[a,b]$, then we get from (2.18) and (2.22) the following inequalities for the Riemann integral

$$\begin{aligned}\Big|f\left(e^{is}\right)(b-a)-\int_a^b f\left(e^{it}\right)dt\Big| &\le 4L\Big[\sin^2\Big(\frac{s-a}{4}\Big)+\sin^2\Big(\frac{b-s}{4}\Big)\Big]\\ &\le 8L\sin^2\Big(\frac{b-a}{4}\Big)\end{aligned} \tag{2.26}$$

for any $s\in[a,b]$ and

$$\Big|f\Big(e^{\frac{a+b}{2}i}\Big)(b-a)-\int_a^b f\left(e^{it}\right)dt\Big| \le 8L\sin^2\Big(\frac{b-a}{8}\Big) \tag{2.27}$$

provided that $f\colon \mathcal{C}(0,1)\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on the circle $\mathcal{C}(0,1)$.

Remark 6. If $w\colon [a,b]\subseteq[0,2\pi]\to\mathbb{C}$ is essentially bounded on $[a,b]$ and $f\colon \mathcal{C}(0,1)\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on the circle $\mathcal{C}(0,1)$, then we have the following weighted integral inequality

$$\begin{aligned}\left|f\left(e^{is}\right)\int_a^b w(t)\,dt-\int_a^b f\left(e^{it}\right)w(t)\,dt\right| &\le 4L\|w\|_\infty\left[\sin^2\left(\frac{s-a}{4}\right)+\sin^2\left(\frac{b-s}{4}\right)\right]\\ &\le 8L\|w\|_\infty\sin^2\left(\frac{b-a}{4}\right)\end{aligned}\tag{2.28}$$

for any $s\in[a,b]$ where $\|w\|_\infty:=\operatorname{ess\,sup}_{t\in[a,b]}|w(t)|$.

In particular, we have

$$\left|f\left(e^{\frac{a+b}{2}i}\right)\int_a^b w(t)\,dt-\int_a^b f\left(e^{it}\right)w(t)\,dt\right|\le 8L\|w\|_\infty\sin^2\left(\frac{b-a}{8}\right).\tag{2.29}$$

The case of monotonic integrators is as follows:

Theorem 5. *Assume that $f\colon\mathcal{C}(0,1)\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on the circle $\mathcal{C}(0,1)$. If $[a,b]\subseteq[0,2\pi]$ and the function $u\colon [a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then*

$$\begin{aligned}\left|f\left(e^{is}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)du(t)\right| &\le 2L\left[\sin\left(\frac{b-s}{2}\right)u(b)-\sin\left(\frac{s-a}{2}\right)u(a)\right]\\ &\quad+L\int_a^b\operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt\end{aligned}\tag{2.30}$$

for any $s\in[a,b]$.

In particular, we have

$$\begin{aligned}\left|f\left(e^{\frac{a+b}{2}i}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)du(t)\right| &\le 2L\sin\left(\frac{b-a}{4}\right)[u(b)-u(a)]\\ &\quad+L\int_a^b\operatorname{sgn}\left(\frac{a+b}{2}-t\right)\cos\left(\frac{\frac{a+b}{2}-t}{2}\right)u(t)\,dt.\end{aligned}\tag{2.31}$$

Proof. It is well known that if $p\colon [a,b]\to\mathbb{C}$ is a continuous function and $v\colon [a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\le\int_a^b|p(t)|\,dv(t).\tag{2.32}$$

Utilising the property (2.32), we have from (2.3) that

$$\begin{aligned}\left|f\left(e^{is}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)du(t)\right| &= \left|\int_a^b\left[f\left(e^{is}\right)-f\left(e^{it}\right)\right]du(t)\right|\\ &\le\int_a^b\left|f\left(e^{is}\right)-f\left(e^{it}\right)\right|du(t)\le L\int_a^b\left|e^{is}-e^{it}\right|du(t)\end{aligned}\tag{2.33}$$

for any $s\in[a,b]$.

Since, by (2.6), $\left|e^{is}-e^{it}\right| = 2\left|\sin\left(\frac{s-t}{2}\right)\right|$ for any $t, s \in \mathbb{R}$, then

$$\begin{aligned} \int_a^b \left|e^{is}-e^{it}\right| du(t) &= 2\int_a^b \left|\sin\left(\frac{s-t}{2}\right)\right| du(t) \\ &= 2\Big[\int_a^s \sin\left(\frac{s-t}{2}\right) du(t) + \int_s^b \sin\left(\frac{t-s}{2}\right) du(t)\Big] \end{aligned} \tag{2.34}$$

for any $s \in [a,b] \subseteq [0, 2\pi]$.

