ARCHIVUM MATHEMATICUM (BRNO)
Tomus 47 (2011), 263–278

PERIODIC SOLUTIONS FOR n-TH ORDER DELAY DIFFERENTIAL EQUATIONS WITH DAMPING TERMS

Lijun Pan

Abstract. By using the coincidence degree theory of Mawhin, we study the existence of periodic solutions for n th order delay differential equations with damping terms $x^{(n)}(t)=\sum\limits_{i=1}^{s} b_i[x^{(i)}(t)]^{2k-1}+f(x(t-\tau(t)))+p(t)$. Some new results on the existence of periodic solutions of the investigated equation are obtained.

1. Introduction

In this paper, we are concerned with the existence of periodic solutions of the n th order delay differential equation

$$x^{(n)}(t)=\sum_{i=1}^{s} b_i[x^{(i)}(t)]^{2k-1}+f\big(x(t-\tau(t))\big)+p(t)\,, \tag{1.1}$$

where n is a positive integer, $s\le n-1$ a positive integer, $b_i (i=1,\cdots,s)$ are constants and $k>1$ is an integer, $f\in C(R,R)$ for $\forall x\in R$, $p\in C(R,R)$ with $p(t+T)=p(t)$.

In recent years, some researchers used the coincidence degree theory of Mawhin to study the existence of periodic solutions of first, second or third order differential equations [5, 6], [9][15]–[19], [22, 23], [25, 26]. For example, in [16], Lu and Ge studied the following delay differential equation

$$x^{''}(t)=f\big(t,x(t),x(t-\tau(t),x^{'}(t))\big)+e(t)\,. \tag{1.2}$$

The authors established the theorems of the existence of periodic solutions of Eq. (1.2), one of the theorems was related to the deviating argument $\tau(t)$. In [26], Zhang and Wang studied the following differential equation

$$x^{'''}(t)+ax^{''2k-1}(t)+bx^{'2k-1}(t)+cx^{2k-1}(t)+g\big(t,x(t-\tau_1,x^{'}(t-\tau_2))\big)=p(t)\,. \tag{1.3}$$

The authors established the existence of periodic solutions of Eq. (1.3) under some conditions on a, b, c and $2k-1$.

Periodic solutions for n, $2n$ and $2n+1$ th order differential equations were discussed in [1]–[4], [8] [11]–[14], [20, 21], [24]. For example, in [11], by means of

2010 *Mathematics Subject Classification*: primary 34C25.
Key words and phrases: delay differential equations, periodic solution, coincidence degree.
Received September 11, 2010, revised April 2011. Editor O. Došlý.

the theory of topological degree, the authors obtained the existence of periodic solutions of the following differential equation without delay

$$x^{(n)}(t) + \sum_{i=2}^{n-1} a_i x^{(i)}(t) + f_1\big(x(t)\big)|x^{'}(t)|^2 + f_2\big(x(t)\big)x^{'}(t) + g\big(t, x(t)\big) = e(t)\,. \tag{1.4}$$

In [20, 1], the existence of periodic solutions of higher order differential equation of the form

$$x^{(n)}(t) = \sum_{i=1}^{n-1} b_i x^{(i)}(t) + f\big(t, x(t), x(t-\tau_1(t)), \dots, x(t-\tau_m(t))\big) + p(t)\,, \tag{1.5}$$

was studied. The authors obtained the results based on the damping terms $x^{(i)}(t)$ and the delay $\tau_i(t)$. In [21], a class of n-th order functional differential equations with damping terms $[x^{(i)}(t)]^k$ $(i = 1, \dots, s)$, $k \geq 1$ were dicussed, but the results of [21] were not related to the delay τ.

In the present paper, by using Mawhin's continuation theorem, we will establish some theorems on the existence of periodic solutions of Eq. (1.1). The results are related not only to b_i and $f(x)$ but also to positive integers s, k. In addition, the delay $\tau(t)$ plays an important role in our theorems. We also give an example to illustrate our new results.

2. Some lemmas

We establish the theorems based on the following Lemmas.

Lemma 2.1 ([16])**.** *Let $n_1 > 1$, $\alpha \in [0, +\infty)$ be constants, $s \in C(R, R)$ with $s(t+T) = s(t)$, and $s(t) \in [-\alpha, \alpha]$, $\forall t \in [0, T]$. Then $\forall x \in C^1(R, R)$ with $x(t+T) = x(t)$, we have*

$$\int_0^T |x(t) - x(t - s(t))|^{n_1}\, dt \leq 2\alpha^{n_1} \int_0^T |x'(t)|^{n_1}\, dt\,. \tag{2.1}$$

Lemma 2.2 ([10])**.** *Suppose $x \in C^1(R, R)$, and $x(t+T) = x(t)$. Then for $\xi \in [0, T]$, we have*

$$|x(t)|_\infty \leq |x(\xi)| + \frac{1}{2}\int_0^T |x^{'}(t)|\, dt\,. \tag{2.2}$$

Lemma 2.3 ([10])**.** *Suppose $x \in C^2(R, R)$, and $x(t+T) = x(t)$. Then*

$$|x^{'}(t)|_\infty \leq \frac{1}{2}\int_0^T |x^{''}(t)|\, dt\,. \tag{2.3}$$

Lemma 2.4. *If $k \geq 1$ is an integer, $x \in C^n(R, R)$, and $x(t+T) = x(t)$. Then*

$$\Big(\int_0^T |x'(t)|^k\, dt\Big)^{\frac{1}{k}} \leq \frac{T}{2}\Big(\int_0^T |x''(t)|^k\, dt\Big)^{\frac{1}{k}} \leq \dots \leq \frac{T^{n-1}}{2^{n-1}}\Big(\int_0^T |x^{(n)}(t)|^k\, dt\Big)^{\frac{1}{k}}\,. \tag{2.4}$$

Proof. From Lemma 2.3, using the Hölder inequality, we obtain

$$\Big(\int_0^T |x'(t)|^k dt\Big)^{\frac{1}{k}} \le T^{\frac{1}{k}}|x'(t)|_\infty \le \frac{1}{2}T^{\frac{1}{k}}\int_0^T |x''(t)|\,dt \le \frac{T}{2}\Big(\int_0^T |x''(t)|^k\,dt\Big)^{\frac{1}{k}}. \tag{2.5}$$

By induction, we have

$$\Big(\int_0^T |x^{'}(t)|^k\,dt\Big)^{\frac{1}{k}} \le \frac{T}{2}\Big(\int_0^T |x''(t)|^k\,dt)^{\frac{1}{k}} \le \cdots \le \frac{T^{n-1}}{2^{n-1}}\Big(\int_0^T |x^{(n)}(t)|^k\,dt\Big)^{\frac{1}{k}}. \tag{2.6}$$

We first introduce Mawhin's continuation theorem.

