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VARIATIONAL METHOD AND CONJUGACY CRITERIA FOR HALF-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. We establish new conjugacy criteria for half-linear second order differential equations. These criteria are based on the relationship between conjugacy of the investigated equation and nonpositivity of the associated energy functional.

1. Introduction

The variational method, consisting in the relationship between disconjugacy of a differential equation and positivity of its associated energy functional, is one of the basic methods of the oscillation theory of a given differential equation. In our paper we deal with the second order half-linear differential equation

$$-\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0\,, \quad \Phi(x) := |x|^{p-1}\operatorname{sgn} x\,,\ \ p > 1\,, \tag{1}$$

where r, c are continuous functions and $r(t) > 0$, $t \in [-a, a]$, and the associated energy functional (equation (1) is its Euler-Lagrange equation)

$$\mathcal{F}(y; -a, a) := \int_{-a}^{a} \big[r(t)|y'|^p + c(t)|y|^p\big]\,\mathrm{d}t$$

considered over the class of (sufficiently smooth) functions satisfying $y(-a) = 0 = y(a)$.

The problem of (dis)conjugacy of the *linear* Sturm-Liouville second order differential equation

$$-\big(r(t)x'\big)' + c(t)x = 0 \tag{2}$$

(which is the special case $p = 2$ in (1)) is relatively deeply developed since it is closely related to the classical Jacobi condition in the calculus of variations. We refer to the classical paper [16] and to the later papers [3, 4, 9, 10, 13, 14] and the references given therein.

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Concerning (dis)conjugacy criteria for (1), similarly to (2), in addition to the variational method, the second basic method is the so-called Riccati technique consisting in the relationship between (1) and the associated Riccati type differential equation (related to (1) by the substitution $w = r\Phi(x'/x) = 0$)

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0\,, \quad q := \frac{p}{p-1}\,.$$

However, there are linear methods which do not extend to (1), a typical example is the transformation method, hence (dis)conjugacy theory of (1) is less developed than that of linear equation (2). We refer to [7, Sec. 1.3] for more details. We also refer to the papers [1, 2, 5, 6, 11, 15] for a brief survey of the known half-linear conjugacy criteria.

Our paper is motivated by [12] where conjugacy of (2) on a compact interval is investigated using the variational method. Here we show that some criteria of that paper, properly modified, can be extended to (1). The paper is organized as follows. In the next section we recall necessary concepts and results which we need to prove our main statements. The conjugacy criteria for (1) proved using the variational method are presented in Section 3 of our paper.

2. Preliminaries

The variational method in the half-linear oscillation theory is based on the following statement. Its proof is the equivalence of the statements (i) and (iv) in [7, Theorem 1.2.2].

Proposition 1. *Equation* (1) *is conjugate in an interval* $[a, b]$*, i.e., there exists a nontrivial solution of this equation having at least two zeros in this interval, if and only if there exists a continuous piecewise differentiable function* y *with* $y(a) = 0 = y(b)$ *such that* $\mathcal{F}(y; a, b) \le 0$*.*

An important role in the proofs of our conjugacy criteria is played by the half-linear trigonometric functions and their inverse functions. These functions are explicitly defined in [8] but implicitly already appear in some earlier papers. We refer to the book [7, Sec. 1.1] for a more detailed treatment of the half-linear trigonometric functions.

Consider the special half-linear differential equation

$$\big(\Phi(x')\big)' + (p-1)\Phi(x) = 0 \tag{3}$$

and denote by

$$\pi_p := 2\int_0^1 (1-s^p)^{-\frac{1}{p}}\,\mathrm{d}s = \frac{2}{p}\int_0^1 (1-u)^{-\frac{1}{p}}u^{-\frac{1}{q}}\,\mathrm{d}u = \frac{2}{p}B\Big(\frac{1}{p}, \frac{1}{q}\Big)\,,$$

where $q = \frac{p}{p-1}$ is the conjugate exponent of p and

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$$

is the Euler beta-function. Using the formulas

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,,\quad \Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x}$$