Utilising the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\begin{aligned} \int_a^s \sin\left(\frac{s-t}{2}\right) du(t) &= \sin\left(\frac{s-t}{2}\right)u(t)\Big|_a^s + \frac{1}{2}\int_a^s \cos\left(\frac{s-t}{2}\right)u(t)\, dt \\ &= -\sin\left(\frac{s-a}{2}\right)u(a) + \frac{1}{2}\int_a^s \cos\left(\frac{s-t}{2}\right)u(t)\, dt \end{aligned}$$

and

$$\begin{aligned} \int_s^b \sin\left(\frac{t-s}{2}\right) du(t) &= \sin\left(\frac{t-s}{2}\right)u(t)\Big|_s^b - \frac{1}{2}\int_s^b \cos\left(\frac{t-s}{2}\right)u(t)\, dt \\ &= \sin\left(\frac{b-s}{2}\right)u(b) - \frac{1}{2}\int_s^b \cos\left(\frac{t-s}{2}\right)u(t)\, dt\,, \end{aligned}$$

which, by (2.34), produce the equality

$$\begin{aligned} \int_a^b \left|e^{is}-e^{it}\right| du(t) &= 2\Big[\sin\left(\frac{b-s}{2}\right)u(b) - \sin\left(\frac{s-a}{2}\right)u(a)\Big] \\ &\quad + \int_a^s \cos\left(\frac{s-t}{2}\right)u(t)\, dt - \int_s^b \cos\left(\frac{t-s}{2}\right)u(t)\, dt \\ &= 2\Big[\sin\left(\frac{b-s}{2}\right)u(b) - \sin\left(\frac{s-a}{2}\right)u(a)\Big] \\ &\quad + \int_a^b \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\, dt\,. \end{aligned} \tag{2.35}$$

Utilising (2.33) we deduce the desired result (2.30). □

Remark 7. We remark that if $a = 0$ and $b = 2\pi$, then we get from (2.30) and (2.31) that

$$\begin{aligned} \left| f\left(e^{is}\right)\left[u(2\pi) - u(0)\right] - \int_0^{2\pi} f\left(e^{it}\right) du(t) \right| &\le 2L\sin\left(\frac{s}{2}\right)\left[u(2\pi) - u(0)\right] \\ &\quad + L\int_0^{2\pi} \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\, dt \end{aligned} \tag{2.36}$$

for any $s \in [a,b]$.

In particular, we have

$$\left|f(-1)\left[u(2\pi)-u(0)\right]-\int_0^{2\pi} f\left(e^{it}\right)\,du(t)\right| \le \sqrt{2}L\left[u(2\pi)-u(0)\right]$$
$$+L\int_0^{2\pi}\operatorname{sgn}(\pi-t)\sin\left(\frac{t}{2}\right)u(t)\,dt\,. \tag{2.37}$$

Corollary 3. *Assume that f and u are as in Theorem 5 then for any $[a,b]\subset[0,2\pi]$ with $0<b-a\le\pi$ we have the sequence of inequalities*

$$\left|f(e^{is})[u(b)-u(a)]-\int_a^b f(e^{it})\,du(t)\right| \le 2L\left[\sin\left(\frac{b-s}{2}\right)u(b)-\sin\left(\frac{s-a}{2}\right)u(a)\right]$$
$$+L\int_a^b\operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt$$
$$\le 2L\left[\sin\left(\frac{b-s}{2}\right)\left[u(b)-u(s)\right]+\sin\left(\frac{s-a}{2}\right)\left[u(s)-u(a)\right]\right]=:B(s) \tag{2.38}$$

where

$$B(s)\le 2L\times\begin{cases}\sin\left[\frac{1}{4}(b-a)+\frac{1}{2}\left|s-\frac{a+b}{2}\right|\right]\left[u(b)-u(a)\right]\\ 2\sin\left(\frac{b-a}{4}\right)\cos\left(\frac{s-\frac{a+b}{2}}{2}\right)\left[\frac{u(b)-u(a)}{2}+\left|u(s)-\frac{u(b)+u(a)}{2}\right|\right]\end{cases}$$

for any $s\in[a,b]$.

In particular, we have

$$\left|f\left(e^{\frac{a+b}{2}i}\right)\left[u(b)-u(a)\right]-\int_a^b f\left(e^{it}\right)\,du(t)\right|\le 2L\sin\left(\frac{b-a}{4}\right)\left[u(b)-u(a)\right]$$
$$+L\int_a^b\operatorname{sgn}\left(\frac{a+b}{2}-t\right)\cos\left(\frac{\frac{a+b}{2}-t}{2}\right)u(t)\,dt=:M \tag{2.39}$$

where

$$M\le 2L\sin\left(\frac{b-a}{4}\right)\left[u(b)-u(a)\right]\,.$$

Proof. Since $0<b-a\le\pi$, then $\frac{|s-t|}{2}\le\frac{\pi}{2}$ for $s,t\in[a,b]$. Utilising the fact that u is monotonic nondecreasing on $[a,b]$ and $\cos\left(\frac{|s-t|}{2}\right)\ge 0$ for $s,t\in[a,b]$, then

$$\int_a^s\cos\left(\frac{s-t}{2}\right)u(t)\,dt\le u(s)\int_a^s\cos\left(\frac{s-t}{2}\right)dt=2u(s)\sin\left(\frac{s-a}{2}\right) \tag{2.40}$$

and

$$\int_s^b\cos\left(\frac{s-t}{2}\right)u(t)\,dt\ge u(s)\int_s^b\cos\left(\frac{s-t}{2}\right)dt=2u(s)\sin\left(\frac{b-s}{2}\right)$$

i.e.,

$$-\int_s^b\cos\left(\frac{s-t}{2}\right)u(t)\,dt\le -2u(s)\sin\left(\frac{b-s}{2}\right). \tag{2.41}$$

Summing (2.40) with (2.41) we deduce that

$$\int_a^b \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt \le 2u(s)\sin\left(\frac{s-a}{2}\right) - 2u(s)\sin\left(\frac{b-s}{2}\right)$$

giving that

$$2L\left[\sin\left(\frac{b-s}{2}\right)u(b) - \sin\left(\frac{s-a}{2}\right)u(a)\right] + L\int_a^b \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt$$
$$\le 2L\left[\sin\left(\frac{b-s}{2}\right)[u(b)-u(s)] + \sin\left(\frac{s-a}{2}\right)[u(s)-u(a)]\right],$$

which proves the second inequality in (2.38).