Let X and Y be Banach spaces, $L\colon D(L)\subset X\to Y$ be a Fredholm operator of index zero, here $D(L)$ denotes the domain of L. $P\colon X\to X$, $Q\colon Y\to Y$ be projectors such that

$$\operatorname{Im}P=\operatorname{Ker}L,\ \operatorname{Ker}Q=\operatorname{Im}L,\ X=\operatorname{Ker}L\oplus\operatorname{Ker}P,\ Y=\operatorname{Im}L\oplus\operatorname{Im}Q\,.$$

It follows that

$$L|_{D(L)\cap\operatorname{Ker}P}\colon D(L)\cap\operatorname{Ker}P\to\operatorname{Im}L$$

is invertible, we denote the inverse of that map by K_p. Let Ω be an open bounded subset of X, $D(L)\cap\overline{\Omega}\ne$ ‰$_0$, the map $N\colon X\to Y$ will be called L-compact in $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N\colon\overline{\Omega}\to X$ is compact. □

Lemma 2.5 ([7])**.** *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lx\ne\lambda Nx$, $\forall x\in\partial\Omega\cap D(L)$, $\lambda\in(0,1)$;

(ii) $QNx\ne 0$, $\forall x\in\partial\Omega\cap\operatorname{Ker}L$;

(iii) $deg\{QNx,\Omega\bigcap\operatorname{Ker}L,0\}\ne 0$.

Then the equation $Lx=Nx$ has at least one solution in $\overline{\Omega}\bigcap D(L)$.

Now, we define $Y=\{x\in C(R,R)\mid x(t+T)=x(t)\}$ with the norm $|x|_\infty=\max_{t\in[0,T]}\{|x(t)|\}$ and $X=\{x\in C^{n-1}(R,R)\mid x(t+T)=x(t)\}$ with norm $\|x\|=\max\{|x|_\infty,|x^{'}|_\infty\dots,|x^{(n-1)}|_\infty\}$, we can easily see that X,Y are two Banach spaces. We also define the operators L and N as follows:

$$\begin{gathered} L\colon D(L)\subset X\to Y\,,\quad Lx=x^{(n)}\,,\\ D(L)=\{x|x\in C^n(R,R),x(t+T)=x(t)\}\end{gathered} \tag{2.7}$$

$$N\colon X\to Y\,,\quad Nx=\sum_{i=1}^{n-1}b_i[x^{(i)}(t)]^k+f\big(x(t-\tau(t))\big)+p(t)\,. \tag{2.8}$$

It is easy to see that Eq. (1.1) can be converted to the abstract equation $Lx=Nx$. Moreover, from the definition of L, we see that $\ker L=R$, $\dim(\ker L)=1$, $\operatorname{Im}L=\{y|y\in Y,\int_0^T y(s)\,ds=0\}$ is closed, and $\dim(Y\setminus\operatorname{Im}L)=1$, we have $\operatorname{codim}(\operatorname{Im}L)=\dim(\ker L)$. So L is a Fredholm operator with index zero. Let

$$P\colon X\longrightarrow\operatorname{Ker}L,\quad Px=x(0),\quad Q\colon Y\longrightarrow Y\setminus\operatorname{Im}L,\quad Qy=\frac{1}{T}\int_0^T y(t)\,dt$$

and let

$$L|_{D(L)\cap \operatorname{Ker} P}: D(L)\cap \operatorname{Ker} P \to \operatorname{Im} L\,.$$

Then $L|_{D(L)\cap \operatorname{Ker} P}$ has a unique continuous inverse K_p. One can easily find that N is L-compact in $\overline{\Omega}$, where $\overline{\Omega}$ is an open bounded subset of X.

3. Main results

Let

$$A(s)=\begin{cases}\sum_{i=1}^{s-2}|b_i|\left(\frac{T^{s-i}}{2^{s-i}}\right)^{2k-1}, & s>2\\ 0\,, & s=2\end{cases}$$

$$B(s)=\frac{1}{T^{\frac{1}{2k}}\gamma_1^{\frac{1}{2k-1}}}\sum_{i=1}^{s}|b_i|^{\frac{1}{2k-1}}\frac{T^{s-i}}{2^{s-i}}+\frac{T^{s-\frac{1}{2k}}}{2^s}$$

$$C(s)=\frac{T^{s-\frac{1}{k}}}{2^{s-2+\frac{1}{2k}}}(2k-1)\beta|\tau(t)|_\infty$$

We will need several conditions on $f(x)$:

(H_1) there exist constants $\gamma\ge 0$, $\gamma_1>0$ and $D>0$ such that

$$|f(x)|\ge \gamma+\gamma_1|x|^{2k-1}\,,\quad |x|>D\,,$$

(H_2) there exists a constant $\beta>0$ such that

$$|f(x)-f(y)|\le \beta|x^{2k-1}-y^{2k-1}|\,,$$

(H_3) $f\in C^1(R,R)$, and there exists a constant $\gamma_2>0$ such that

$$\lim_{x\to\infty}\left|\frac{f^{'}(x)}{x^{2k-2}}\right|\le \gamma_2\,.$$

Theorem 3.1. *Suppose that $n=2m$, $m>0$ an integer, $s>1$ an odd, and conditions (H_1)–(H_3) hold. If*

$$A(s)+B^{2k-2}(s)C(s)+\gamma_2 B^{2k-2}(s)\frac{T^{s-\frac{1}{k}}}{2^s}<|b_s|\,, \tag{3.1}$$

then Eq. (1.1) *has at least one T-periodic solution.*

Proof. Consider the equation

$$Lx=\lambda Nx\,,\quad \lambda\in(0,1),$$

where L and N are defined by (2.7) and (2.8). Let

$$\Omega_1=\{x\in D(L)/\operatorname{Ker} L, Lx=\lambda Nx \text{ for some } \lambda\in(0,1)\}\,.$$

For $x\in\Omega_1$, we have

$$x^{(n)}(t)=\lambda\sum_{i=1}^{s}b_i[x^{(i)}(t)]^{2k-1}+\lambda f(x(t-\tau(t)))+\lambda p(t),\lambda\in(0,1). \tag{3.2}$$