with the Euler gamma function $\Gamma(x)=\int_0^\infty t^{x-1}\,e^{-t}\,\mathrm{d}t$, we have

$$\pi_p=\frac{2\pi}{p\sin\frac{\pi}{p}}\,.$$

It can be shown that the solution of (3) given by the initial condition $x(0)=0$, $x'(0)=1$ can be extended over the whole real line as the odd $2\pi_p$ periodic function, this function we denote by $\sin_p t$ and it is called the *half-linear sine function.* Its derivative $\cos_p t:=\big(\sin_p t\big)'$ defines the *half-linear cosine function.* These functions satisfy the half-linear Pythagorean identity

$$|\sin_p t|^p+|\cos_p t|^p=1\,,\quad t\in\mathbb{R}\,. \tag{4}$$

The inverse functions to $\sin_p$ and $\cos_p$ defined on $\big[-\frac{\pi_p}{2},\frac{\pi_p}{2}\big]$ and $[0,\pi_p]$, respectively, define the functions $\operatorname{arcsin}_p t$ and $\operatorname{arccos}_p t$ in a natural way. We have

$$(\operatorname{arcsin}_p t)'=\frac{1}{(1-|t|^p)^{\frac{1}{p}}}\,. \tag{5}$$

The half-linear trigonometric functions and their inverse functions satisfy many extensions of the identities for the classical trigonometric functions and their inverse functions, but, for example, a half-linear version of the "linear" identity $\arcsin t+\arccos t=\frac{\pi}{2}$ is missing.

At the end of this section let us recall, for the sake of the later comparison, the results of [12] which we are going to extend to (1) in our paper.

Proposition 2. *Equation* (2) *is conjugate in the interval* $[-a,a]$ *whenever at least one of the following two conditions holds:*

(i) *There exists* $s\in[0,a)$ *such that*

$$\frac{3}{(a-s)(a+2s)}R(s)+C(s)\le 0\,, \tag{6}$$

where for $0\le s<a$

$$C(s)=\sup_{s<h<a}\frac{1}{2h}\int_{-h}^{h}c(t)\,\mathrm{d}t\,,\quad R(s)=\frac{1}{2(a-s)}\Big(\int_{-a}^{-s}r(t)\,\mathrm{d}t+\int_s^a r(t)\,\mathrm{d}t\Big)\,.$$

(ii) *It holds*

$$\frac{\pi^2}{4a^2}R+C\le 0\,, \tag{7}$$

where

$$R=\sup_{0<h<a}\frac{1}{2(a-h)}\Big(\int_{-a}^{-h}r(t)\,\mathrm{d}t+\int_h^a r(t)\,\mathrm{d}t\Big)\,,\quad C=\sup_{0<h<a}\frac{1}{2h}\int_{-h}^{h}c(t)\,\mathrm{d}t\,.$$

In both cases (i) *and* (ii) *the constant by* R *in* (6) *and* (7) *is the best possible.*

3. Conjugacy criteria

Recall that points t_1, $t_2 \in [-a,a]$, $-a \le t_1 < t_2 \le a$ are said to be *conjugate with respect to equation* (1) if there exists a nontrivial solution x of this equation such that $x(t_1) = 0 = x(t_2)$. Equation (1) is said to be *conjugate* in an interval $I \subset \mathbb{R}$ if there exists a pair of conjugate points in this interval. If $I = [a,b]$ is a closed bounded interval, conjugacy of (1) in I is equivalent to the fact that the nontrivial solution of this equation given by the initial condition $x(a) = 0$ has a zero in $(a,b]$.

The following statement is a generalization of the linear criterion proved in [12] as Theorem 2, see also part (ii) of Proposition 2.

