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MEAN OSCILLATION AND BOUNDEDNESS OF MULTILINEAR INTEGRAL OPERATORS WITH GENERAL KERNELS

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Abstract. In this paper, the boundedness properties for some multilinear operators related to certain integral operators from Lebesgue spaces to Orlicz spaces are proved. The integral operators include singular integral operator with general kernel, Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

1. Introduction and results

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [3]–[7], [18]–[20]). Let T be the Calderón-Zygmund singular integral operator and $b \in \mathrm{BMO}(R^n)$, a classical result of Coifman, Rochberg and Weiss (see [6]) stated that the commutator $[b,T](f) = T(bf) - bT(f)$ is bounded on $L^p(R^n)$ for $1 < p < \infty$. The purpose of this paper is to introduce some multilinear operator associated to certain integral operators with general kernels (see [1, 10, 15]) and prove the boundedness properties of the multilinear operators from Lebesgue spaces to Orlicz spaces.

In this paper, we are going to consider some integral operators as following (see [1]).

Let l and m_j be the positive integers ($j = 1, \dots, l$), $m_1 + \dots + m_l = m$ and b_j be the functions on R^n ($j = 1, \dots, l$). Set, for $1 \le j \le l$,

$$R_{m_j+1}(b_j; x, y) = b_j(x) - \sum_{|\alpha| \le m_j} \frac{1}{\alpha!} D^\alpha b_j(y)(x-y)^\alpha\,.$$

Definition 1. Let $T\colon S \to S'$ be a linear operator such that T is bounded on $L^2(R^n)$ and has a kernel K, that is there exists a locally integrable function $K(x,y)$

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on $R^n \times R^n \setminus \{(x,y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x,y)f(y)dy$$

for every bounded and compactly supported function f, where K satisfies:

$$|K(x,y)| \le C|x-y|^{-n},$$

$$\int_{2|y-z|<|x-y|} (|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)|)\,dx \le C,$$

and there is a sequence of positive constant numbers $\{C_k\}$ such that for any $k \ge 1$,

$$\Big(\int_{2^k|z-y|\le|x-y|<2^{k+1}|z-y|} (|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)|)^q\,dy\Big)^{1/q} \le C_k(2^k|z-y|)^{-n/q'},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$. The multilinear operator related to the operator T is defined by

$$T^b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j;x,y)}{|x-y|^m} K(x,y)f(y)\,dy.$$

Definition 2. Let $F(x,y,t)$ define on $R^n \times R^n \times [0,+\infty)$, we denote that

$$F_t(f)(x) = \int_{R^n} F(x,y,t)f(y)\,dy$$

and

$$F_t^b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j;x,y)}{|x-y|^m} F(x,y,t)f(y)\,dy$$

for every bounded and compactly supported function f. Let H be the Banach space $H = \{h\colon \|h\| < \infty\}$. For each fixed $x \in R^n$, we view $F_t(f)(x)$ and $F_t^b(f)(x)$ as a mapping from $[0,+\infty)$ to H. Then, the multilinear operators related to F_t is defined by

$$S^b(f)(x) = \|F_t^b(f)(x)\|,$$

where F_t satisfies:

$$\|F(x,y,t)\| \le C|x-y|^{-n},$$

$$\int_{2|y-z|<|x-y|} (\|F(x,y,t) - F(x,z,t)\| + \|F(y,x,t) - F(z,x,t)\|)\,dx \le C,$$

and there is a sequence of positive constant numbers $\{C_k\}$ such that for any $k \ge 1$,

$$\Big(\int_{2^k|z-y|\le|x-y|<2^{k+1}|z-y|} (\|F(x,y,t) - F(x,z,t)\| + \|F(y,x,t) - F(z,x,t)\|)^q\,dy\Big)^{1/q} \le C_k(2^k|z-y|)^{-n/q'},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$. We also define that $S(f)(x) = \|F_t(f)(x)\|$.

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 1 (see [8, 19, 20, 22, 23]) and that T^b and S^b are just the commutators of T and S with b if $m = 0$ (see [6, 9, 11, 19, 20]). While when $m > 0$, it is non-trivial generalizations of the commutators. Let T be the Calderón-Zygmund singular integral operator, a classical result of Coifman, Rochberg and Weiss (see [6]) states that the commutator $[b, T] = T(bf) - bTf$ (where $b \in \mathrm{BMO}(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$, Chanillo (see [2]) proves a similar result when T is replaced by the fractional integral operator. In [9], Janson proved boundedness properties for the commutators related to the Calderón-Zygmund singular integral operators from Lebesgue spaces to Orlicz spaces. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [3]–[5], [7]). The main purpose of this paper is to prove the boundedness properties for the multilinear operators T^b and S^b from Lebesgue spaces to Orlicz spaces.

Let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy \,,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) \, dx$. It is well-known that (see [8, 22])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| \, dy \,.$$

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy \,.$$

We write that $M_p f = (M(f^p))^{1/p}$ for $0 < p < \infty$. For $1 \le r < \infty$ and $0 < \beta < n$, let

$$M_{\beta,r}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1 - r\beta/n}} \int_Q |f(y)|^r dy \right)^{1/r} .$$

We say that f belongs to $\mathrm{BMO}(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$ and $\|f\|_{\mathrm{BMO}} = \|f^{\#}\|_{L^\infty}$. More generally, let ρ be a non-decreasing positive function on $[0, +\infty)$ and define $\mathrm{BMO}_\rho(R^n)$ as the space of all functions f such that

$$\frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y) - f_Q| \, dy \le C\rho(r) \,.$$

For $\beta > 0$, the Lipschitz space $\mathrm{Lip}_\beta(R^n)$ is the space of functions f such that

$$\|f\|_{\mathrm{Lip}_\beta} = \sup_{x \ne y} |f(x) - f(y)| / |x - y|^\beta < \infty \,.$$

For f, m_f denotes the distribution function of f, that is $m_f(t) = |\{x \in R^n : |f(x)| > t\}|$.