Integrating (3.2) on $[0,T]$, we have

$$\sum_{i=1}^{s} b_i \int_0^T \left[x^{(i)}(t)\right]^{2k-1} dt + \int_0^T f\big(x(t-\tau(t))\big)\,, dt + \int_0^T p(t)\,dt = 0\,. \tag{3.3}$$

We will prove that there exists $t_1 \in [0,T]$ such that

$$|x(t_1)| \le \frac{1}{T^{\frac{1}{2k}}\gamma_1^{\frac{1}{2k-1}}} \sum_{i=1}^{s} |b_i|^{\frac{1}{2k-1}} \frac{T^{n-i}}{2^{n-i}} \Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{2k}} + D^*\,, \tag{3.4}$$

where $D^* = \max\big\{D, \big(\frac{||p(t)|_\infty - \gamma|}{\gamma_1}\big)^{\frac{1}{2k-1}}\big\}$. In fact, if there exists some $\xi \in [0,T]$ such that $|x(\xi - \tau(\xi))| \le D$, since $\xi - \tau(\xi) \in R$, then there exist some integer j and some $t_1 \in [0,T]$ such that $\xi - \tau(\xi) = jT + t_1$. So we have

$$\begin{aligned} |x(t_1)| &= |x(\xi - \tau(\xi))| \le D \\ &\le \frac{1}{T^{\frac{1}{2k}}\gamma_1^{\frac{1}{2k-1}}} \sum_{i=1}^{s} |b_i|^{\frac{1}{2k-1}} \frac{T^{n-i}}{2^{n-i}} \Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{2k}} + D^*\,. \end{aligned} \tag{3.5}$$

If $\forall t \in [0,T]$, $|x(t-\tau(t))| > D$, then from (3.3) and applying Lemma 2.4, there exists a $\xi \in [0,T]$ such that

$$\begin{aligned} |f\big(x(\xi - \tau(\xi))\big)| &\le \frac{1}{T}\sum_{i=1}^{s} |b_i| \int_0^T |x^{(i)}(t)|^{2k-1}\,dt + \frac{1}{T}\int_0^T |p(t)| \\ &\le \frac{1}{T}\sum_{i=1}^{s} |b_i| \Big(\frac{T^{s-i}}{2^{s-i}}\Big)^{2k-1} \int_0^T |x^{(s)}(t)|^{2k-1}\,dt + |p(t)|_\infty\,. \end{aligned} \tag{3.6}$$

In view of condition (H_1), it follows that

$$\begin{aligned} \gamma + \gamma_1 |x(\xi - \tau(\xi))|^{2k-1} &\le |f(x(\xi - \tau(\xi)))| \\ &\le \frac{1}{T}\sum_{i=1}^{s} |b_i| (\frac{T^{s-i}}{2^{s-i}})^{2k-1} \int_0^T |x^{(s)}(t)|^{2k-1}\,dt + |p(t)|_\infty\,. \end{aligned} \tag{3.7}$$

So

$$\begin{aligned} &|x(\xi - \tau(\xi))|^{2k-1} \\ &\qquad \le \frac{1}{T\gamma_1}\sum_{i=1}^{s} |b_i| (\frac{T^{s-i}}{2^{s-i}})^{2k-1} \int_0^T |x^{(s)}(t)|^{2k-1}\,dt + \frac{||p(t)|_\infty - \gamma|}{\gamma_1}\,. \end{aligned} \tag{3.8}$$

Using inequality

$$(a+b)^l \le a^l + b^l \quad \text{for} \quad a \ge 0,\ b \ge 0 \quad \text{and} \quad 0 \le l \le 1\,, \tag{3.9}$$

it follows from (3.8) that

$$|x(\xi-\tau(\xi))| \le \Big(\frac{1}{T\gamma_1}\Big)^{\frac{1}{2k-1}} \sum_{i=1}^{s} |b_i|^{\frac{1}{2k-1}} \frac{T^{s-i}}{2^{s-i}} \Big(\int_0^T |x^{(s)}(t)|^{2k-1}\,dt\Big)^{\frac{1}{2k-1}}$$

$$+\Big(\frac{||p(t)|_\infty-\gamma|}{\gamma_1}\Big)^{\frac{1}{2k-1}}. \tag{3.10}$$

Using inequality

$$\Big(\frac{1}{T}\int_0^T |x(t)|^r|\Big)^{\frac{1}{r}} \le \Big(\frac{1}{T}\int_0^T |x(t)|^l|\Big)^{\frac{1}{l}} \quad \text{for} \quad 0\le r\le l \quad \text{and} \quad \forall x\in R, \tag{3.11}$$

from (3.10), we obtain

$$|x(\xi-\tau(\xi))| \le \frac{1}{T^{\frac{1}{2k}}\gamma_1^{\frac{1}{2k-1}}} \sum_{i=1}^{s} |b_i|^{\frac{1}{2k-1}} \frac{T^{s-i}}{2^{s-i}} \Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{2k}} + D^*. \tag{3.12}$$

Then there exist some integer j and some $t_1\in[0,T]$ such that $\xi-\tau(\xi)=jT+t_1$. So we have

$$\begin{aligned}|x(t_1)| &= |x(\xi-\tau(\xi))|\\ &\le \frac{1}{T^{\frac{1}{2k}}\gamma_1^{\frac{1}{2k-1}}} \sum_{i=1}^{s} |b_i|^{\frac{1}{2k-1}} \frac{T^{s-i}}{2^{s-i}} \Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{2k}} + D^*.\end{aligned} \tag{3.13}$$

From Lemma 2.2 and Lemma 2.4, using the Hölder inequality, we obtain

$$\begin{aligned}|x(t)|_\infty &\le |x(t_1)| + \frac{1}{2}\int_0^T |x^{'}(t)|\,dt \le |x(t_1)| + \frac{1}{2}T^{1-\frac{1}{2k}}\int_0^T (|x^{'}(t)|^{2k}dt)^{\frac{1}{2k}}\\ &\le \Big[\frac{1}{T^{\frac{1}{2k}}\gamma_1^{\frac{1}{2k-1}}} \sum_{i=1}^{s} |b_i|^{\frac{1}{2k-1}} \frac{T^{s-i}}{2^{s-i}} + \frac{T^{s-\frac{1}{2k}}}{2^s}\Big]\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt)^{\frac{1}{2k}+D^*}\\ &= B(s)\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{2k}} + D^*.\end{aligned} \tag{3.14}$$