Denote

$$R = \sup_{0<h<a} \frac{1}{2(a-h)} \Big(\int_{-a}^{-h} r(t)\,\mathrm{d}t + \int_{h}^{a} r(t)\,\mathrm{d}t \Big)\,, \quad C = \sup_{0<h<a} \frac{1}{2h} \int_{-h}^{h} c(t)\,\mathrm{d}t\,. \tag{8}$$

Theorem 1. *If*

$$(p-1)\Big(\frac{\pi_p}{2a}\Big)^p R + C \le 0\,, \tag{9}$$

then (1) *is conjugate in* $[-a,a]$. *The constant* $(p-1)\big(\frac{\pi_p}{2a}\big)^p$ *by* R *is the best possible one.*

Proof. Consider the function $v(t) = \sin_p\big(\frac{\pi_p}{2a}(t+a)\big)$, then $v(-a) = 0 = v(a)$. We will show that $\mathcal{F}(v; -a, a) \le 0$. To this end, denote

$$\gamma(t) = \Big|\sin_p \frac{\pi_p}{2a}(t+a)\Big|^p, \quad \sigma(t) = \Big|\cos_p \frac{\pi_p}{2a}(t+a)\Big|^p, \quad t \in [-a,a]\,,$$

$$k(y) = a - \frac{2a}{\pi_p} \arcsin_p(\sqrt[p]{y})\,, \quad h(y) = -a + \frac{2a}{\pi_p} \arccos_p(-\sqrt[p]{y})\,, \quad y \in [0,1]\,.$$

First we perform some preliminary computations of integrals associated with the functions h and k (which are simple for the classical cyclometric functions, but one needs to overcome certain technical difficulties in the half-linear case). The calculation of the integral $\int_0^1 \arcsin_p(\sqrt[p]{y})\mathrm{d}y$ can be done by means of integration by parts and Euler's gamma and beta function as follows

$$\begin{aligned}
\int_0^1 \arcsin_p(\sqrt[p]{y})\mathrm{d}y &= y \arcsin_p(\sqrt[p]{y})\big|_0^1 - \frac{1}{p}\int_0^1 \sqrt[p]{\frac{y}{1-y}}\mathrm{d}y \\
&= \frac{\pi_p}{2} - \frac{1}{p} B\Big(1+\frac{1}{p}, 1-\frac{1}{p}\Big) \\
&= \frac{\pi_p}{2} - \frac{1}{p}\frac{\Gamma(1+\frac{1}{p})\Gamma(1-\frac{1}{p})}{\Gamma(2)} \\
&= \frac{\pi_p}{2} - \frac{1}{p}\frac{\frac{1}{p}\Gamma(\frac{1}{p})\Gamma(1-\frac{1}{p})}{1} = \frac{\pi_p}{2} - \frac{1}{p}\frac{\frac{1}{p}\frac{\pi}{\sin(\frac{\pi}{p})}}{1} = \frac{\pi_p}{2} - \frac{1}{p}\frac{\pi_p}{2} = \frac{1}{q}\frac{\pi_p}{2}\,,
\end{aligned}$$

where $q = \frac{p}{p-1}$ is the conjugate exponent of p.

The integral $\int_0^1 \arccos_p(\sqrt[p]{y})\mathrm{d}y$ can be calculated via the previous integral as follows. We have, using the geometric meaning of the calculated integral (it is the area of a plane figure) and Pythagorean identity (4)

$$\begin{aligned}\int_0^1 \arccos_p(-\sqrt[p]{y})\mathrm{d}y &= \pi_p - \int_0^{\frac{\pi_p}{2}} \big|\cos_p t\big|^p \mathrm{d}y = \pi_p - \int_0^{\frac{\pi_p}{2}} \big[1 - \big|\sin_p t\big|^p\big]\mathrm{d}y \\ &= \frac{\pi_p}{2} + \int_0^{\frac{\pi_p}{2}} \big|\sin_p t\big|^p \mathrm{d}y = \frac{\pi_p}{2} + \Big[\frac{\pi_p}{2} - \int_0^1 \arcsin_p\big(\sqrt[p]{y}\big)\mathrm{d}y\Big] \\ &= \pi_p - \int_0^1 \arcsin_p\big(\sqrt[p]{y}\big)\mathrm{d}y = \pi_p - \frac{1}{q}\frac{\pi_p}{2}\,.\end{aligned}$$

