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INEQUALITIES BETWEEN THE SUM OF POWERS AND THE EXPONENTIAL OF SUM OF POSITIVE AND COMMUTING SELFADJOINT OPERATORS

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Abstract. Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators acting in Hilbert space $\mathcal{H}$ and $\mathcal{B}^+(\mathcal{H})$ the set of all positive selfadjoint elements of $\mathcal{B}(\mathcal{H})$. The aim of this paper is to prove that for every finite sequence $(A_i)_{i=1}^n$ of selfadjoint, commuting elements of $\mathcal{B}^+(\mathcal{H})$ and every natural number $p \geq 1$, the inequality

$$\frac{e^p}{p^p}\Big(\sum_{i=1}^{n} A_i^p\Big) \leq \exp\Big(\sum_{i=1}^{n} A_i\Big)\,,$$

holds.

1. Preliminaries and main results

Our starting result in this paper is the following theorem established in [3] for $p = 2$ and extended to case $p \geq 1$ in [2].

Theorem 1.0.1. *Let $(x_i)_{i=1}^n$ be a sequence of nonnegative real numbers. Then for every real $p \geq 1$, inequality*

$$\frac{e^p}{p^p}\sum_{i=1}^{n} x_i^p \leq \exp\Big(\sum_{i=1}^{n} x_i\Big)\,, \tag{1.0.1}$$

holds. Equality in (1.0.1) holds if $x_i = p$ for a certain $1 \leq i \leq n$ and $x_j = 0$ for $j \neq i$. So the constant $\frac{e^p}{p^p}$ is the best possible.

Our goal is to obtain a similar result for sequences of positive operators in Hilbert space.

Let $\mathcal{H}$ be a complex Hilbert space with inner scalar product $\langle\cdot,\cdot\rangle$ and $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators acting in Hilbert space $\mathcal{H}$. $I_{\mathcal{H}}$ will denote the unity in $\mathcal{B}(\mathcal{H})$. An element A of $\mathcal{B}(\mathcal{H})$ is said to be positive if $\langle Ax, x\rangle \geq 0$ for all elements $x \in \mathcal{H}$. Let A and B be two positive elements of $\mathcal{B}(\mathcal{H})$. Then $A \geq B$ means that $\langle Ax, x\rangle - \langle Bx, x\rangle \geq 0$ for every $x \in \mathcal{H}$. We need the following properties of positive operators [1, 4].

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Theorem 1.0.2. *Let $\mathcal{B}^+(\mathcal{H})$ be the set of all positive elements of $\mathcal{B}(\mathcal{H})$. Then,*

a) *$\mathcal{B}^+(\mathcal{H})$ is a closed cone,*

b) *$A, B \in \mathcal{B}^+(\mathcal{H})$ and commute $\Longrightarrow AB \in \mathcal{B}^+(\mathcal{H})$,*

c) *$A \in \mathcal{B}^+(\mathcal{H}) \Longleftrightarrow A = B^2$, ($B$ is a positive selfadjoint operator).*

d) *$A \in \mathcal{B}^+(\mathcal{H})$ and selfadjoint if and only if, A has a spectral representation of the form:*

$$A = \int_m^{M+\epsilon} \lambda dE_\lambda \,, \tag{1.0.2}$$

where, ϵ is any positive real number,

$$0 \leq m = \inf_{\|x\|=1} \langle Ax, x\rangle \leq M = \sup_{\|x\|=1} \langle Ax, x\rangle < +\infty \,.$$

e) *If A is selfadjoint with spectral representation (1.0.2), then for every real function f continuous on $[m, M+\epsilon]$,*

$$f(A) = \int_m^{M+\epsilon} f(\lambda) dE_\lambda \,, \tag{1.0.3}$$

and $f(A) = 0$ (resp. $f(A) \geq 0$) if and only if, $f(\lambda) = 0$ (resp. $f(\lambda) \geq 0$) on $[m, M+\epsilon]$.

Note that m and M in precedent theorem are respectively the smallest and biggest values of the spectrum of A.

Definition 1.0.3. Let $A \in \mathcal{B}^+(\mathcal{H})$. $\exp(A)$ is the element of $\mathcal{B}(\mathcal{H})$ given by formula,

$$\exp(A) = \sum_{k=0}^{+\infty} \frac{A^k}{k!} \,. \tag{1.0.4}$$

It is easy to check that $\exp(z.I_{\mathcal{H}}) = \exp(z) \cdot I_{\mathcal{H}}$ for any complex z. Moreover, if A, B are two commuting elements of $\mathcal{B}(\mathcal{H})$ then,

$$\exp(A+B) = \exp(A)\exp(B) = \exp(B)\exp(A) \,.$$

If A is a selfadjoint element of $\mathcal{B}^+(\mathcal{H})$ with spectral representation (1.0.2) and p a natural number, then according to Theorem 1.0.2, we have representations

$$A^p = \int_m^{M+\epsilon} \lambda^p dE_\lambda \quad \text{and} \quad \exp(A) = \int_m^{M+\epsilon} \exp(\lambda) dE_\lambda \,, \tag{1.0.5}$$

which we will frequently use throughout this paper.