Let ρ be a non-decreasing convex function on $[0,+\infty)$ with $\rho(0)=0$. ρ^{-1} denotes the inverse function of ρ. The Orlicz space $L_\rho(R^n)$ is defined by the set of functions f such that $\int_{R^n}\rho(\lambda|f(x)|)dx<\infty$ for some $\lambda>0$. The Luxemburg norm is given by (see [21])

$$\|f\|_{L_\rho}=\inf_{\lambda>0}\lambda^{-1}\Big(1+\int_{R^n}\rho(\lambda|f(x)|)\,dx\Big).$$

We shall prove the following theorems in Section 2.

Theorem 1. *Let $0<\beta\le 1$, $q'<p<n/l\beta$ and φ, ψ be two non-decreasing positive functions on $[0,+\infty)$ with $(\psi^l)^{-1}(t)=t^{1/p}\varphi^l(t^{-1/n})$. Suppose that ψ is convex, $\psi(0)=0$, $\psi(2t)\le C\psi(t)$. Let T be the same as in Definition 1 and the sequence $\{k^lC_k\}\in l^1$. Then T^b is bounded from $L^p(R^n)$ to $L_{\psi^l}(R^n)$ if $D^\alpha b_j\in \mathrm{BMO}(R^n)$ for all α with $|\alpha|=m_j$ and $j=1,\dots,l$.*

Theorem 2. *Let $0<\beta\le 1$, $q'<p<n/m\beta$ and φ, ψ be two non-decreasing positive functions on $[0,+\infty)$ with $(\psi^l)^{-1}(t)=t^{1/p}\varphi^l(t^{-1/n})$. Suppose that ψ is convex, $\psi(0)=0$, $\psi(2t)\le C\psi(t)$. Let S be the same as in Definition 2 and the sequence $\{C_k\}\in l^1$. Then S^b is bounded from $L^p(R^n)$ to $L_{\psi^l}(R^n)$ if $D^\alpha b_j\in \mathrm{BMO}(R^n)$ for all α with $|\alpha|=m_j$ and $j=1,\dots,l$.*

Remark. **(a)** If $l=1$ and $\psi^{-1}(t)=t^{1/p}\varphi(t^{-1/n})$, then T^b and S^b are all bounded on from $L^p(R^n)$ to $L_\psi(R^n)$ under the conditions of Theorems 1 and 2.

(b) If $l=1$, $\varphi(t)\equiv 1$ and $\psi(t)=t^p$ for $1<p<\infty$, then T^b and S^b are all bounded on $L^p(R^n)$ if $D^\alpha b\in \mathrm{BMO}_\varphi(R^n)$ for all α with $|\alpha|=m$.

(c) If $l=1$, $\psi(t)=t^s$ and $\varphi(t)=t^{n(1/p-1/s)}$ for $1<p<s<\infty$, then, by $\mathrm{BMO}_{t^\beta}(R^n)=\mathrm{Lip}_\beta(R^n)$ (see [9, Lemma 4]), T^b and S^b are all bounded from $L^p(R^n)$ to $L^s(R^n)$ if $D^\alpha b\in \mathrm{Lip}_{n(1/p-1/s)}(R^n)$ for all α with $|\alpha|=m$.

2. Proof of theorems

We begin with the following preliminary lemmas.

Lemma 1 (see [1]). *Let T and S be the the same as Definitions 1 and 2, the sequence $\{C_k\}\in l^1$. Then T and S are bounded on $L^p(R^n)$ for $1<p<\infty$.*

Lemma 2 (see [9]). *Let ρ be a non-decreasing positive function on $[0,+\infty)$ and η be an infinitely differentiable function on R^n with compact support such that $\int_{R^n}\eta(x)\,dx=1$. Denote that $b^t(x)=\int_{R^n}b(x-ty)\eta(y)\,dy$. Then $\|b-b^t\|_{\mathrm{BMO}}\le C\rho(t)\|b\|_{\mathrm{BMO}_\rho}$.*

Lemma 3 (see [1]). *Let $0<\beta<1$ or $\beta=1$ and ρ be a non-decreasing positive function on $[0,+\infty)$. Then $\|b^t\|_{\mathrm{Lip}_\beta}\le Ct^{-\beta}\rho(t)\|b\|_{\mathrm{BMO}_\rho}$.*

Lemma 4 (see [1]). *Suppose $1\le p_2<p<p_1<\infty$, ρ is a non-increasing function on R^+, B is a linear or sublinear operator such that $m_{B(f)}(t^{1/p_1}\rho(t))\le Ct^{-1}$ if $\|f\|_{L^{p_1}}\le 1$ and $m_{B(f)}(t^{1/p_2}\rho(t))\le Ct^{-1}$ if $\|f\|_{L^{p_2}}\le 1$. Then $\int_0^\infty m_{B(f)}(t^{1/p}\rho(t))\,dt\le C$ if $\|f\|_{L^p}\le (p/p_1)^{1/p}$.*

Lemma 5 (see [2]). *Suppose that $0<\beta<n$, $1\le r<p<n/\beta$ and $1/s=1/p-\beta/n$. Then $\|M_{\beta,r}(f)\|_{L^s}\le C\|f\|_{L^p}$.*

Lemma 6 (see [5]). *Let b be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(b; x, y)| \le C|x-y|^m \sum_{|\alpha|=m} \Big(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} |D^\alpha b(z)|^q\,dz\Big)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