The bounds for $B(s)$ follows from the elementary property stating that

$$\alpha x + \beta y \le \max\{\alpha, \beta\}(x+y)$$

where α, β, x, $y \ge 0$. □

3. A quadrature rule

We consider the following partition of the interval $[a,b]$

$$\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and the intermediate points $\xi_k \in [x_k, x_{k+1}]$ where $0 \le k \le n-1$. Define $h_k := x_{k+1} - x_k$, $0 \le k \le n-1$ and $\nu(\Delta_n) = \max\{h_k : 0 \le k \le n-1\}$ the norm of the partition Δ_n.

For the continuous function $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ and the function $u\colon\ [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ of bounded variation on $[a,b]$, define the quadrature rule

$$O_n(f,u,\Delta_n,\xi) := \sum_{k=0}^{n-1} f\left(e^{i\xi_k}\right)[u(x_{k+1}) - u(x_k)] \tag{3.1}$$

and the remainder $R_n(f,u,\Delta_n,\xi)$ in approximating the Riemann-Stieltjes integral $\int_a^b f\left(e^{it}\right) du(t)$ by $O_n(f,u,\Delta_n,\xi)$. Then we have

$$\int_a^b f\left(e^{it}\right) du(t) = O_n(f,u,\Delta_n,\xi) + R_n(f,u,\Delta_n,\xi)\ . \tag{3.2}$$

The following result provides a priory bounds for $R_n(f,u,\Delta_n,\xi)$ in several instances of f and u as above.

Proposition 1. *Assume that $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the following Hölder's type condition*

$$|f(z) - f(w)| \le H|z-w|^r$$

for any w, $z \in \mathcal{C}(0,1)$, where $H > 0$ and $r \in (0,1]$ are given.

If $[a,b] \subseteq [0,2\pi]$ and the function $u\colon\ [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then for any partition $\Delta_n\colon a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ with the norm

$\nu(\Delta_n) \leq \pi$ we have the error bound

$$|R_n(f,u,\Delta_n,\xi)| \leq 2^r H \sum_{k=0}^{n-1} \sin^r \left[\frac{1}{4}(x_{k+1}-x_k) + \frac{1}{2}\left|\xi_k - \frac{x_k+x_{k+1}}{2}\right|\right] \bigvee_{x_k}^{x_{k+1}}(u)$$
$$\leq 2^r H \sum_{k=0}^{n-1} \sin^r \left[\frac{1}{2}(x_{k+1}-x_k)\right] \bigvee_{x_k}^{x_{k+1}}(u)$$
$$\leq H \sum_{k=0}^{n-1} (x_{k+1}-x_k)^r \bigvee_{x_k}^{x_{k+1}}(u) \leq H\nu^r(\Delta_n) \bigvee_a^b (u) \tag{3.3}$$

for any intermediate points $\xi_k \in [x_k, x_{k+1}]$ where $0 \leq k \leq n-1$.

Proof. Since $\nu(\Delta_n) \leq \pi$, then on writing inequality (2.9) on each interval $[x_k, x_{k+1}]$ and for any intermediate points $\xi_k \in [x_k, x_{k+1}]$ where $0 \leq k \leq n-1$, we have

$$\left| f(e^{i\xi_k})[u(x_{k+1}) - u(x_k)] - \int_{x_k}^{x_{k+1}} f\left(e^{it}\right) du(t) \right|$$
$$\leq 2^r H \sin^r \left[\frac{1}{4}(x_{k+1}-x_k) + \frac{1}{2}\left|\xi_k - \frac{x_k+x_{k+1}}{2}\right|\right] \bigvee_{x_k}^{x_{k+1}}(u)$$
$$\leq 2^r H \sin^r \left[\frac{1}{2}(x_{k+1}-x_k)\right] \bigvee_{x_k}^{x_{k+1}}(u) \leq H(x_{k+1}-x_k)^r \bigvee_{x_k}^{x_{k+1}}(u) \tag{3.4}$$

where for the last inequality we have used the fact that $\sin x \leq x$ for $x \in \left[0, \frac{\pi}{2}\right]$.

Summing over k from 0 to $n-1$ in (3.4) and utilizing the generalized triangle inequality, we deduce the first part of (3.3). The second part is obvious. □

Corollary 4. *Assume that f, u and Δ_n are as in Proposition 1. Define the midpoint trapezoid type quadrature rule by*

$$T_n(f,u,\Delta_n) := \sum_{k=0}^{n-1} f\left(e^{\frac{x_{k+1}+x_k}{2}i}\right)[u(x_{k+1}) - u(x_k)] \tag{3.5}$$

and the error $E_n(f,u,\Delta_n)$ by

$$\int_a^b f\left(e^{it}\right) du(t) = T_n(f,u,\Delta_n) + E_n(f,u,\Delta_n) . \tag{3.6}$$

Then we have the error bounds

$$|E_n(f,u,\Delta_n)| \leq 2^r H \sum_{k=0}^{n-1} \sin^r \left[\frac{1}{4}(x_{k+1}-x_k)\right] \bigvee_{x_k}^{x_{k+1}}(u)$$
$$\leq \frac{1}{2^r} H \sum_{k=0}^{n-1} (x_{k+1}-x_k)^r \bigvee_{x_k}^{x_{k+1}}(u) \leq \frac{1}{2^r} H\nu^r(\Delta_n) \bigvee_a^b (u) . \tag{3.7}$$