We have the following main results:

Theorem 1.0.4. *Let $A \in \mathcal{B}^+(\mathcal{H})$. Then for every natural $p \geq 1$,*

$$\frac{e^p}{p^p} A^p \leq \exp(A) \,. \tag{1.0.6}$$

Moreover, if $A = p \cdot I_{\mathcal{H}}$ then, we have equality in (1.0.6) and the constant $\frac{e^p}{p^p}$ is the best possible.

Theorem 1.0.5. *Let $(A_i)_{i=1}^n$ be a finite sequence of commuting, selfadjoint elements of $\mathcal{B}^+(\mathcal{H})$. Then for every natural $p \geq 1$,*

$$\frac{e^p}{p^p}\sum_{i=1}^n A_i^p \leq \exp\Big(\sum_{i=1}^n A_i\Big). \tag{1.0.7}$$

Moreover, if for a certain $1 \leq i \leq n$, $A_i = p \cdot I_{\mathcal{H}}$ and $A_j = 0$ for $j \neq i$, then, we have equality in (1.0.7) and the constant $\frac{e^p}{p^p}$ is the best possible.

Remark 1.0.6. If A_i are roots of polynomial $P_n(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 I_{\mathcal{H}}$ and all operators $A_i - A_j$ $(i \neq j)$ are invertible then, for every natural $p \geq 1$,

$$\frac{e^p}{p^p}\sum_{i=1}^n A_i^p \leq \frac{e^p}{p^p}(-a_{n-1})^p \cdot I_{\mathcal{H}} \leq e^{-a_{n-1}} \cdot I_{\mathcal{H}} \tag{1.0.8}$$

Indeed, as in the scalar case, we have $A_1 + A_2 + \cdots + A_n = -a_{n-1} \cdot I_{\mathcal{H}}$.

Theorem 1.0.7. *Let $p \geq 1$ be a natural number and $(A_i)_{i=1}^n$ a finite sequence of invertible, commuting and selfadjoint elements of $\mathcal{B}^+(\mathcal{H})$ such that $0 < A_i \leq p \cdot I_{\mathcal{H}}$ for every $1 \leq i \leq n$. Then,*

$$\exp\Big(\sum_{i=1}^n A_i\Big) \leq \frac{p^p}{n}e^{np}\Big(\sum_{i=1}^n A_i^{-p}\Big). \tag{1.0.9}$$

Moreover, if $1 \leq i \leq n$, $A_i = p \cdot I_{\mathcal{H}}$ for every $1 \leq i \leq n$ then, we have equality in (1.0.9) and the constant $\frac{p^p}{n}e^{np}$ is the best possible.

From Theorems 1.0.5 and 1.0.7, follows,

Corollary 1.0.8. *Let $p \geq 1$ be a natural number and $(A_i)_{i=1}^n$ a finite sequence of invertible, commuting and selfadjoint elements of $\mathcal{B}^+(\mathcal{H})$ such that $0 < A_i \leq p \cdot I_{\mathcal{H}}$ for every $1 \leq i \leq n$. Then,*

$$\frac{e^p}{p^p}\sum_{i=1}^n A_i^p \leq \exp\Big(\sum_{i=1}^n A_i\Big) \leq \frac{p^p}{n}e^{np}\Big(\sum_{i=1}^n A_i^{-p}\Big). \tag{1.0.10}$$

Constants $\frac{p^p}{n}$ and $\frac{p^p}{n}e^{np}$, are the best possible.

2. Proofs of main results

2.1. Theorem 1.0.4.

Proof. Consider in the space $\mathcal{H}$ the selfadjoint operator $B = \exp(A) - e^p p^{-p} A^p$. We need to prove that B is positive. Let

$$A = \int_m^{M+\epsilon} \lambda dE_\lambda$$

be the spectral representation of A. By Theorem 1.0.2,

$$B = \int_m^{M+\epsilon} (e^\lambda - e^p p^{-p}\lambda^p) dE_\lambda\,. \tag{2.1.1}$$

According to [2], we have

$$\lambda \geq 0 \Longrightarrow e^{\lambda} - e^{p}p^{-p}\lambda^{p} \geq 0\,. \tag{2.1.2}$$

Hence, operator B is positive. It is easy to check that if $A = p \cdot I_{\mathcal{H}}$ then, we have equality in (1.0.6). Let now α be a constant such that $\alpha A^p \leq \exp(A)$ for all $A \in \mathcal{B}^+(\mathcal{H})$. Setting $A = p \cdot I_{\mathcal{H}}$, we obtain that $\alpha \leq e^p p^{-p}$. This finishes the proof. □

2.2. **Theorem 1.0.5.** To prove this theorem we need the following lemma.