Now we compute the energy functional for the function v. By means of Fubini's theorem we obtain

$$\begin{aligned}\mathcal{F}(v,-a,a) &= \int_{-a}^a \big[r(t)|v'(t)|^p + c(t)|v(t)|^p\big]\,\mathrm{d}t \\ &= \Big(\frac{\pi_p}{2a}\Big)^p \int_{-a}^a r(t)\Big|\cos_p \frac{\pi_p}{2a}(t+a)\Big|^p \mathrm{d}t + \int_{-a}^a c(t)\Big|\sin_p \frac{\pi_p}{2a}(t+a)\Big|^p \mathrm{d}t \\ &= \Big(\frac{\pi_p}{2a}\Big)^p \int_{-a}^a r(t)\int_0^{\sigma(t)} \mathrm{d}y\,\mathrm{d}t + \int_{-a}^a c(t)\int_0^{\gamma(t)} \mathrm{d}y\,\mathrm{d}t \\ &= \Big(\frac{\pi_p}{2a}\Big)^p \int_0^1 \Big(\int_{-a}^{-h(y)} r(t)\,\mathrm{d}t + \int_{h(y)}^a r(t)\,\mathrm{d}t\Big)\mathrm{d}y + \int_0^1 \Big(\int_{-k(y)}^{k(y)} c(t)\,\mathrm{d}t\Big)\mathrm{d}y \\ &= \Big(\frac{\pi_p}{2a}\Big)^p 2\int_0^1 \frac{1}{2(a-h(y))}\Big(\int_{-a}^{-h(y)} r(t)\,\mathrm{d}t + \int_{h(y)}^a r(t)\mathrm{d}t\Big)(a-h(y))\mathrm{d}y \\ &\quad + 2\int_0^1 \frac{1}{2k(y)}\Big(\int_{-k(y)}^{k(y)} c(t)\mathrm{d}t\Big)k(y)\mathrm{d}y \\ &\le \Big(\frac{\pi_p}{2a}\Big)^p 2R\int_0^1 \big(a-h(y)\big)\mathrm{d}y + 2C\int_0^1 k(y)\,\mathrm{d}y \\ &= 2a\Big[\Big(\frac{\pi_p}{2a}\Big)^p \frac{R}{q} + \frac{C}{p}\Big] \\ &= \frac{2a}{p}\Big[(p-1)\Big(\frac{\pi_p}{2a}\Big)^p R + C\Big]\,.\end{aligned}$$

Therefore, from (9) we get $\mathcal{F}(v,-a,a) \le 0$ and by Proposition 1 equation (1) is conjugate in $[-a,a]$.

To show that the constant $(p-1)\big(\frac{\pi_p}{2a}\big)^p$ in (9) is the best possible observe that the function $x(t) = \sin_p\big(\frac{\pi_p}{2\beta}(t+a)\big)$, where $\beta > a$, is the positive solution to the equation

$$(\Phi(x'))' - (p-1)\Big(\frac{\pi}{2\beta}\Big)^p \Phi(x) = 0\,, \qquad t \in [-a,a]\,, \tag{10}$$

as can be verified by a direct computation using the fact that $\sin_p t$ is a solution of (3). Consider now the functions $r(t) \equiv 1$ and $c(t) \equiv -(p-1)\big(\frac{\pi}{2\beta}\big)^p$. Substituting these functions into (8) we find that they satisfy the equality $(p-1)\big(\frac{\pi}{2\beta}\big)^p R+C=0$. Consequently, by using the first part of our criterion, there exists a pair of conjugate points relative to (10). On the other hand, the function $\sin_p\big(\frac{\pi_p}{2\beta}(t+a)\big)$ is a positive solution of this equation for $t\in[-a,a]$, hence this equation is discojugate in $[-a,a]$. Therefore, the constant $(p-1)\big(\frac{\pi_p}{2a}\big)^p$ is the best possible one since it cannot be replaced by the constant $(p-1)\big(\frac{\pi_p}{2\beta}\big)^p<(p-1)\big(\frac{\pi_p}{2a}\big)^p$. $\square$