To prove the theorems of the paper, we need the following

Key Lemma. *Let T and S be the same as in Definitions 1 and 2. Suppose that $Q = Q(x_0, d)$ is a cube with* $\operatorname{supp} f \subset (2Q)^c$ *and $x, \tilde{x} \in Q$.*

(I) *If the sequence $\{k^l C_k\} \in l^1$ and $D^\alpha b_j \in \mathrm{BMO}(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$, then*

$$|T^b(f)(x) - T^b(f)(x_0)| \le C \prod_{j=1}^{l} \Big(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}}\Big) M_r(f)(\tilde{x}) \quad \textit{for any} \quad r > q';$$

(II) *If the sequence $\{C_k\} \in l^1$, $0 < \beta \le 1$ and $D^\alpha b_j \in \mathrm{Lip}_\beta(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$, then*

$$|T^b(f)(x) - T^b(f)(x_0)| \le C \prod_{j=1}^{l} \Big(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta}\Big) M_{l\beta,r}(f)(\tilde{x}) \quad \textit{for any} \quad r > q';$$

(III) *If the sequence $\{k^l C_k\} \in l^1$ and $D^\alpha b_j \in \mathrm{BMO}(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$, then*

$$\|F_t^b(f)(x) - F_t^b(f)(x_0)\| \le C \prod_{j=1}^{l} \Big(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}}\Big) M_r(f)(\tilde{x}) \quad \textit{for any} \quad r > q';$$

(IV) *If the sequence $\{k^l C_k\} \in l^1$, $0 < \beta \le 1$ and $D^\alpha b_j \in \mathrm{Lip}_\beta(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$, then*

$$\|F_t^b(f)(x) - F_t^b(f)(x_0)\| \le C \prod_{j=1}^{l} \Big(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta}\Big) M_{l\beta,r}(f)(\tilde{x}) \quad \textit{for any} \quad r > q'.$$

Proof. Without loss of generality, we may assume $l = 2$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha b_j)_{\tilde{Q}} x^\alpha$, then $R_m(b_j; x, y) = R_m(\tilde{b}_j; x, y)$ and $D^\alpha \tilde{b}_j =$

$D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $\operatorname{supp} f \subset (2Q)^c$ and $x, \tilde{x} \in Q$,

$$
\begin{aligned}
T^b(f)(x) - T^b(f)(x_0) &= \int_{R^n} \Big(\frac{K(x,y)}{|x-y|^m} - \frac{K(x_0,y)}{|x_0-y|^m} \Big) \prod_{j=1}^{2} R_{m_j}(\tilde{b}_j; x, y) f(y)\, dy \\
&+ \int_{R^n} \big(R_{m_1}(\tilde{b}_1; x, y) - R_{m_1}(\tilde{b}_1; x_0, y) \big) \frac{R_{m_2}(\tilde{b}_2; x, y)}{|x_0-y|^m} K(x_0,y) f(y)\, dy \\
&+ \int_{R^n} \big(R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; x_0, y) \big) \frac{R_{m_1}(\tilde{b}_1; x_0, y)}{|x_0-y|^m} K(x_0,y) f(y)\, dy \\
&- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \Big[\frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x,y) - \frac{R_{m_2}(\tilde{b}_2; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} K(x_0,y) \Big] \\
&\times D^{\alpha_1} \tilde{b}_1(y) f(y)\, dy \\
&- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \Big[\frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x,y) - \frac{R_{m_1}(\tilde{b}_1; x_0, y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} K(x_0,y) \Big] \\
&\times D^{\alpha_2} \tilde{b}_2(y) f(y)\, dy \\
&+ \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \Big[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x,y) - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} K(x_0,y) \Big] \\
&\times D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y) f(y)\, dy \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6\,.
\end{aligned}
$$

(I). By Lemma 6 and the following inequality (see [10]), for $b \in \mathrm{BMO}(R^n)$,

$$
|b_{Q_1} - b_{Q_2}| \le C \log(|Q_2|/|Q_1|) \|b\|_{\mathrm{BMO}} \quad \text{for} \quad Q_1 \subset Q_2\,,
$$

we know that, for $x \in Q$ and $y \in 2^{k+1}Q \setminus 2^k Q$ with $k \ge 1$,

$$
\begin{aligned}
|R_m(\tilde{b}; x, y)| &\le C|x-y|^m \sum_{|\alpha|=m} \big(\|D^\alpha b\|_{\mathrm{BMO}} + |(D^\alpha b)_{\tilde{Q}(x,y)} - (D^\alpha b)_{\tilde{Q}}| \big) \\
&\le Ck|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}}\,.
\end{aligned}
$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, by the conditions on K and recalling $r > q'$, we obtain

$$
\begin{aligned}
|I_1| &\le \int_{R^n \setminus 2Q} \Big| \frac{1}{|x-y|^m} - \frac{1}{|x_0-y|^m} \Big| |K(x,y)| \prod_{j=1}^{2} |R_{m_j}(\tilde{b}_j; x, y)|\, |f(y)|\, dy \\
&\quad + \int_{R^n \setminus 2Q} |K(x,y) - K(x_0,y)|\, |x_0-y|^{-m} \prod_{j=1}^{2} |R_{m_j}(\tilde{b}_j; x, y)|\, |f(y)|\, dy
\end{aligned}
$$

$$
\begin{aligned}
&\le \sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\Big|\frac{1}{|x-y|^m}-\frac{1}{|x_0-y|^m}\Big|\,|K(x,y)|\prod_{j=1}^{2}|R_{m_j}(\tilde b_j;x,y)||f(y)|\,dy\\
&\quad+\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|K(x,y)-K(x_0,y)|\,|x_0-y|^{-m}\prod_{j=1}^{2}|R_{m_j}(\tilde b_j;x,y)|\,|f(y)|\,dy\\
&\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)\sum_{k=1}^{\infty}k^2\int_{2^{k+1}Q\setminus 2^kQ}\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|\,dy\\
&\quad+C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)\sum_{k=1}^{\infty}k^2\Big(\int_{2^{k+1}Q\setminus 2^kQ}|f(y)|^{q'}\,dy\Big)^{1/q'}\\
&\quad\times\Big(\int_{2^{k+1}Q\setminus 2^kQ}|K(x,y)-K(x_0,y)|^q\,dy\Big)^{1/q}\\
&\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)\sum_{k=1}^{\infty}k^2(2^{-k}+C_k)\Big(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^r dy\Big)^{1/r}\\
&\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)M_r(f)(\tilde x)\,.
\end{aligned}
$$