The case of both integrator and integrand being Lipschitzian is incorporated in the following result:

Proposition 2. *Assume that $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on the circle $\mathcal{C}(0,1)$. If $[a,b] \subseteq [0,2\pi]$ and the function $u\colon [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $K > 0$ on $[a,b]$, then for any partition $\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ we have the error bound*

$$
\begin{aligned}
|R_n(f,u,\Delta_n,\xi)| &\le 4LK\sum_{k=0}^{n-1}\left[\sin^2\left(\frac{\xi_k - x_k}{4}\right) + \sin^2\left(\frac{x_{k+1}-\xi_k}{4}\right)\right] \\
&\le 8LK\sum_{k=0}^{n-1}\sin^2\left(\frac{x_{k+1}-x_k}{4}\right) \le \frac{1}{2}LK\sum_{k=0}^{n-1}(x_{k+1}-x_k)^2 \\
&\le \frac{1}{2}LK(b-a)\,\nu(\Delta_n) \qquad (3.8)
\end{aligned}
$$

for any intermediate points $\xi_k \in [x_k, x_{k+1}]$ where $0 \le k \le n-1$.

In particular, we have

$$
\begin{aligned}
|E_n(f,u,\Delta_n)| &\le 8LK\sum_{k=0}^{n-1}\sin^2\left(\frac{x_{k+1}-x_k}{8}\right) \\
&\le \frac{1}{8}LK\sum_{k=0}^{n-1}(x_{k+1}-x_k)^2 \le \frac{1}{8}LK(b-a)\,\nu(\Delta_n)\,. \qquad (3.9)
\end{aligned}
$$

The proof follows by Theorem 4 and the details are omitted.

Proposition 3. *Assume that $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on the circle $\mathcal{C}(0,1)$. If $[a,b] \subseteq [0,2\pi]$ and the function $u\colon [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then for any partition $\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ with the norm $\nu(\Delta_n) \le \pi$ we have the error bound*

$$
\begin{aligned}
|R_n(f,u,\Delta_n,\xi)| \le{}& 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{x_{k+1}-\xi_k}{2}\right)u(x_{k+1}) - \sin\left(\frac{\xi_k - x_k}{2}\right)u(x_k)\right] \\
&+ L\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}\operatorname{sgn}(\xi_k - t)\cos\left(\frac{\xi_k - t}{2}\right)u(t)\,dt \\
\le{}& 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{x_{k+1}-\xi_k}{2}\right)[u(x_{k+1}) - u(\xi_k)]\right. \\
&\left. + \sin\left(\frac{\xi_k - x_k}{2}\right)[u(\xi_k) - u(x_k)]\right] \\
\le{}& 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{4}(x_{k+1}-x_k) + \frac{1}{2}\left|\xi_k - \frac{x_k + x_{k+1}}{2}\right|\right][u(x_{k+1}) - u(x_k)] \\
\le{}& 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{2}(x_{k+1}-x_k)\right][u(x_{k+1}) - u(x_k)]
\end{aligned}
$$

$$\leq L \sum_{k=0}^{n-1} (x_{k+1} - x_k) [u(x_{k+1}) - u(x_k)] \leq \nu(\Delta_n) L [u(b) - u(a)] \tag{3.10}$$

for any intermediate points $\xi_k \in [x_k, x_{k+1}]$ *where* $0 \leq k \leq n-1$.

In particular, we have

$$\begin{aligned} |E_n(f,u,\Delta_n)| &\leq 2L \sum_{k=0}^{n-1} \sin\left(\frac{x_{k+1}-x_k}{4}\right) [u(x_{k+1}) - u(x_k)] \\ &\quad + L \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \operatorname{sgn}\left(\frac{x_k + x_{k+1}}{2} - t\right) \cos\left(\frac{\frac{x_k+x_{k+1}}{2} - t}{2}\right) u(t)\, dt \\ &\leq 2L \sum_{k=0}^{n-1} \sin\left(\frac{x_{k+1}-x_k}{4}\right) [u(x_{k+1}) - u(x_k)] \\ &\leq \frac{1}{2} L \sum_{k=0}^{n-1} (x_{k+1} - x_k) [u(x_{k+1}) - u(x_k)] \leq \frac{1}{2} L\nu(\Delta_n) [u(b) - u(a)] . \end{aligned} \tag{3.11}$$

The proof follows by Corollary 3 and the details are omitted.

4. Applications for functions of unitary operators

We recall that the bounded linear operator U on the Hilbert space H is unitary iff $U^* = U^{-1}$.

It is well known that (see for instance [5, p. 275-p. 276]), if U is a unitary operator, then there exists a family of projections $\{E_\lambda\}_{\lambda \in [0,2\pi]}$, called the spectral family of U with the following properties

a) $E_\lambda \leq E_\mu$ for $0 \leq \lambda \leq \mu \leq 2\pi$;

b) $E_0 = 0$ and $E_{2\pi} = 1_H$ (the identity operator on H);

c) $E_{\lambda+0} = E_\lambda$ for $0 \leq \lambda < 2\pi$;

d) $U = \int_0^{2\pi} e^{i\lambda} dE_\lambda$ where the integral is of Riemann-Stieltjes type.