On the other hand, multiplying both sides of (3.2) by $x^{(s)}(t)$, and integrating on $[0,T]$, we have

$$\begin{aligned}\int_0^T x^{(n)}(t)x^{(s)}(t)\,dt &= \lambda\sum_{i=1}^{s} b_i\int_0^T [x^{(i)}(t)]^{2k-1}x^{(s)}(t)\,dt\\ &\quad +\lambda\int_0^T f(x(t-\tau(t)))x^{(s)}(t)\,dt + \lambda\int_0^T p(t)x^{(s)}(t)\,dt.\end{aligned} \tag{3.15}$$

Since $n=2m$ and s is odd, then

$$\int_0^T x^{(2m)}(t)x^{(s)}(t)\,dt = 0, \qquad \int_0^T [x^{(s-1)}(t)]^{2k-1}x^{(s)}(t)\,dt = 0. \tag{3.16}$$

It follows from (3.15) that

$$
\begin{aligned}
b_s\int_0^T |x^{(s)}(t)|^{2k}\,dt &= -\sum_{i=1}^{s-2} b_i\int_0^T [x^{(i)}(t)]^{2k-1}x^{(s)}(t)\,dt\\
&\quad -\int_0^T f(x(t-\tau(t)))x^{(s)}(t)\,dt-\int_0^T p(t)x^{(s)}(t)\,dt\\
&= -\sum_{i=1}^{s-2} b_i\int_0^T [x^{(i)}(t)]^{2k-1}x^{(s)}(t)\,dt\\
&\quad +\int_0^T [f(x(t))-f(x(t-\tau(t)))]x^{(s)}(t)\,dt\\
&\quad -\int_0^T f(x(t))x^{(s)}(t)\,dt-\int_0^T p(t)x^{(s)}(t)\,dt\,. \qquad (3.17)
\end{aligned}
$$

Noting that

$$
\int_0^T f(x(t))x^{(s)}(t)\,dt = -\int_0^T f^{'}(x(t))x^{(s-1)}(t)x^{'}(t)\,dt\,, \qquad (3.18)
$$

by using the Hölder inequality and Lemma 2.4, we have

$$
\begin{aligned}
|b_s|\int_0^T |x^{(s)}(t)|^{2k}\,dt &\le \int_0^T |x^{(s)}(t)|\Big[\sum_{i=1}^{s-2}|b_i|\,|x^{(i)}(t)|^{2k-1}\\
&\quad +|f(x(t))-f(x(t-\tau(t)))|+|p(t)|\Big]\,dt+\Big|\int_0^T f(x(t))x^s(t)\,dt\Big|\\
&\le \Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{2k}}\Big[A(s)\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{1-\frac{1}{2k}}\\
&\quad +\Big(\int_0^T |f(x(t))-f(x(t-\tau))|^{\frac{2k}{2k-1}}\,dt\Big)^{1-\frac{1}{2k}}\\
&\quad +|p(t)|_\infty T^{1-\frac{1}{2k}}\Big]+\int_0^T |f^{'}(x(t))|\,|x^{(s-1)}(t)|\,|x^{'}(t)|\,dt\,. \qquad (3.19)
\end{aligned}
$$

Set $s(t)=[x(t)]^{2k-1}$, then $s'(t)=(2k-1)[x(t)]^{2k-2}x'(t)$. Hence applying Lemma 2.1 and from condition (H_2), we have

$$
\begin{aligned}
&\int_0^T |f(x(t))-f(x(t-\tau(t)))|^{\frac{2k}{2k-1}}\,dt\\
&\qquad \le \beta^{\frac{2k}{2k-1}}\int_0^T |[x(t)]^{2k-1}-[x(t-\tau(t))]^{2k-1}|^{\frac{2k}{2k-1}}\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= \beta^{\frac{2k}{2k-1}} \int_0^T |s(t) - s(t-\tau(t))|^{\frac{2k}{2k-1}}\, dt \\
&\le 2(\beta|\tau(t)|_\infty)^{\frac{2k}{2k-1}} \int_0^T |s^{'}(t)|^{\frac{2k}{2k-1}}\, dt \\
&= 2[(2k-1)\beta|\tau(t)|_\infty]^{\frac{2k}{2k-1}} \int_0^T |[x(t)]^{2k-2}x^{'}(t)|^{\frac{2k}{2k-1}}\, dt \\
&\le 2[(2k-1)\beta|\tau(t)|_\infty]^{\frac{2k}{2k-1}} |x(t)|_\infty^{\frac{(4k-4)k}{2k-1}} \int_0^T |x^{'}(t)|^{\frac{2k}{2k-1}}\, dt\,. \qquad (3.20)
\end{aligned}
$$

Hence, using inequality (3.11) and Lemma 2.4, we get

$$
\begin{aligned}
&\Big(\int_0^T |f(x(t)) - f(x(t-\tau))|^{\frac{2k}{2k-1}}\, dt\Big)^{1-\frac{1}{2k}} \\
&\quad\le 2^{1-\frac{1}{2k}}(2k-1)\beta|\tau(t)|_\infty |x(t)|_\infty^{2k-2}\Big(\int_0^T |x^{'}(t)|^{\frac{2k}{2k-1}}\, dt\Big)^{1-\frac{1}{2k}} \\
&\quad\le 2^{1-\frac{1}{2k}}T^{1-\frac{1}{k}}(2k-1)\beta|\tau(t)|_\infty |x(t)|_\infty^{2k-2}\Big(\int_0^T |x^{'}(t)|^{2k}\, dt\Big)^{\frac{1}{2k}} \\
&\quad\le \frac{T^{s-\frac{1}{k}}}{2^{s-2+\frac{1}{2k}}}(2k-1)\beta|\tau(t)|_\infty |x(t)|_\infty^{2k-2}\Big(\int_0^T |x^{(s)}(t)|^{2k}\, dt\Big)^{\frac{1}{2k}} \\
&\quad= C(s)|x(t)|_\infty^{2k-2}\Big(\int_0^T |x^{(s)}(t)|^{2k}\, dt\Big)^{\frac{1}{2k}}\,. \qquad (3.21)
\end{aligned}
$$

Choose a constant $\varepsilon > 0$ such that

$$
A(s) + B^{2k-2}(s)C(s) + (\gamma_2 + \varepsilon)B^{2k-2}(s)\frac{T^{s+1-\frac{1}{k}}}{2^s} < |b_s|\,. \qquad (3.22)
$$

For the above constant $\varepsilon > 0$, we see from condition (H_3) that there is a constant $\delta > 0$ such that

$$
|f^{'}(x(t))| < (\gamma_2 + \varepsilon)|x(t)|^{2k-2}\,, \quad \text{for} \quad |x(t)| > \delta,\ t \in [0, T]\,. \qquad (3.23)
$$