For I_2, by the formula (see [5]):

$$
R_m(\tilde b;x,y)-R_m(\tilde b;x_0,y)=\sum_{|\gamma|<m}\frac{1}{\gamma!}R_{m-|\gamma|}(D^\gamma\tilde b;x,x_0)(x-y)^\gamma
$$

and Lemma 6, we have

$$
|R_m(\tilde b;x,y)-R_m(\tilde b;x_0,y)|\le C\sum_{|\gamma|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\gamma|}|x-y|^{|\gamma|}\|D^\alpha b\|_{\mathrm{BMO}},
$$

thus

$$
\begin{aligned}
|I_2|&\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}k\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|\,dy\\
&\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)\sum_{k=1}^{\infty}k2^{-k}\Big(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^r dy\Big)^{1/r}\\
&\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)M_r(f)(\tilde x)\,.
\end{aligned}
$$

Similarly,

$$
|I_3|\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)M_r(f)(\tilde x)\,.
$$

For I_4, similar to the proof of I_1 and I_2, taking $1 < p < \infty$ such that $1/p+1/q+1/r = 1$, we get

$$\begin{aligned}
|I_4| \le{}& C \sum_{|\alpha_1|=m_1} \int_{R^n\setminus 2Q} \Big|\frac{(x-y)^{\alpha_1}}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1}}{|x_0-y|^m}\Big|\,|K(x,y)| \\
&\times |R_{m_2}(\tilde b_2; x,y)|\,|D^{\alpha_1}\tilde b_1(y)|\,|f(y)|\,dy \\
&+ C \sum_{|\alpha_1|=m_1} \int_{R^n\setminus 2Q} |R_{m_2}(\tilde b_2; x,y) - R_{m_2}(\tilde b_2; x_0,y)| \\
&\times \frac{|(x_0-y)^{\alpha_1}K(x,y)|}{|x_0-y|^m}|D^{\alpha_1}\tilde b_1(y)|\,|f(y)|\,dy \\
&+ C \sum_{|\alpha_1|=m_1} \int_{R^n\setminus 2Q} |K(x,y)-K(x_0,y)|\Big|\frac{(x_0-y)^{\alpha_1}}{|x_0-y|^m}\Big| \\
&\times |R_{m_2}(\tilde b_2; x_0,y)|\,|D^{\alpha_1}\tilde b_1(y)|\,|f(y)|\,dy \\
\le{}& C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}b_2||_{\mathrm{BMO}} \sum_{k=1}^{\infty} k2^{-k} \sum_{|\alpha_1|=m_1} \Big(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q} |D^{\alpha_1}\tilde b_1(y)|^{r'}dy\Big)^{1/r'} \\
&\times \Big(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q} |f(y)|^r dy\Big)^{1/r} \\
&+ C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2}b_2\|_{\mathrm{BMO}} \sum_{|\alpha_1|=m_1} \sum_{k=1}^{\infty} \\
&\times k\Big(\int_{2^{k+1}Q\setminus 2^kQ} |K(x,y)-K(x_0,y)|^q dy\Big)^{1/q} \\
&\times \Big(\int_{2^{k+1}Q\setminus 2^kQ} |D^{\alpha_1}\tilde b_1(y)|^p dy\Big)^{1/p}\Big(\int_{2^{k+1}Q\setminus 2^kQ} |f(y)|^r dy\Big)^{1/r} \\
\le{}& C \prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big) \sum_{k=1}^{\infty} k^2(2^{-k}+C_k) \\
&\times \Big(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q} |f(y)|^r dy\Big)^{1/r} \\
\le{}& C \prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big) M_r(f)(\tilde x)\,.
\end{aligned}$$

Similarly,

$$|I_5| \le C \prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big) M_r(f)(\tilde x)\,.$$

For I_6, taking $1 < r_1, r_2 < \infty$ such that $1/q + 1/p + 1/r_1 + 1/r_2 = 1$, then

$$
\begin{aligned}
|I_6| \le C &\sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}} \int_{R^n\setminus 2Q} \Big| \frac{(x-y)^{\alpha_1+\alpha_2}K(x,y)}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1+\alpha_2}K(x_0,y)}{|x_0-y|^m} \Big| \\
&\times |D^{\alpha_1}\tilde{b}_1(y)||D^{\alpha_2}\tilde{b}_2(y)||f(y)|\,dy \\
\le C &\sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}} \sum_{k=1}^{\infty} (2^{-k}+C_k) \Big(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)|^r dy \Big)^{1/r} \\
&\times \Big(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |D^{\alpha_1}\tilde{b}_1(y)|^{r_1} dy \Big)^{1/r_1} \Big(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |D^{\alpha_2}\tilde{b}_2(y)|^{r_2} dy \Big)^{1/r_2} \\
\le C &\prod_{j=1}^{2} \Big(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}} \Big) \sum_{k=1}^{\infty} k^2 (2^{-k}+C_k) M_r(f)(\tilde{x}) \\
\le C &\prod_{j=1}^{2} \Big(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}} \Big) M_r(f)(\tilde{x})\,.
\end{aligned}
$$