Moreover, if $\{F_\lambda\}_{\lambda \in [0,2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U, then $F_\lambda = E_\lambda$ for all $\lambda \in [0, 2\pi]$.

Also, for every continuous complex valued function $f : \mathcal{C}(0,1) \to \mathbb{C}$ on the complex unit circle, we have

$$f(U) = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda \tag{4.1}$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$f(U)x = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda x\,, \tag{4.2}$$

$$\langle f(U)x, y\rangle = \int_0^{2\pi} f\left(e^{i\lambda}\right) d\langle E_\lambda x, y\rangle \tag{4.3}$$

and

$$\left\| f\left(U\right) x\right\|^{2}=\int_{0}^{2\pi}\left|f\left(e^{i\lambda}\right)\right|^{2} d\left\|E_{\lambda}x\right\|^{2}, \tag{4.4}$$

for any $x,\ y\in H$.

We consider the following partition of the interval $[a,b]$

$$\Delta_n : 0=\lambda_0<\lambda_1<\cdots<\lambda_{n-1}<\lambda_n=2\pi$$

and the intermediate points $\xi_k\in[\lambda_k,\lambda_{k+1}]$ where $0\le k\le n-1$. Define $h_k := \lambda_{k+1}-\lambda_k$, $0\le k\le n-1$ and $\nu\left(\Delta_n\right)=\max\left\{h_k : 0\le k\le n-1\right\}$ the norm of the partition Δ_n.

If U is a unitary operator on the Hilbert space H and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, the spectral family of U, then we can introduce the following sums

$$O_n\left(f,U,\Delta_n,\xi;x,y\right):=\sum_{k=0}^{n-1} f\left(e^{i\xi_k}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,y\right\rangle \tag{4.5}$$

and

$$T_n\left(f,U,\Delta_n;x,y\right):=\sum_{k=0}^{n-1} f\left(e^{\frac{\lambda_{k+1}+\lambda_k}{2}i}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,y\right\rangle \tag{4.6}$$

where $x,\ y\in H$.

Theorem 6. *With the above assumptions for $U,\{E_\lambda\}_{\lambda\in[0,2\pi]},\Delta_n$ with $\nu\left(\Delta_n\right)\le\pi$ and if $f\colon\mathcal{C}(0,1)\to\mathbb{C}$ satisfies the Hölder's type condition $\left|f(z)-f(w)\right|\le H\left|z-w\right|^r$ for any $w,\ z\in\mathcal{C}(0,1)$, where $H>0$ and $r\in(0,1]$ are given, then we have the representation*

$$\left\langle f\left(U\right)x,y\right\rangle=O_n\left(f,U,\Delta_n,\xi;x,y\right)+R_n\left(f,U,\Delta_n,\xi;x,y\right) \tag{4.7}$$

with the error $R_n\left(f,U,\Delta_n,\xi;x,y\right)$ satisfying the bounds

$$\begin{aligned}
&\left|R_n(f,U,\Delta_n,\xi;x,y)\right|\\
&\le 2^r H\sum_{k=0}^{n-1}\sin^r\left[\frac{1}{4}\left(\lambda_{k+1}-\lambda_k\right)+\frac{1}{2}\left|\xi_k-\frac{\lambda_k+\lambda_{k+1}}{2}\right|\right]\bigvee_{\lambda_k}^{\lambda_{k+1}}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\
&\le 2^r H\sum_{k=0}^{n-1}\sin^r\left[\frac{1}{2}\left(\lambda_{k+1}-\lambda_k\right)\right]\bigvee_{\lambda_k}^{\lambda_{k+1}}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\
&\le H\sum_{k=0}^{n-1}\left(\lambda_{k+1}-\lambda_k\right)^r\bigvee_{\lambda_k}^{\lambda_{k+1}}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\le H\nu^r\left(\Delta_n\right)\bigvee_{0}^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\
&\le H\nu^r\left(\Delta_n\right)\left\|x\right\|\left\|y\right\| \qquad (4.8)
\end{aligned}$$

for any $x,\ y\in H$ and the intermediate points $\xi_k\in[\lambda_k,\lambda_{k+1}]$ where $0\le k\le n-1$.

In particular we have

$$\left\langle f\left(U\right)x,y\right\rangle=T_n\left(f,U,\Delta_n;x,y\right)+E_n\left(f,U,\Delta_n;x,y\right) \tag{4.9}$$

with the error

$$\begin{aligned}
&\left|E_n\left(f,U,\Delta_n;x,y\right)\right| \\
&\quad\leq 2^r H \sum_{k=0}^{n-1} \sin^r\left[\frac{1}{4}\left(\lambda_{k+1}-\lambda_k\right)\right] \bigvee_{\lambda_k}^{\lambda_{k+1}}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\quad\leq \frac{1}{2^r} H \sum_{k=0}^{n-1} \left(\lambda_{k+1}-\lambda_k\right)^r \bigvee_{\lambda_k}^{\lambda_{k+1}}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\quad\leq \frac{1}{2^r} H \nu^r\left(\Delta_n\right) \bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{1}{2^r} H \nu^r\left(\Delta_n\right)\|x\|\,\|y\| \qquad (4.10)
\end{aligned}$$

for any x, $y \in H$.