Denote

$$
\Delta_1 = \{t \in [0, T] : |x(t)| \le \delta\}\,, \quad \Delta_2 = \{t \in [0, T] : |x(t)| > \delta\}\,. \qquad (3.24)
$$

Applying Lemma 2.4 and the Hölder inequality, we have

$$
\begin{aligned}
&\int_0^T |f^{'}(x(t))|\,|x^{(s-1)}(t)||x^{'}(t)|\, dt \le |x^{'}(t)|_\infty \int_0^T |f^{'}(x(t))|\,|x^{(s-1)}(t)|\, dt \\
&\quad\le |x^{'}(t)|_\infty\Big(\int_0^T |f^{'}(x(t))|^{\frac{2k}{2k-1}}\, dt\Big)^{1-\frac{1}{2k}}\Big(\int_0^T |x^{(s-1)}(t)|^{2k}\, dt\Big)^{\frac{1}{2k}} \\
&\quad\le \frac{T}{2}|x^{'}(t)|_\infty\Big(\int_0^T |f^{'}(x(t))|^{\frac{2k}{2k-1}}\, dt\Big)^{1-\frac{1}{2k}}\Big(\int_0^T |x^{(s)}(t)|^{2k}\, dt\Big)^{\frac{1}{2k}}\,. \qquad (3.25)
\end{aligned}
$$

Since

$$\Big(\int_0^T |f'(x(t))|^{\frac{2k}{2k-1}}\,dt\Big)^{1-\frac{1}{2k}} \le \Big[\int_{\Delta_1} |f'(x(t))|^{\frac{2k}{2k-1}}\,dt + \int_{\Delta_2} |f'(x(t))|^{\frac{2k}{2k-1}}\,dt\Big]^{1-\frac{1}{2k}}$$

$$\le f'_\delta T^{1-\frac{1}{2k}} + T^{1-\frac{1}{2k}}(\gamma_2+\varepsilon)|x(t)|_\infty^{2k-2}\,, \tag{3.26}$$

where $f'_\delta = \max_{|x|\le\delta}|f'(x)|$, using the Hölder inequality and Lemma 2.3, we have

$$|x'(t)|_\infty \le \frac{1}{2}\int_0^T |x''(t)|\,dt \le \frac{1}{2}T^{1-\frac{1}{2k}}\Big(\int_0^T |x''(t)|^{2k}\,dt\Big)^{\frac{1}{2k}}$$

$$\le \frac{T^{s-1-\frac{1}{2k}}}{2^{s-1}}\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{2k}}\,. \tag{3.27}$$

Hence we obtain

$$\int_0^T |f'(x(t))||x^{(s-1)}(t)||x'(t)|\,dt \le \frac{T^{s+1-\frac{1}{k}}}{2^s}f'_\delta\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{k}}$$

$$+(\gamma_2+\varepsilon)\frac{T^{s+1-\frac{1}{k}}}{2^s}|x(t)|_\infty^{2k-2}\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{k}}\,. \tag{3.28}$$

Substituting the above formula and (3.21) into (3.19), we have

$$|b_s|\int_0^T |x^{(s)}(t)|^{2k}\,dt \le \Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{2k}}\Big[A(s)\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{1-\frac{1}{2k}}$$

$$+ C(s)|x(t)|_\infty^{2k-2}\Big(\int_0^T |x^{s}(t)|^{2k}\,dt\Big)^{\frac{1}{2k}} + |p(t)|_\infty T^{1-\frac{1}{2k}}\Big]$$

$$+\frac{T^{s+1-\frac{1}{k}}}{2^s}f'_\delta\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{k}}$$

$$+(\gamma_2+\varepsilon)\frac{T^{s+1-\frac{1}{k}}}{2^s}|x(t)|_\infty^{2k-2}\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{k}}\,. \tag{3.29}$$

Then, we have

$$[|b_s|-A(s)]\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{1-\frac{1}{2k}}$$

$$\le \Big[C(s)+(\gamma_2+\varepsilon)\frac{T^{s+1-\frac{1}{k}}}{2^s}\Big]|x(t)|_\infty^{2k-2}\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{2k}}$$

$$+\frac{T^{s+1-\frac{1}{k}}}{2^s}f'_\delta\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{2k}} + u_1\,, \tag{3.30}$$

where u_1 is a positive constant. We can prove that there is a constant $M_1>0$ such that

$$\int_0^T |x^{(s)}(t)|^{2k}dt \le M_1\,. \tag{3.31}$$

For some nonnegative integer l, there is a constant $0<h<1$ such that

$$(1+x)^l < 1+(l+1)x\,,\quad x\in(0,h)\,. \tag{3.32}$$

For the above h, if $\big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\big)^{\frac{1}{2k}} \le \frac{D^*}{B(s)h}$, then it is easy to see that there is a constant $N_1 > 0$ such that

$$\int_0^T |x^{(s)}(t)|^{2k}\,dt \le N_1\,. \tag{3.33}$$

If $\big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\big)^{\frac{1}{2k}} > \frac{D^*}{B(s)h}$, from (3.14), we get

$$\begin{aligned}
|x(t)|_\infty^{2k-2} &\le \Big[D^* + B(s)\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{2k}}\Big]^{2k-2}\\
&= B^{2k-2}(s)\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{1-\frac{1}{k}}\Big[1+\frac{D^*}{B(s)(\int_0^T |x^{(s)}(t)|^{2k}\,dt)^{\frac{1}{2k}}}\Big]^{2k-2}\\
&\le B^{2k-2}(s)\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{1-\frac{1}{k}}\Big[1+\frac{D^*(2k-1)}{B(s)(\int_0^T |x^{(s)}(t)|^{2k}\,dt)^{\frac{1}{2k}}}\Big]\\
&= B^{2k-2}(s)\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{1-\frac{1}{k}}\\
&\quad + B^{2k-3}(s)D^*(2k-1)\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{1-\frac{3}{2k}}\,.
\end{aligned} \tag{3.34}$$

Substituting the above formula into (3.30), we have

$$\begin{aligned}
&\Big[|b_s| - A(s) - B^{2k-2}(s)C(s) - (\gamma_2+\varepsilon)B^{2k-2}(s)\frac{T^{s+1-\frac{1}{k}}}{2^s}\Big]\\
&\quad\times\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{1-\frac{1}{2k}} \le \Big[C(s) + (\gamma_2+\varepsilon)\frac{T^{s+1-\frac{1}{k}}}{2^s}\Big]\\
&\quad\times B^{2k-3}(s)D^*(2k-1)\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{1-\frac{1}{k}}\\
&\quad+\frac{T^{s+1-\frac{1}{k}}}{2^s}f_\delta'\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt\Big)^{\frac{1}{k}} + u_1\,.
\end{aligned} \tag{3.35}$$