Thus

$$
|T^b(f)(x) - T^b(f)(x_0)| \le C \prod_{j=1}^{2} \Big(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}} \Big) M_r(f)(\tilde{x})\,.
$$

(II). By Lemma 6 and the following inequality, for $b \in \mathrm{Lip}_\beta(R^n)$,

$$
|b(x) - b_Q| \le \frac{1}{|Q|} \int_Q \|b\|_{\mathrm{Lip}_\beta} |x-y|^\beta dy \le C \|b\|_{\mathrm{Lip}_\beta} (|x-x_0|+d)^\beta\,,
$$

we get

$$
|R_m(\tilde{b}; x, y)| \le C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta} (|x-y|+d)^{m+\beta}
$$

and

$$
|R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y)| \le C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta} (|x-y|+d)^{m+\beta}\,,
$$

then

$$
\begin{aligned}
|I_1| \le &\sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^k Q} \Big| \frac{1}{|x-y|^m} - \frac{1}{|x_0-y|^m} \Big| |K(x,y)| \prod_{j=1}^{2} |R_{m_j}(\tilde{b}_j; x, y)|\,|f(y)|\,dy \\
&+ \sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^k Q} |K(x,y) - K(x_0,y)|\,|x_0-y|^{-m} \prod_{j=1}^{2} |R_{m_j}(\tilde{b}_j; x, y)|\,|f(y)|\,dy \\
\le C &\prod_{j=1}^{2} \Big(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta} \Big) \sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^k Q} \frac{|x-x_0|}{|x_0-y|^{n+1-2\beta}} |f(y)|\,dy
\end{aligned}
$$

$$
\begin{aligned}
&+ C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)\sum_{k=1}^{\infty}|2^{k+1}Q|^{2\beta/n}\Big(\int_{2^{k+1}Q\setminus 2^kQ}|f(y)|^{q'}dy\Big)^{1/q'}\\
&\times\Big(\int_{2^{k+1}Q\setminus 2^kQ}|K(x,y)-K(x_0,y)|^q\,dy\Big)^{1/q}\\
&\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)\sum_{k=1}^{\infty}(2^{-k}+C_k)\Big(\frac{1}{|2^{k+1}Q|^{1-2\beta r/n}}\int_{2^{k+1}Q}|f(y)|^r dy\Big)^{1/r}\\
&\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)M_{2\beta,r}(f)(\tilde{x}),
\end{aligned}
$$

$$
\begin{aligned}
|I_2+I_3| &\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)\sum_{k=1}^{\infty}(2^{-k}+C_k)\Big(\frac{1}{|2^{k+1}Q|^{1-2\beta r/n}}\int_{2^{k+1}Q}|f(y)|^r dy\Big)^{1/r}\\
&\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)M_{2\beta,r}(f)(\tilde{x}),
\end{aligned}
$$

$$
\begin{aligned}
|I_4| \le\ & C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus 2Q}\Big|\frac{(x-y)^{\alpha_1}}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}}{|x_0-y|^m}\Big|\,|K(x,y)|\,|R_{m_2}(\tilde{b}_2;x,y)|\\
&\qquad\times|D^{\alpha_1}\tilde{b}_1(y)|\,|f(y)|\,dy\\
&+C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus 2Q}|R_{m_2}(\tilde{b}_2;x,y)-R_{m_2}(\tilde{b}_2;x_0,y)|\frac{|(x_0-y)^{\alpha_1}K(x,y)|}{|x_0-y|^m}\\
&\qquad\times|D^{\alpha_1}\tilde{b}_1(y)|\,|f(y)|\,dy\\
&+C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus 2Q}|K(x,y-K(x_0,y)|\Big|\frac{(x_0-y)^{\alpha_1}}{|x_0-y|^m}\Big|\,|R_{m_2}(\tilde{b}_2;x_0,y)|\\
&\qquad\times|D^{\alpha_1}\tilde{b}_1(y)|\,|f(y)|\,dy\\
&\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)\sum_{k=1}^{\infty}2^{-k}\Big(\frac{1}{|2^{k+1}Q|^{1-2\beta/n}}\int_{2^{k+1}Q}|f(y)|\,dy\Big)\\
&+C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)\sum_{k=1}^{\infty}|2^{k+1}Q|^{2\beta/n}\Big(\int_{2^{k+1}Q\setminus 2^kQ}|f(y)|^{q'}dy\Big)^{1/q'}\\
&\times\Big(\int_{2^{k+1}Q\setminus 2^kQ}|K(x,y)-K(x_0,y)|^q\,dy\Big)^{1/q}
\end{aligned}
$$

$$\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)\sum_{k=1}^{\infty}(2^{-k}+C_k)\Big(\frac{1}{|2^{k+1}Q|^{1-2\beta r/n}}\int_{2^{k+1}Q}|f(y)|^r dy\Big)^{1/r}$$

$$\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)M_{2\beta,r}(f)(\tilde{x})\,,$$

$$|I_5|\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)M_{2\beta,r}(f)(\tilde{x})\,,$$

$$|I_6|\le C\sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}}\int_{R^n\setminus 2Q}\Big|\frac{(x-y)^{\alpha_1+\alpha_2}K(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}K(x_0,y)}{|x_0-y|^m}\Big|$$

$$\times|D^{\alpha_1}\tilde{b}_1(y)|\,|D^{\alpha_2}\tilde{b}_2(y)|\,|f(y)|\,dy$$

$$\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)\sum_{k=1}^{\infty}(2^{-k}+C_k)\Big(\frac{1}{|2^{k+1}Q|^{1-2\beta r/n}}\int_{2^{k+1}Q}|f(y)|^r\,dy\Big)^{1/r}$$

$$\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)M_{2\beta,r}(f)(\tilde{x})\,.$$

Thus

$$|T^b(f)(x)-T^b(f)(x_0)|\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)M_{2\beta,r}(f)(\tilde{x})\,.$$

□

A same argument as in the proof of (I) and (II) will give the proof of (III) and (VI), we omit the details.