Proof. For given $x, y \in H$, define the function $u(\lambda) := \langle E_\lambda x, y\rangle$, $\lambda \in [0,2\pi]$. We will show that u is of bounded variation and

$$\bigvee_0^{2\pi}(u) =: \bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \|x\|\,\|y\| \ . \qquad (4.11)$$

It is well known that, if P is a nonnegative selfadjoint operator on H, i.e., $\langle Px, x\rangle \geq 0$ for any $x \in H$, then the following inequality is a *generalization of the Schwarz inequality* in H

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle\,\langle Py, y\rangle \ , \qquad (4.12)$$

for any x, $y \in H$.

Now, if d: $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 2\pi$ is an arbitrary partition of the interval $[0, 2\pi]$, then we have by Schwarz's inequality for nonnegative operators (4.12) that

$$\begin{aligned}
&\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) = \sup_d\left\{\sum_{i=0}^{n-1}\left|\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,y\right\rangle\right|\right\} \\
&\leq \sup_d\left\{\sum_{i=0}^{n-1}\left[\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle^{1/2}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle^{1/2}\right]\right\} := I\,. \qquad (4.13)
\end{aligned}$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}
I &\leq \sup_d\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2} \\
&\leq \sup_d\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\sup_d\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2} \\
&= \left[\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,x\right\rangle\right)\right]^{1/2}\left[\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}y,y\right\rangle\right)\right]^{1/2} = \|x\|\,\|y\| \qquad (4.14)
\end{aligned}$$

for any x, $y \in H$.

On making use of (4.13) and (4.14) we deduce the desired result (4.11).

Now, applying Proposition 1 to the spectral representation (4.3) we deduce the desired result (4.7) with the error bound (4.8). The details are omitted. □

Remark 8. In the case when the partition reduces to the whole interval $[0, 2\pi]$, then utilizing the inequality (2.7) we deduce the bound

$$\left|f\left(e^{is}\right)\langle x,y\rangle-\langle f(U)x,y\rangle\right|\leq 2^{r}H\bigvee_{0}^{2\pi}\left(\langle E_{(\cdot)}x,y\rangle\right)\leq 2^{r}H\|x\|\|y\| \tag{4.15}$$

for any $s\in[0,2\pi]$ and any vectors $x, y\in H$.

In the case when the division is

$$\Delta_2 : 0=\lambda_0<\lambda_1=\pi<\lambda_2=2\pi$$

and we take the intermediate points $u\in[0,\pi]$ and $v\in[\pi,2\pi]$, then we get from Theorem 6 that

$$\begin{aligned}&\left|f(e^{iu})\langle E_{\pi}x,y\rangle+f(e^{iv})\langle(I-E_{\pi})x,y\rangle-\langle f(U)x,y\rangle\right|\\&\qquad\leq 2^{r}H\Big[\sin^{r}\left[\frac{1}{4}\pi+\frac{1}{2}\Big|u-\frac{\pi}{2}\Big|\right]\bigvee_{0}^{\pi}(\langle E_{(\cdot)}x,y\rangle)\\&\qquad\qquad+\sin^{r}\left[\frac{1}{4}\pi+\frac{1}{2}\Big|v-\frac{3\pi}{2}\Big|\right]\bigvee_{\pi}^{2\pi}\left(\langle E_{(\cdot)}x,y\rangle\right)\Big]\end{aligned} \tag{4.16}$$

for any vectors $x, y\in H$.

The best inequality we can get from (4.17) is obtained for $u=\frac{\pi}{2}$ and $v=\frac{3\pi}{2}$, namely

$$\begin{aligned}&\left|f(i)\langle E_{\pi}x,y\rangle+f(-i)\langle(1_H-E_{\pi})x,y\rangle-\langle f(U)x,y\rangle\right|\\&\qquad\leq 2^{\frac{r}{2}}H\bigvee_{0}^{2\pi}\left(\langle E_{(\cdot)}x,y\rangle\right)\leq 2^{\frac{r}{2}}H\|x\|\|y\|\end{aligned} \tag{4.17}$$

for any vectors $x, y\in H$.

If U is a unitary operator on the Hilbert space H and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, the spectral family of U, then we can introduce the following sums depending only of one vector $x\in H$

$$\tilde{O}_n(f,U,\Delta_n,\xi;x):=\sum_{k=0}^{n-1}f\left(e^{i\xi_k}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \tag{4.18}$$

and

$$\tilde{T}_n(f,U,\Delta_n;x,y):=\sum_{k=0}^{n-1}f\left(e^{\frac{\lambda_{k+1}+\lambda_k}{2}i}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle . \tag{4.19}$$

Theorem 7. *With the above assumptions for* $U, \{E_\lambda\}_{\lambda\in[0,2\pi]}, \Delta_n$ *with* $\nu(\Delta_n)\leq\pi$ *and, if* $f\colon\mathcal{C}(0,1)\to\mathbb{C}$ *is Lipschitzian with the constant* $L>0$ *on the circle* $\mathcal{C}(0,1)$, *then we have the representation*

$$\langle f(U)x,x\rangle = \tilde{O}_n(f,U,\Delta_n,\xi;x) + \tilde{R}_n(f,U,\Delta_n,\xi;x) \tag{4.20}$$