So there is a constant $N_2 > 0$ such that

$$\int_0^T |x^{(s)}(t)|^{2k}\,dt \le N_2\,. \tag{3.36}$$

Let $M_1 = \max\{N_1, N_2\}$. Then from (3.33) and (3.36), we have

$$\int_0^T |x^{(s)}(t)|^{2k}dt \le M_1\,. \tag{3.37}$$

From (3.14), there is a constant $M_2 > 0$ such that

$$|x(t)|_\infty \le M_2\,. \tag{3.38}$$

Integrating (3.2) on $[0,T]$, using the Hölder inequality and Lemma 2.4, we have

$$\begin{aligned}\int_0^T |x^{(n)}(t)|\,dt &\le \sum_{i=1}^{s}|b_i|\int_0^T |x^{(i)}(t)|^{2k-1}\,dt\\ &\quad+\int_0^T |f(x(t-\tau(t)))|\,dt+\int_0^T |p(t)|\,dt\\ &\le \Big[\sum_{i=1}^{s}|b_i|T^{(s-i)(2k-1)+\frac{1}{2k}}\Big]\Big(\int_0^T |x^{(s)}(t)|^{2k}\,dt)^{1-\frac{1}{2k}}+(|p(t)|_\infty+f_{M_2})T\\ &\le \Big[\sum_{i=1}^{s}|b_i|T^{(s-i)(2k-1)+\frac{1}{2k}}\Big](M_1)^{\frac{2k-1}{2k}}\\ &\quad+(|p(t)|_\infty+f_{M_2})T=M\,. \end{aligned} \tag{3.39}$$

where $f_{M_2}=\max_{|x|\le M_2}|f(x)|$, M is a positive constant. We claim that

$$|x^{(i)}(t)|_\infty \le \frac{T^{n-i-1}}{2^{n-i}}\int_0^T |x^{(n)}(t)|\,dt\,,\quad i=1,2,\dots,n-1\,. \tag{3.40}$$

In fact, applying Lemma 2.3, we obtain

$$|x^{(n-1)}(t)|_\infty \le \frac{1}{2}\int_0^T |x^{(n)}(t)|\,dt\,. \tag{3.41}$$

Similarily, applying Lemma 2.3 again, it follows that

$$\begin{aligned}|x^{(n-2)}(t)|_\infty &\le \frac{1}{2}\int_0^T |x^{(n-1)}(t)|\,dt \le \frac{1}{2}T|x^{(n-1)}(t)|_\infty\\ &\le \frac{T}{2^2}\int_0^T |x^{(n)}(t)|\,dt\,. \end{aligned} \tag{3.42}$$

By induction, we have

$$|x^{(i)}(t)|_\infty \le \frac{T^{n-i-1}}{2^{n-i}}\int_0^T |x^{(n)}(t)|\,dt\,,\quad i=1,2,\dots,n-1\,. \tag{3.43}$$

Furthermore, we have

$$|x^{(i)}(t)|_\infty \le \frac{T^{n-i-1}}{2^{n-i}}\int_0^T |x^{(n)}(t)|\,dt \le \frac{T^{n-i-1}}{2^{n-i}}M\,,\quad i=1,2,\dots,n-1\,. \tag{3.44}$$

It follows that there is a constant $A>0$ such that $\|x\|\le A$, Thus Ω_1 is bounded. □

Let $\Omega_2=\{x\in \operatorname{Ker} L, QNx=0\}$. Suppose $x\in\Omega_2$, then $x(t)=d\in R$ and satisfies

$$QNx=\frac{1}{T}\int_0^T [f(d)+p(t)]\,dt=0\,. \tag{3.45}$$

We will prove that there exists a constant $B>0$ such that $|d|\le B$. If $|d|\le D$, taking $D=B$, we get $|d|\le B$. If $|d|>D$, from (3.1), we have

$$\gamma+\gamma_1|d|^{2k-1}\le |f(d)|\le |p(t)|_\infty\,. \tag{3.46}$$

Thus

$$|d| \le \Big[\frac{||p(t)|_\infty - \gamma|}{\gamma_1}\Big]^{\frac{1}{2k-1}} . \tag{3.47}$$

Taking $\big[\frac{||p(t)|_\infty - \gamma|}{\gamma_1}\big]^{\frac{1}{2k-1}} = B$, we have $|d| \le B$, which implies Ω_2 is bounded. Let Ω be a non-empty open bounded subset of X such that $\Omega \supset \overline{\Omega_1} \cup \overline{\Omega_2} \cup \overline{\Omega_3}$, where $\Omega_3 = \{x \in X, |x| < \max\{D+1, [\frac{||p(t)|_\infty - \gamma|}{\gamma_1}]^{\frac{1}{2k-1}} + 1\}$. We can easily see that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. Then by the above argument we have

(i) $\quad Lx \ne \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0,1)$;

(ii) quad $QNx \ne 0$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$.

At last we will prove that condition (iii) of Lemma 2.5 is satisfied. We take

$$\begin{aligned} &H\colon (\Omega \cap \operatorname{Ker} L) \times [0,1] \to \operatorname{Ker} L \\ &H(x,\mu) = \mu x + \frac{1-\mu}{T}\int_0^T [f(x) + p(t)]\,dt \end{aligned} \tag{3.48}$$

From assumptions (H_1), we can easily obtain $H(x,\mu) \ne 0$, $\forall (x,\mu) \in \partial\Omega \cap \operatorname{Ker} L \times [0,1]$, which results in

$$\begin{aligned} \deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\{H(x,0), \Omega \cap \operatorname{Ker} L, 0\} \\ &= \deg\{H(x,1), \Omega \cap \operatorname{Ker} L, 0\} \ne 0 . \end{aligned} \tag{3.49}$$

Hence, by using Lemma 2.5, we know that Eq. (1.1) has at least one T-periodic solution.