Now we are in position to prove our theorems.

Proof of Theorem 1. Without loss of generality, we may assume $l=2$. We prove the theorem in several steps. First, we prove, if $D^\alpha b_j\in\mathrm{BMO}(R^n)$ for all α with $|\alpha|=m_j$ and $j=1,\dots,l$,

$$(T^b(f))^{\#}\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)M_r(f) \tag{1}$$

for any r with $q'<r<\infty$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Let $\tilde{Q}=5\sqrt{n}Q$ and $\tilde{b}_j(x)=b_j(x)-\sum\limits_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha b_j)_{\tilde{Q}}x^\alpha$, then $R_m(b_j;x,y)=R_m(\tilde{b}_j;x,y)$ and $D^\alpha\tilde{b}_j=D^\alpha b_j-(D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha|=m_j$. We write, for $f_1=f\chi_{\tilde{Q}}$ and $f_2=f\chi_{R^n\setminus\tilde{Q}}$,

$$
\begin{aligned}
T^b(f)(x) &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m} K(x,y) f_1(y)\,dy \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1} D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m} K(x,y) f_1(y)\,dy \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2} D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K(x,y) f_1(y)\,dy \\
&\quad + \sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K(x,y) f_1(y)\,dy \\
&\quad + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(b_j;x,y)}{|x-y|^m} K(x,y) f_2(y)\,dy \\
&= T\Big(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_1\Big) \\
&\quad - T\Big(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m} f_1\Big) \\
&\quad - T\Big(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\Big) \\
&\quad + T\Big(\sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}} \frac{1}{\alpha_1!\alpha_2!} \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\Big) + T^b(f_2)(x)\,,
\end{aligned}
$$

then

$$
\begin{aligned}
|T^b(f)(x) - T^b(f_2)(x_0)| &\le \Big|T\Big(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_1\Big)\Big| \\
&\quad + \Big|T\Big(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m} f_1\Big)\Big| \\
&\quad + \Big|T\Big(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\Big)\Big| \\
&\quad + \Big|T\Big(\sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}} \frac{1}{\alpha_1!\alpha_2!} \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\Big)\Big| \\
&\quad + |T^b(f_2)(x) - T^b(f_2)(x_0)| \\
&= L_1(x) + L_2(x) + L_3(x) + L_4(x) + L_5(x)
\end{aligned}
$$

and

$$\begin{aligned}\frac{1}{|Q|}\int_Q |T^b(f)(x)-T^b(f_2)(x_0)|\,dx &\le \frac{1}{|Q|}\int_Q L_1(x)\,dx+\frac{1}{|Q|}\int_Q L_2(x)\,dx\\ &\quad+\frac{1}{|Q|}\int_Q L_3(x)\,dx+\frac{1}{|Q|}\int_Q L_4(x)\,dx+\frac{1}{|Q|}\int_Q L_5(x)\,dx\\ &= L_1+L_2+L_3+L_4+L_5\,.\end{aligned}$$

Now, for L_1, if $x\in Q$ and $y\in 2Q$, by using Lemma 6, we get

$$R_m(\tilde{b};x,y)\le C|x-y|^m\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{BMO}},$$

thus, by the L^r boundedness of T (see Lemma 1) and Hölder's inequality, we obtain

$$\begin{aligned}L_1 &\le C\prod_{j=1}^2\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)\frac{1}{|Q|}\int_Q |T(f_1)(x)|\,dx\\ &\le C\prod_{j=1}^2\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)\Big(\frac{1}{|Q|}\int_{R^n}|T(f_1)(x)|^r\,dx\Big)^{1/r}\\ &\le C\prod_{j=1}^2\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)\Big(\frac{1}{|Q|}\int_{R^n}|f_1(x)|^r\,dx\Big)^{1/r}\\ &\le C\prod_{j=1}^2\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)\Big(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^r\,dx\Big)^{1/r}\\ &\le C\prod_{j=1}^2\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)M_r(f)(\tilde{x})\,.\end{aligned}$$

For L_2, denoting $r=uv$ for $1<u,v<\infty$ and $1/v+1/v'=1$, we have

$$\begin{aligned}L_2 &\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{\mathrm{BMO}}\sum_{|\alpha_1|=m_1}\sum_{|\alpha|=m}\frac{1}{|Q|}\int_Q|T(D^{\alpha_1}\tilde{b}_1f_1)(x)|\,dx\\ &\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{\mathrm{BMO}}\sum_{|\alpha_1|=m_1}\Big(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{b}_1f_1)(x)|^u\,dx\Big)^{1/u}\\ &\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{\mathrm{BMO}}\sum_{|\alpha_1|=m_1}\Big(\frac{1}{|Q|}\int_{R^n}|D^{\alpha_1}\tilde{b}_1(x)|\,|f_1(x)|^u\,dx\Big)^{1/u}\\ &\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{\mathrm{BMO}}\Big(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^{uv}\,dx\Big)^{1/uv}\\ &\quad\times\sum_{|\alpha_1|=m_1}\Big(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(x)-(D^{\alpha_1}\tilde{b}_1)_{\tilde{Q}}|^{uv'}\,dx\Big)^{1/uv'}\end{aligned}$$

$$\leq C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)M_r(f)(\tilde{x})\,.$$

For L_3, similar to the proof of L_2, we get

$$L_3\leq C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)M_r(f)(\tilde{x}\,.$$