with the error $\tilde{R}_n(f,U,\Delta_n,\xi;x)$ *satisfying the bounds*

$$\begin{aligned}
&\left|\tilde{R}_n(f,U,\Delta_n,\xi;x)\right| \\
&\le 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{\lambda_{k+1}-\xi_k}{2}\right)\langle E_{\lambda_{k+1}}x,x\rangle - \sin\left(\frac{\xi_k-\lambda_k}{2}\right)\langle E_{\lambda_k}x,x\rangle\right] \\
&\quad + L\sum_{k=0}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}}\operatorname{sgn}(\xi_k-t)\cos\left(\frac{\xi_k-t}{2}\right)\langle E_t x,x\rangle\,dt \\
&\le 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{\lambda_{k+1}-\xi_k}{2}\right)\left[\left\langle\left(E_{\lambda_{k+1}}-E_{\xi_k}\right)x,x\right\rangle\right]\right. \\
&\quad \left. + \sin\left(\frac{\xi_k-\lambda_k}{2}\right)\langle(E_{\xi_k}-E_{\lambda_k})x,x\rangle\right] \\
&\le 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{4}(\lambda_{k+1}-\lambda_k)+\frac{1}{2}\left|\xi_k-\frac{\lambda_k+\lambda_{k+1}}{2}\right|\right]\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \\
&\le 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{2}(\lambda_{k+1}-\lambda_k)\right]\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle
\end{aligned}$$

$$\le L\sum_{k=0}^{n-1}(\lambda_{k+1}-\lambda_k)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \le \nu(\Delta_n)L\|x\|^2 \tag{4.21}$$

for any $x\in H$ *and the intermediate points* $\xi_k\in[\lambda_k,\lambda_{k+1}]$ *where* $0\le k\le n-1$.

In particular we have

$$\langle f(U)x,x\rangle = \tilde{T}_n(f,U,\Delta_n;x) + \tilde{E}_n(f,U,\Delta_n;x) \tag{4.22}$$

with the error

$$\begin{aligned}
|\tilde{E}_n(f,U,\Delta_n;x)| &\le 2L\sum_{k=0}^{n-1}\sin\left(\frac{\lambda_{k+1}-\lambda_k}{4}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \\
&\quad + L\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}\operatorname{sgn}\left(\frac{\lambda_k+\lambda_{k+1}}{2}-t\right)\cos\left(\frac{\frac{\lambda_k+\lambda_{k+1}}{2}-t}{2}\right)\langle E_t x,x\rangle\,dt \\
&\le 2L\sum_{k=0}^{n-1}\sin\left(\frac{\lambda_{k+1}-\lambda_k}{4}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle
\end{aligned}$$

$$\le \frac{1}{2}L\sum_{k=0}^{n-1}(\lambda_{k+1}-\lambda_k)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \le \frac{1}{2}L\nu(\Delta_n)\|x\|^2 \tag{4.23}$$

for any $x\in H$.

The proof follows by Proposition 3 applied for the monotonic nondecreasing function $u(t) := \langle E_t x, x\rangle$, $t \in [0, 2\pi]$.

Remark 9. We remark that if the partition reduces to the whole interval $[0, 2\pi]$ then we get from (2.36) that

$$\begin{aligned} \left| f(e^{is})\|x\|^2 - \langle f(U)x, x\rangle \right| &\leq 2L \sin\left(\frac{s}{2}\right)\|x\|^2 \\ &+ L\int_0^{2\pi} \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)\langle E_t x, x\rangle\, dt \end{aligned} \tag{4.24}$$

for any $s \in [a, b]$ and $x \in H$.

In particular, we have

$$\begin{aligned} \left| f(-1)\|x\|^2 - \langle f(U)x, x\rangle \right| &\leq \sqrt{2}L\|x\|^2 \\ &+ L\int_0^{2\pi} \operatorname{sgn}(\pi-t)\sin\left(\frac{t}{2}\right)\langle E_t x, x\rangle\, dt \end{aligned} \tag{4.25}$$

for any $x \in H$.

Example 1. In order to provide some simple examples for the inequalities above we choose two complex functions as follows.

a) Consider the power function $f\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}$, $f(z) = z^m$ where m is a nonzero integer. Then, obviously, for any z, w belonging to the unit circle $\mathcal{C}(0,1)$ we have the inequality

$$|f(z) - f(w)| \leq |m|\,|z-w|$$

which shows that f is Lipschitzian with the constant $L = |m|$ on the circle $\mathcal{C}(0,1)$.

Then from (4.15), we get for any unitary operator U that

$$\left| e^{ims}\langle x, y\rangle - \langle U^m x, y\rangle \right| \leq 2|m| \bigvee_0^{2\pi}\left(\langle E_{(\cdot)}x, y\rangle\right) \leq 2|m|\,\|x\|\,\|y\| \tag{4.26}$$

for any $s \in [0, 2\pi]$ and $x, y \in H$.

Also, from (4.16) and the intermediate points $u \in [0, \pi]$ and $v \in [\pi, 2\pi]$, we have for any unitary operator U

$$\begin{aligned} &\left| e^{imu}\langle E_\pi x, y\rangle + e^{imv}\langle (1_H - E_\pi)x, y\rangle - \langle U^m x, y\rangle \right| \\ &\quad\leq 2|m|\Big[\sin\left[\frac{1}{4}\pi + \frac{1}{2}\left|u - \frac{\pi}{2}\right|\right]\bigvee_0^{\pi}\left(\langle E_{(\cdot)}x, y\rangle\right) \\ &\quad\quad + \sin\left[\frac{1}{4}\pi + \frac{1}{2}\left|v - \frac{3\pi}{2}\right|\right]\bigvee_\pi^{2\pi}\left(\langle E_{(\cdot)}x, y\rangle\right)\Big] \end{aligned} \tag{4.27}$$

for any vectors $x, y \in H$, where $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U.