Theorem 3.2. *Suppose that $n = 2m$, $m > 0$ an integer, $s = 1$, and the conditions (H_1)–(H_2) hold. If*

$$B^{2k-2}(1)C(1) < |b_1| , \tag{3.50}$$

then Eq. (1.1) *has at least one T-periodic solution.*

Proof. From the proof of Theorem 3.1, we have

$$|x(t)|_\infty \le B(1)\Big(\int_0^T |x'(t)|^{2k}\,dt\Big)^{\frac{1}{2k}} + D^* . \tag{3.51}$$

Multiplying both sides of (3.2) by $x'(t)$, and integrating on $[0,T]$, we have

$$\begin{aligned} \int_0^T x^{(n)}(t)x'(t)\,dt &= \lambda b_1 \int_0^T [x'(t)]^{2k-1}x'(t)\,dt \\ &\quad + \lambda \int_0^T f(x(t-\tau(t)))x^{'}(t)\,dt + \lambda \int_0^T p(t)x^{'}(t)\,dt . \end{aligned} \tag{3.52}$$

Since $\int_0^T f(x(t))x'(t)\,dt = 0$ and $\int_0^T x^{(2m)}(t)x'(t)\,dt = 0$, we obtain

$$\begin{aligned}|b_1|\int_0^T |x'(t)|^{2k}\,dt &\le \int_0^T |x'(t)|\big[|f(x(t)) - f(x(t-\tau(t)))| + |p(t)|\big]\,dt\\ &\le \Big(\int_0^T |x'(t)|^{2k} dt\Big)^{\frac{1}{2k}}\Big[\Big(\int_0^T |f(x(t)) - f(x(t-\tau(t)))|^{\frac{2k}{2k-1}} dt\Big)^{1-\frac{1}{2k}}\\ &\quad + |p(t)|_\infty T^{1-\frac{1}{2k}}\Big]. \qquad (3.53)\end{aligned}$$

Applying the above method, we have

$$\begin{aligned}&(|b_1| - B^{2k-2}(1)C(1))\Big(\int_0^T |x'(t)|^{2k}\,dt\Big)^{1-\frac{1}{2k}}\\ &\quad\le B^{2k-3}(1)C(1)D^*(2k-1)\Big(\int_0^T |x'(t)|^{2k}\,dt\Big)^{1-\frac{1}{k}}\\ &\qquad + \Big(\int_0^T |x'(t)|^{2k}\,dt\Big)^{\frac{1}{2k}} + u_2\,, \qquad (3.54)\end{aligned}$$

where u_2 is a positive constant. Hence there is a constant $M_3 > 0$ such that

$$\int_0^T |x'(t)|^{2k} dt \le M_3\,. \qquad (3.55)$$

The remainder can be proved in the same way as in the proof of Theorem 3.1. □

Theorem 3.3. *Suppose that $n = 4m$, $m > 0$ an integer, $s = 4l$, $l > 0$ an integer, and conditions (H_1)–(H_3) hold. If*

$$A(s) + B^{2k-2}(s)C(s) + \gamma_2 B^{2k-2}(s)\frac{T^{s-\frac{1}{k}}}{2^s} < -b_s\,, \qquad (3.56)$$

then Eq. (1.1) has at least one T-periodic solution.

Theorem 3.4. *Suppose that $n = 4m$, $m > 0$ an integer, $s = 4l - 2$, $l > 0$ an integer, and conditions (H_1)–(H_3) hold. If*

$$A(s) + B^{2k-2}(s)C(s) + \gamma_2 B^{2k-2}(s)\frac{T^{s-\frac{1}{k}}}{2^s} < b_s\,, \qquad (3.57)$$

then Eq. (1.1) has at least one T-periodic solution.

Theorem 3.5. *Suppose that $n = 4m + 2$, $m > 0$ an integer, $s = 4l + 2$, $l > 0$ an integer, and conditions (H_1)–(H_3) hold. If*

$$A(s) + B^{2k-2}(s)C(s) + \gamma_2 B^{2k-2}(s)\frac{T^{s-\frac{1}{k}}}{2^s} < -b_s\,, \qquad (3.58)$$

then Eq. (1.1) has at least one T-periodic solution.

Theorem 3.6. *Suppose that $n = 2m + 1$, $m > 0$ an integer, $s = 2l$, $l > 0$ an integer, and conditions (H_1)–(H_3) hold. If*

$$A(s) + B^{2k-2}(s)C(s) + \gamma_2 B^{2k-2}(s)\frac{T^{s-\frac{1}{k}}}{2^s} < |b_s|, \qquad (3.59)$$

then Eq. (1.1) has at least one T-periodic solution.

Theorem 3.7. *Suppose that* $n = 4m + 1$, $m > 0$ *an integer,* $s = 4l + 1$, $l > 0$ *an integer, and conditions* (H_1)*–*(H_3) *hold. If*

$$A(s) + B^{2k-2}(s)C(s) + \gamma_2 B^{2k-2}(s)\frac{T^{s-\frac{1}{k}}}{2^s} < -b_s\,, \tag{3.60}$$

then Eq. (1.1) *has at least one* T*-periodic solution.*

Theorem 3.8. *Suppose that* $n = 4m + 1$, $m > 0$ *an integer,* $s = 1$, *and the conditions* (H_1)*–*(H_2) *hold. If*

$$B^{2k-2}(1)C(1) < -b_1\,, \tag{3.61}$$

then Eq. (1.1) *has at least one* T*-periodic solution.*

Theorem 3.9. *Suppose that* $n = 4m + 1$, $m > 0$ *an integer,* $s = 4l - 1$, $l > 0$ *an integer, and conditions* (H_1)*–*(H_3) *hold. If*

$$A(s) + B^{2k-2}(s)C(s) + \gamma_2 B^{2k-2}(s)\frac{T^{s-\frac{1}{k}}}{2^s} < b_s, \tag{3.62}$$

then Eq. (1.1) *has at least one* T*-periodic solution.*

Theorem 3.10. *Suppose that* $n = 4m + 3$, $m > 0$ *an integer,* $s = 4l + 1$, $l > 0$ *an integer, and conditions* (H_1)*–*(H_3) *hold. If*

$$A(s) + B^{2k-2}(s)C(s) + \gamma_2 B^{2k-2}(s)\frac{T^{s-\frac{1}{k}}}{2^s} < b_s, \tag{3.63}$$

then Eq. (1.1) *has at least one* T*-periodic solution.*

Theorem 3.11. *Suppose that* $n = 4m + 3$, $m \geq 0$ *an integer,* $s = 1$, *and the conditions* (H_1) *and* (H_2) *hold. If*

$$B^{2k-2}(1)C(1) < b_1, \tag{3.64}$$

then Eq. (1.1) *has at least one* T*-periodic solution.*

Theorem 3.12. *Suppose that* $n = 4m + 3$, $m > 0$ *an integer,* $s = 4l + 3$, $l \geq 0$ *an integer, and conditions* (H_1)*–*(H_3) *hold. If*

$$A(s) + B^{2k-2}(s)C(s) + \gamma_2 B^{2k-2}(s)\frac{T^{s-\frac{1}{k}}}{2^s} < -b_s, \tag{3.65}$$

then Eq. (1.1) *has at least one* T*-periodic solution.*

The proof of Theorems 3.3–3.7, 3.9–3.10, 3.12 are similar to that of Theorem 3.1, and the proof of Theorems 3.8, 3.10 are similar to that of Theorem 3.2, which are omited here.