Similarly, for L_4, denoting $r=uw$ for $1<u,v_1,v_2,w<\infty$ and $1/v_1+1/v_2+1/w=1$, we obtain, by Hölder'inequality,

$$\begin{aligned}
L_4&\leq C\sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}}\frac{1}{|Q|}\int_Q|T(D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2f_1)(x)|\,dx\\
&\leq C\sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}}\Big(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2f_1)(x)|^u\,dx\Big)^{1/u}\\
&\leq C\sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}}|Q|^{-1/u}\Big(\int_{R^n}|D^{\alpha_1}\tilde{b}_1(x)D^{\alpha_2}\tilde{b}_2(x)f_1(x)|^u\,dx\Big)^{1/u}\\
&\leq C\sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}}\Big(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(x)|^{uv_1}\,dx\Big)^{1/uv_1}\Big(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|D^{\alpha_2}\tilde{b}_2(x)|^{uv_2}\,dx\Big)^{1/uv_2}\\
&\quad\times\Big(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^{uw}\,dx\Big)^{1/uw}\\
&\leq C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)M_r(f)(\tilde{x})\,.
\end{aligned}$$

For L_5, by using **Key Lemma**, we have

$$L_3\leq C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)M_r(f)(\tilde{x})\,.$$

We now put these estimates together and take the supremum over all Q such that $\tilde{x}\in Q$, we obtain

$$(T^b(f))^{\#}(\tilde{x})\leq C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)M_r(f)(\tilde{x})\,.$$

Thus, taking r such that $q'<r<p$, we obtain

$$\|T^b(f)\|_{L^p} \le C\|(T^b(f))^{\#}\|_{L^p} \le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)\|M_r(f)\|_{L^p}$$

$$\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\Big)\|f\|_{L^p}. \tag{2}$$

Secondly, we prove that, if $D^\alpha b_j \in \mathrm{Lip}_\beta(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$,

$$(T^b(f))^{\#} \le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)M_{2\beta,r}(f) \tag{3}$$

for any r with $q' < r < n/2\beta$. In fact, by Lemma 6, we have, for $x \in Q$ and $y \in 2Q$

$$\begin{aligned}|R_m(\tilde{b};x,y)| &\le C|x-y|^m\sum_{|\alpha|=m}\sup_{z\in 2Q}|D^\alpha b(z) - (D^\alpha b)_Q|\\ &\le C|x-y|^m|Q|^{\beta/n}\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{Lip}_\beta}\,,\end{aligned}$$

similar to the proof of (1) and by **Key Lemma**, we obtain

$$\begin{aligned}&\frac{1}{|Q|}\int_Q|T^b(f)(x) - T^b(f)(x_0)|\,dx\\
&\le \frac{1}{|Q|}\int_Q\Big|T\Big(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_1\Big)\Big|\,dx\\
&\quad + C\frac{1}{|Q|}\int_Q\Big|T\Big(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}f_1\Big)\Big|\,dx\\
&\quad + C\frac{1}{|Q|}\int_Q\Big|T\Big(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\Big)\Big|\,dx\\
&\quad + C\frac{1}{|Q|}\int_Q\Big|T\Big(\sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\Big)\Big|\,dx\\
&\quad + \frac{1}{|Q|}\int_Q|T^b(f_2)(x) - T^b(f_2)(x_0)|\,dx\\
&\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)\frac{1}{|Q|^{1/r-2\beta/n}}\Big(\int_{\tilde{Q}}|f(x)|^r\,dx\Big)^{1/r}\\
&\quad + \frac{1}{|Q|}\int_Q|T^b(f_2)(x) - T^b(f_2)(x_0)|\,dx\\
&\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)M_{2\beta,r}(f)(\tilde{x})\,.\end{aligned}$$

Thus, (3) holds. We take $q' < r < p < n/2\beta$, $1/w = 1/p - 2\beta/n$ and obtain, by Lemma 5,

$$\|T^b(f)\|_{L^w} \le C\|(T^b(f))^{\#}\|_{L^w} \le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)\|M_{2\beta,r}(f)\|_{L^w}$$

$$\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)\|f\|_{L^p}\,. \tag{4}$$

Now we verify that T^b satisfies the conditions of Lemma 3. In fact, for any $1 < p_i < n/2\beta$, $1/w_i = 1/p_i - 2\beta/n (i = 1, 2)$ and $\|f\|_{L^{p_i}} \le 1$, note that $T^b(f)(x) = T^{b-b^s}(f)(x) + T^{b^s}(f)(x)$ and $D^\alpha(b^s) = (D^\alpha b)^s$ with $D^\alpha(b_j - b_j^s) \in \mathrm{BMO}(R^n)$ and $D^\alpha b_j^s \in Lip_\beta(R^n)$, by (2) and Lemma 2, we obtain

$$\begin{aligned}\|T^{b-b^s}(f)\|_{L^{p_i}} &\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}(b_j - b_j^s)\|_{\mathrm{BMO}}\Big)\|f\|_{L^{p_i}}\\ &\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j - (D^{\alpha_j}b_j)^s\|_{\mathrm{BMO}}\Big)\\ &\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}_\varphi}\Big)\varphi^2(s)\,,\end{aligned}$$

and by (4) and Lemma 3, we obtain

$$\begin{aligned}\|T^{b^s}(f)\|_{L^{w_i}} &\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{Lip}_\beta}\Big)\|f\|_{L^{p_i}}\\ &\le Cs^{-2\beta}\varphi^2(s)\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}_\varphi}\Big)\,.\end{aligned}$$