Lemma 2.2.1. *Let $p \geq 1$ be a natural number and $(A_i)_{i=1}^n$ a finite sequence of commuting, selfadjoint elements of $\mathcal{B}^+(\mathcal{H})$. Then,*

$$\sum_{i=1}^{n} A_i^p \leq \Big(\sum_{i=1}^{n} A_i\Big)^p\,. \tag{2.2.1}$$

Proof. If $n = 2$ then by Theorem 1.0.2, operator $A_1^k A_2^{p-k}$ is positive for every $0 \leq k \leq p$. By the binomial theorem, we have:

$$(A_1 + A_2)^p = \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} A_1^k A_2^{p-k} = A_1^p + A_2^p + \sum_{k=1}^{p-1} \frac{p!}{k!(p-k)!} A_1^k A_2^{p-k}\,.$$

Since operator

$$\sum_{k=1}^{p-1} \frac{p!}{k!(p-k)!} A_1^k A_2^{p-k}$$

is positive (as a sum of positive operators), it follows from the last equality that

$$A_1^p + A_2^p \leq (A_1 + A_2)^p\,.$$

Suppose now that

$$\sum_{i=1}^{n} A_i^p \leq \Big(\sum_{i=1}^{n} A_i\Big)^p\,.$$

Then,

$$\begin{aligned}\Big(\sum_{i=1}^{n+1} A_i\Big)^p &= \Big(\sum_{i=1}^{n} A_i + A_{n+1}\Big)^p \geq \Big(\sum_{i=1}^{n} A_i\Big)^p + A_{n+1}^p \\ &\geq \sum_{i=1}^{n} A_i^p + A_{n+1}^p = \sum_{i=1}^{n+1} A_i^p\,.\end{aligned}$$

□

Let us now prove Theorem 1.0.5.

Proof. According to lemma and Theorem 1.0.4, we have:

$$e^p p^{-p} \sum_{i=1}^{n} A_i^p \leq e^p p^{-p} \Big(\sum_{i=1}^{n} A_i\Big)^p \leq \exp\Big(\sum_{i=1}^{n} A_i\Big)\,. \tag{2.2.2}$$

On the other hands, it is easy to check that we have equalities in this last formula if we set $A_i = p \cdot I_{\mathcal{H}}$ for a certain i and null operator for others indices. The same argumentation used in the precedent theorem shows that $e^p p^{-p}$ is the best possible constant. $\square$

2.3. **Theorem 1.0.7.**

Proof. Let us firstly remark that invertibility of operators A_i, $(i = 1, 2, \dots, n)$ implies that

$$A_i = \int_{m_i}^{M_i+\epsilon} \lambda dE_\lambda, \quad m_i > 0$$

and

$$A_i^{-1} = \int_{m_i}^{M_i+\epsilon} \lambda^{-1} dE_\lambda \,.$$

Since $A_i \leq p \cdot I_{\mathcal{H}}$, it follows from the spectral representation

$$A_i - p \cdot I_{\mathcal{H}} = \int_{m_i}^{M_i+\epsilon} (\lambda - p) dE_\lambda$$

that for all $1 \leq i \leq n$,

$$\lambda \in [m_i, M_i + \epsilon] \Longrightarrow \lambda \leq p \,.$$

Consequently, for all $1 \leq i \leq n$,

$$A_i^{-1} - p^{-1} \cdot I_{\mathcal{H}} = \int_{m_i}^{M_i+\epsilon} (\lambda^{-1} - p^{-1}) dE_\lambda \Longrightarrow p^{-1} \cdot I_{\mathcal{H}} \leq A_i^{-1}$$

$$\Longrightarrow \frac{n}{p^p} \cdot I_{\mathcal{H}} \leq \sum_{i=1}^{n} A_i^{-p} \,.$$

Selfadjoint operators $A_1^{-1}, A_2^{-1}, \dots, A_n^{-1}$ are bounded, commuting and positive. Using Theorem 1.0.5, we obtain

$$\frac{e^p}{p^p} \sum_{i=1}^{n} A_i^{-p} = \frac{e^p}{p^p} \sum_{i=1}^{n} (A_i^{-1})^p \leq \exp\Big(\sum_{i=1}^{n} A_i^{-1}\Big) \,. \tag{2.3.1}$$

Since, $\exp(A) \leq \exp(B)$ for $A \leq B$ then, we have finally

$$\exp\Big(\sum_{i=1}^{n} A_i\Big) \leq \exp\Big(\sum_{i=1}^{n} p \cdot I_{\mathcal{H}}\Big) = e^{np} \frac{p^p}{n} \frac{n}{p^p} \cdot I_{\mathcal{H}} \leq e^{np} \frac{p^p}{n} \sum_{i=1}^{n} A_i^{-p} \,.$$

It is clear that for equality holds for $A_i = p \cdot I_{\mathcal{H}}$ for every $1 \leq i \leq n$. This finishes the proof. $\square$

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