Example 1. Consider the following equation

$$\begin{aligned} &x^{(4)}(t) + 1000[x'''(t)]^3 + \frac{1}{50}[x''(t)]^3 + \frac{1}{100}[x'(t)]^3 \\ &+ \frac{1}{300}[x(t - \frac{1}{100}\sin t)]^3 = \cos t\,, \end{aligned} \tag{3.66}$$

where $n = 4$, $s = 3$, $k = 2$, $b_3 = 1000$, $b_2 = \frac{1}{50}$, $b_1 = \frac{1}{100}$, $f(x) = \frac{1}{300}x^3$, $p(t) = \cos t$, $\tau(t) = \frac{1}{100}\sin t$. Thus, $T = 2\pi$, $f(x)$ satisfies conditions (H_1)–(H_3), $T = 2\pi$, $\gamma_1 = \beta = \frac{1}{300}$, $\gamma_2 = \frac{1}{100}$, and

$$A(3) + A_1^2(3)A_2(3) + \gamma_2 A_1^2(3)\frac{T^{3-\frac{1}{2}}}{2^3} < |b_3| . \tag{3.67}$$

By Theorem 3.1, we know that Eq. (3.66) has at least one 2π-periodic solution.

References

[1] Chen, X. R., Pan, L. J., *Existence of periodic solutions for nth order differential equations with deviating argument*, Int. J. Pure. Appl. Math. **55** (2009), 319–333.

[2] Cong, F., *Periodic solutions for 2kth order ordinary differential equations with resonance*, Nonlinear Anal. **32** (1998), 787–793.

[3] Cong, F., *Existence of $2k+1$th order ordinary differential equations*, Appl. Math. Lett. **17** (2004), 727–732.

[4] Cong, F., Huang, Q. D., Shi, S. Y., *Existence and uniqueness of periodic solutions for $2n+1$th order ordinary differential equations*, J. Math. Anal. Appl. **241** (2000), 1–9.

[5] Ezeilo, J. O. C., *On the existence of periodic solutions of a certain third–order differential equation*, Proc. Cambridge Philos. Soc. **56** (1960), 381–389.

[6] Fabry, C., Mawhin, J., Nkashama, M. N., *A multiplicity result for periodic solutions of forced nonlinear second order ordinary differential equations*, Bull. London Math. Soc. **10** (1986), 173–180.

[7] Gaines, R. E., Mawhin, J., *Concidence degree and nonlinear differential equations*, Lecture Notes in Math. **568** (1977), Berlin, New York, Springer–Verlag.

[8] Jiao, G., *Periodic solutions of $2n+1$th order ordinary differential equations*, J. Math. Anal. Appl. **272** (2004), 691–699.

[9] Kiguradze, I. T., Půža, B., *On periodic solutions of system of differential equations with deviating arguments*, Nonlinear Anal. **42** (2000), 229–242.

[10] Li, J. W., Wang, G. Q., *Sharp inequalities for periodic functions*, Appl. Math. E-Notes **5** (2005), 75–83.

[11] Liu, B. W., Huang, L. H., *Existence of periodic solutions for nonlinear nth order ordinary differential equations*, Acta Math. Sinica **47** (2004), 1133–1140, in Chinese.

[12] Liu, W. B., Li, Y., *The existence of periodic solutions for high order duffing equations*, Acta Math. Sinica **46** (2003), 49–56, in Chinese.

[13] Liu, Y. J., Yang, P. H., Ge, W. G., *Periodic solutions of high–order delay differential equations*, Nonlinear Anal. **63** (2005), 136–152.

[14] Liu, Z. L., *Periodic solution for nonlinear nth order ordinary differential equation*, J. Math. Anal. Appl. **204** (1996), 46–64.

[15] Lu, S., Ge, W., *On the existence of periodic solutions for Lienard equation with a deviating argument*, J. Math. Anal. Appl. **289** (2004), 241–243.

[16] Lu, S., Ge, W., *Sufficient conditions for the existence of periodic solutions to some second order differential equation with a deviating argument*, J. Math. Anal. Appl. **308** (2005), 393–419.

[17] Mawhin, J., *Degré topologique et solutions périodiques des systémes différentiels nonlineaires*, Bull. Soc. Roy. Sci. Liége **38** (1969), 308–398.

[18] Mawhin, J., *L^2–estimates and periodic solutions of some nonlinear differential equations*, Boll. Unione Mat. Ital. **10** (1974), 343–354.

[19] Omari, P., Villari, G., Zanolin, F., *Periodic solutions of Lienard equation with one–side growth restrictions*, J. Differential Equations **67** (1987), 278–293.

[20] Pan, L. J., *Periodic solutions for higher order differential equations with deviating argument*, J. Math. Anal. Appl. **343** (2008), 904–918.

[21] Pan, L. J., Chen, X. R., *Periodic solutions for nth order functional differential equations*, Bull. Belg. Math. Soc. Simon Stevin **17** (2010), 109–126.

[22] Reissig, R., *Periodic solutions of third–order nonlinear differential equation*, Ann. Mat. Pura Appl. **92** (1972), 193–198.

[23] Ren, J. L., Ge, W. G., *On the existence of periodic solutions for the second order functional differential equation*, Acta Math. Sinica **47** (2004), 569–578, in Chinese.

[24] Srzednicki, S., *On periodic solutions of certain nth differential equations*, J. Math. Anal. Appl. **196** (1995), 666–675.

[25] Wang, G. Q., *Existence theorems of periodic solutions for a delay nonlinear differential equation with piecewise constant argument*, J. Math. Anal. Appl. **298** (2004), 298–307.

[26] Zhang, Z. Q., Wang, Z. C., *Periodic solution of the third order functional differential equation*, J. Math. Anal. Appl. **292** (2004), 115–134.

School of Mathematics, Jia Ying University,
Meizhou Guangdong, 514015, P. R. China
E-mail: plj1977@126.com