Thus, for $s = t^{-1/n}$ and $i = 1, 2$,

$$\begin{aligned}m_{T^b(f)}\big((\psi^2)^{-1}(t)\big) &\le m_{T^b(f)}\big(t^{1/p_i}\varphi^2(t^{-1/n})\big)\\ &\le m_{T^{b-b^s}(f)}\big(t^{1/p_i}\varphi^2(t^{-1/n})/2\big) + m_{T^{b^s}(f)}\big(t^{1/p_i}\varphi^2(t^{-1/n})/2\big)\\ &\le C\Big[\Big(\frac{\varphi^2(s)}{t^{1/p_i}\varphi^2(s)}\Big)^{p_i} + \Big(\frac{s^{-2\beta}\varphi^2(s)}{t^{1/p_i}\varphi^2(s)}\Big)^{w_i}\Big] = Ct^{-1}\,.\end{aligned}$$

Taking $1 < p_2 < p < p_1 < n/2\beta$ and by Lemma 4, we obtain, for $\|f\|_{L^p} \le (p/p_1)^{1/p}$,

$$\int_{R^n}\psi^2\big(|T^b(f)(x)|\big)\,dx = \int_0^\infty m_{T^b(f)}\big((\psi^2)^{-1}(t)\big)\,dt \le C\,,$$

then, $\|T^b(f)\|_{L_{\psi^2}} \le C$.

This completes the proof of Theorem 1. $\square$

By using the same arguments as in the proof of Theorem 1 will give the proof of Theorem 2, we omit the details.

3. Applications

In this section we shall apply the Theorems 1 and 2 to some particular operators such as the Calderón-Zygmund singular integral operator and Littlewood-Paley operator, Marcinkiewicz operator.

Application 1. Calderón-Zygmund singular integral operator.

Let T be the Calderón-Zygmund operator (see [7, 8, 22, 23]), the multilinear operator related to T is defined by

$$T^b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j; x, y)}{|x-y|^m} K(x,y) f(y)\, dy\,.$$

Then it is easily to verify that **Key Lemma** holds for T^b, thus T satisfies the conditions in Theorem 1 and Theorem 1 holds for T^b.

Application 2. Littlewood-Paley operator.

Let $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

(1) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,

(2) $|\psi(x+y) - \psi(x)| \le C|y|^\varepsilon (1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

The multilinear Littlewood-Paley operator is defined by

$$g_\psi^b(f)(x) = \Big(\int_0^\infty |F_t^b(f)(x)|^2 \frac{dt}{t}\Big)^{1/2},$$

where

$$F_t^b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j; x, y)}{|x-y|^m} \psi_t(x-y) f(y)\, dy$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. We write that $F_t(f) = \psi_t * f$. We also define that

$$g_\psi(f)(x) = \Big(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\Big)^{1/2},$$

which is the Littlewood-Paley operator (see [23]).

Let H be the space $H = \big\{h\colon \|h\| = \big(\int_0^\infty |h(t)|^2 dt/t\big)^{1/2} < \infty\big\}$, then, for each fixed $x \in R^n$, $F_t^b(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H, and it is clear that

$$g_\psi(f)(x) = \|F_t(f)(x)\| \quad \text{and} \quad g_\psi^b(f)(x) = \|F_t^b(f)(x)\|\,.$$

It is easily to see that g_ψ^b satisfies the conditions of Theorem 2 (see [11]–[16]), thus Theorem 2 holds for g_ψ^b.

Application 3. Marcinkiewicz operator.

Let Ω be homogeneous of degree zero on R^n and $\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0$. Assume that $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$ for $0 < \gamma \le 1$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x-y|^\gamma$. The multilinear Marcinkiewicz operator is defined by

$$\mu_\Omega^b(f)(x) = \Big(\int_0^\infty |F_t^b(f)(x)|^2 \frac{dt}{t^3} \Big)^{1/2},$$

where

$$F_t^b(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \frac{\prod_{j=1}^l R_{m_j+1}(b_j; x, y)}{|x-y|^m} f(y)\, dy,$$

we write that

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\, dy.$$

We also define that

$$\mu_\Omega(f)(x) = \Big(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \Big)^{1/2},$$

which is the Marcinkiewicz operator (see [24]).

Let H be the space $H = \big\{h\colon \|h\| = \big(\int_0^\infty |h(t)|^2 dt/t^3\big)^{1/2} < \infty\big\}$. Then, it is clear that

$$\mu_\Omega(f)(x) = \|F_t(f)(x)\| \quad \text{and} \quad \mu_\Omega^b(f)(x) = \|F_t^b(f)(x)\|.$$

It is easily to see that μ_Ω^b satisfies the conditions of Theorem 2 (see [12]–[16, 24]), thus Theorem 2 holds for μ_Ω^b.

Application 4. Bochner-Riesz operator.

Let $\delta > (n-1)/2$, $F_t^\delta(\hat{f})(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n} B^\delta(z/t)$ for $t > 0$. The maximal Bochner-Riesz operator is defined by (see [17])

$$B_{\delta,*}(f)(x) = \sup_{t>0} |F_t^\delta(f)(x)|.$$

Set H be the space $H = \{h\colon \|h\| = \sup_{t>0} |h(t)| < \infty\}$. The multilinear operator related to the maximal Bochner-Riesz operator is defined by

$$B_{\delta,*}^b(f)(x) = \sup_{t>0} |F_{\delta,t}^b(f)(x)|,$$

where

$$F_{\delta,t}^b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j; x, y)}{|x-y|^m} B_t^\delta(x-y) f(y)\, dy,$$

We know

$$B_{\delta,*}^b(f)(x) = \|B_{\delta,t}^b(f)(x)\|.$$

It is easily to see that $B_{\delta,*}^b$ satisfies the conditions of Theorem 2 (see [12]–[13, 25]), thus Theorem 2 holds for $B_{\delta,*}^b$.

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