The best inequality we can get from (4.27) is obtained for $u = \frac{\pi}{2}$ and $v = \frac{3\pi}{2}$, namely

$$\begin{aligned} & \left| i^m \langle E_\pi x, y \rangle + (-i)^m \langle (1_H - E_\pi) x, y \rangle - \langle U^m x, y \rangle \right| \\ & \leq \sqrt{2} |m| \bigvee_0^{2\pi} \left(\langle E_{(\cdot)} x, y \rangle \right) \leq \sqrt{2} |m| \|x\| \|y\| , \end{aligned} \tag{4.28}$$

for any vectors $x, y \in H$.

b) For $a \neq \pm 1, 0$ consider the function $f \colon \mathcal{C}(0,1) \to \mathbb{C}$, $f_a(z) = \frac{1}{1-az}$. Observe that

$$|f_a(z) - f_a(w)| = \frac{|a| |z-w|}{|1-az| |1-aw|} \tag{4.29}$$

for any $z, w \in \mathcal{C}(0,1)$.

If $z = e^{it}$ with $t \in [0, 2\pi]$, then we have

$$\begin{aligned} |1-az|^2 &= 1 - 2a \operatorname{Re}(\bar{z}) + a^2 |z|^2 = 1 - 2a \cos t + a^2 \\ &\geq 1 - 2|a| + a^2 = (1-|a|)^2 \end{aligned}$$

therefore

$$\frac{1}{|1-az|} \leq \frac{1}{|1-|a||} \quad \text{and} \quad \frac{1}{|1-aw|} \leq \frac{1}{|1-|a||} \tag{4.30}$$

for any $z, w \in \mathcal{C}(0,1)$.

Utilising (4.29) and (4.30) we deduce

$$|f_a(z) - f_a(w)| \leq \frac{|a|}{(1-|a|)^2} |z-w| \tag{4.31}$$

for any $z, w \in \mathcal{C}(0,1)$, showing that the function f_a is Lipschitzian with the constant $L_a = \frac{|a|}{(1-|a|)^2}$ on the circle $\mathcal{C}(0,1)$.

Applying the inequality (4.15), we get for any unitary operator U that

$$\begin{aligned} & \left| (1-ae^{is})^{-1} \langle x, y \rangle - \left\langle (1_H - aU)^{-1} x, y \right\rangle \right| \\ & \leq \frac{2|a|}{(1-|a|)^2} \bigvee_0^{2\pi} \left(\langle E_{(\cdot)} x, y \rangle \right) \leq \frac{2|a|}{(1-|a|)^2} \|x\| \|y\| \end{aligned} \tag{4.32}$$

for any $s \in [0, 2\pi]$ and $x, y \in H$.

Also, from (4.16) and the intermediate points $u \in [0, \pi]$ and $v \in [\pi, 2\pi]$, we have for any unitary operator U

$$\begin{aligned} & \left| (1-ae^{iu})^{-1} \langle E_\pi x, y \rangle + (1-ae^{iv})^{-1} \langle (1_H - E_\pi) x, y \rangle - \left\langle (1_H - aU)^{-1} x, y \right\rangle \right| \\ & \leq \frac{2|a|}{(1-|a|)^2} \left[\sin \left[\frac{1}{4}\pi + \frac{1}{2} \left| u - \frac{\pi}{2} \right| \right] \bigvee_0^{\pi} \left(\langle E_{(\cdot)} x, y \rangle \right) \right. \\ & \left. + \sin \left[\frac{1}{4}\pi + \frac{1}{2} \left| v - \frac{3\pi}{2} \right| \right] \bigvee_\pi^{2\pi} (\langle E_{(\cdot)} x, y \rangle) \right] \end{aligned} \tag{4.33}$$

for any vectors $x, y \in H$, where $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U.

The best inequality we can get from (4.27) is obtained for $u = \frac{\pi}{2}$ and $v = \frac{3\pi}{2}$, namely

$$\left|(1-ai)^{-1}\langle E_\pi x, y\rangle + (1+ai)^{-1}\langle(1_H - E_\pi)x, y\rangle - \left\langle(1_H - aU)^{-1}x, y\right\rangle\right|$$

$$\leq \frac{\sqrt{2}\,|a|}{(1-|a|)^2}\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \frac{\sqrt{2}\,|a|}{(1-|a|)^2}\|x\|\,\|y\| \tag{4.34}$$

for any vectors $x, y \in H$.

References

[1] Dragomir, S. S., *On the Ostrowski's inequality for Riemann-Stieltjes integral*, Korean J. Appl. Math. **7** (2000), 477–485.

[2] Dragomir, S. S., *On the Ostrowski inequality for Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$ *where f is of Hölder type and u is of bounded variation and applications*, J. KSIAM **5** (1) (2001), 35–45.

[3] Dragomir, S. S., *Ostrowski's type inequalities for continuous functions of selfadjoint operators on Hilbert spaces: a survey of recent results*, Ann. Funct. Anal. **2** (1) (2011), 139–205.

[4] Dragomir, S. S., *Ostrowski's type inequalities for some classes of continuous functions of selfadjoint operators in Hilbert spaces*, Comput. Math. Appl. **62** (12) (2011), 4439–4448.

[5] Helmberg, G., *Introduction to Spectral Theory in Hilbert Space*, John Wiley and Sons, 1969.

[6] Ostrowski, A., *Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert* (German), Comment. Math. Helv. **10** (1) (1937), 226–227.

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