The next criterion is a half-linear extension of Theorem 1 in [12], see the part (i) in Proposition 2. Similarly as in the previous case we denote

$$R(s)=\frac{1}{2(a-s)}\Big(\int_{-a}^{-s} r(t)\,\mathrm{d}t+\int_s^a r(t)\,\mathrm{d}t\Big),\quad C(s)=\sup_{s<h<a}\frac{1}{2h}\int_{-h}^{h} c(t)\,\mathrm{d}t\,. \tag{11}$$

Theorem 2. *If there exists a number $s\in\mathbb{R}$, $0\le s<a$, such that*

$$\frac{p+1}{(a-s)^{p-1}(a+ps)}R(s)+C(s)\le 0 \tag{12}$$

then equation (1) *is conjugate in* $[-a,a]$.

Proof. Consider the test function $v\colon[-a,a]\to\mathbb{R}$ defined as follows

$$v(t)=\begin{cases} a-|t| & \text{for} \quad |t|\in[s,a]\,,\\ a-s & \text{for} \quad t\in[-s,s]\,.\end{cases}$$

Then one can easily get that

$$v'(t)=\begin{cases} -\operatorname{sgn} t & \text{for} \quad |t|\in[s,a]\,,\\ 0 & \text{for} \quad t\in[-s,s]\,.\end{cases}$$

Set $h(y)=a-\sqrt[p]{y}$ and by means of Fubini's theorem we obtain

$$\begin{aligned}
\mathcal{F}(v;-a,a)&=\int_{-a}^{a} r(t)|v'(t)|^p+c(t)|v(t)|^p\,\mathrm{d}t=\int_{-a}^{-s} r(t)\,\mathrm{d}t\\
&\quad+\int_s^a r(t)\,\mathrm{d}t+\int_{-a}^{a}\int_0^{|v(t)|^p} c(t)\,\mathrm{d}y\,\mathrm{d}t\\
&=2(a-s)R(s)+\int_0^{(a-s)^p}\int_{-h(y)}^{h(y)} c(t)\,\mathrm{d}t\,\mathrm{d}y\\
&=2(a-s)R(s)+2\int_0^{(a-s)^p}\Big(\frac{1}{2h(y)}\int_{-h(y)}^{h(y)} c(t)\,\mathrm{d}t\Big)h(y)\,\mathrm{d}y\\
&\le 2(a-s)R(s)+2C(s)\int_0^{(a-s)^p} h(y)\,\mathrm{d}y\\
&=2(a-s)R(s)+2C(s)\Big[a(a-s)^p-\frac{1}{\frac{1}{p}+1}(a-s)^{(\frac{1}{p}+1)p}\Big]\\
&=\frac{2}{p+1}(a-s)^p(a+ps)\Big[\frac{p+1}{(a-s)^{p-1}(a+ps)}R(s)+C(s)\Big]\le 0\,.
\end{aligned}$$

This implies conjugacy of (1) in $[-a,a]$ by Proposition 1. $\square$

Remark 1. In the linear case $p = 2$, it is shown in [12] that the constant $\frac{3}{(a-s)(a+2s)}$ by $R(s)$ in (6) is the best possible via a certain relatively complicated construction of the functions r, c in (2) for which (7) with a constant less than $\frac{3}{(a-s)(a+2s)}$ holds but (2) with these functions is disconjugate in $[-a, a]$. This construction uses properties of the hyperbolic functions $\sinh t$, $\cosh t$ and we were not able to extend this construction to the half-linear case yet. Nevertheless, we conjecture that the constant $\frac{p+1}{(a-s)^{p-1}(a+ps)}$ is the best one also in the general half-linear case. To prove this conjecture is a subject of the present investigation.

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