ARCHIVUM MATHEMATICUM (BRNO)
Tomus 46 (2010), 227–235

MODULE (φ, ψ)-AMENABILITY OF BANACH ALGEBRAS

Abasalt Bodaghi

Abstract. Let S be an inverse semigroup with the set of idempotents E and $S/\approx$ be an appropriate group homomorphic image of S. In this paper we find a one-to-one correspondence between two cohomology groups of the group algebra $\ell^1(S)$ and the semigroup algebra $\ell^1(S/\approx)$ with coefficients in the same space. As a consequence, we prove that S is amenable if and only if $S/\approx$ is amenable. This could be considered as the same result of Duncan and Namioka [5] with another method which asserts that the inverse semigroup S is amenable if and only if the group homomorphic image $S/\sim$ is amenable, where $\sim$ is a congruence relation on S.

1. Introduction

For a discrete semigroup S, $\ell^\infty(S)$ is the Banach algebra of bounded complex-valued functions on S with the supremum norm and pointwise multiplication. For each $a \in S$ and $f \in \ell^\infty(S)$, let $l_a f$ and $r_a f$ denote the left and the right translations of f by a, that is $(l_a f)(s) = f(as)$ and $(r_a f)(s) = f(sa)$, for each $s \in S$. Then a linear functional $m \in (\ell^\infty(S))^*$ is called a *mean* if $\|m\| = \langle m, 1\rangle = 1$; m is called a *left (right) invariant mean* if $m(l_a f) = m(f)$ ($m(r_a f) = m(f)$, respectively) for all $s \in S$ and $f \in \ell^\infty(S)$. A discrete semigroup S is called *amenable* if there exists a mean m on $\ell^\infty(S)$ which is both left and right invariant (see [5]).

A Banach algebra $\mathcal{A}$ is *amenable* if every bounded derivation from $\mathcal{A}$ into any dual Banach A-module is inner, equivalently if $H^1(\mathcal{A}, X^*) = \{0\}$ for every Banach A-module X, where $H^1(\mathcal{A}, X^*)$ is the *first Hochschild cohomology group* of A with coefficients in X^*. This concept was introduced by Barry Johnson in [7]. He showed that discrete group G is amenable if and only if the Banach algebra $\ell^1(G)$ is amenable. This fails to be true for discrete semigroups. M. Amini in [1] introduced the concept of module amenability for a class of Banach algebras, and showed that for an inverse semigroup S, the semigroup algebra $\ell^1(S)$ is module amenable as a Banach module on $\ell^1(E)$, where E is the set of idempotents of S, if and only if S is amenable (see also [3]). If $\mathcal{A}$ and $\mathfrak{A}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-module with compatible actions, then each $\mathfrak{A}$-module endomorphism φ (not necessarily $\mathbb{C}$-linear) on $\mathcal{A}$ induces a continuous endomorphism $\widehat{\varphi}$ on $\mathcal{A}/J$, where J is a closed ideal of $\mathcal{A}$, in particular generated by $\alpha \cdot (ab) - (ab) \cdot \alpha$ for

2000 *Mathematics Subject Classification*: primary 43A07; secondary 46H25.
Key words and phrases: Banach modules, module derivation, module amenability, inverse semigroup.
Received September 11, 2009, revised April 2010. Editor V. Müller.

all $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. In section two, for each pair $\mathfrak{A}$-module endomorphism φ and ψ on $\mathcal{A}$, we define module (φ, ψ)-amenability of Banach algebras, and among other results investigate the relation between module (φ, ψ)-amenability of $\mathcal{A}$ and $(\widehat{\varphi}, \widehat{\psi})$-amenability of $\mathcal{A}/J$. In section three we show that if S is an inverse semigroup with a directed upward set of idempotents E, then there exists a one-to-one correspondence between the quotient spaces $H^{\ell^1(E)}_{(\varphi,\psi)}(\ell^1(S), X^*)$, the first relative (to $\ell^1(E)$) (φ, ψ)-cohomology group of $\ell^1(S)$ with coefficients in X^* and $H_{(\widehat{\varphi},\widehat{\psi})}(\ell^1(S/\approx), X^*)$, the first $(\widehat{\varphi}, \widehat{\psi})$-cohomology group of $\ell^1(S/\approx)$, where $S/\approx$ is the maximal group homomorphic image of S, which $s \approx t$ whenever $\delta_s - \delta_t$ belongs to the closed linear span of the set

$$\{\delta_{\text{set}} - \delta_{st} : s, t \in S, e \in E\}.$$

Finally we show that S is amenable if and only if $S/\approx$ is amenable. The fact S is amenable if and only if $S/\sim$ is amenable, where $\sim$ is a congruence relation on S is first proved by Duncan and Namioka [5] (see also [10, Propositin A.0.5]). Also it was proved in [12] that with a two-sided stable equivalence relation $\simeq$ on S, if the semigroup $S/\simeq$ has a left invariant mean, then so does S.

2. module (φ, ψ)-amenability of Banach algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. We denote by $\text{Hom}(\mathcal{A}, \mathcal{B})$ the metric space of all bounded homomorphisms from $\mathcal{A}$ into $\mathcal{B}$, with the metric derived from $Ł(\mathcal{A}, \mathcal{B})$; the bounded linear operators from $\mathcal{A}$ into $\mathcal{B}$, and denote $\text{Hom}(\mathcal{A}, \mathcal{A})$ by $\text{Hom}(\mathcal{A})$.

Let X be a $\mathcal{A}$-module and let σ and τ be in $\text{Hom}(\mathcal{A})$. A bounded linear mapping $D\colon \mathcal{A} \to X$ is called a (σ, τ)-*derivation* if

$$D(ab) = D(a) \cdot \sigma(b) + \tau(a) \cdot D(b) \qquad (a, b \in \mathcal{A}).$$

A bounded linear mapping $D\colon \mathcal{A} \to X$ is called a (σ, τ)-*inner derivation* if there exists $x \in X$ such that

$$D(a) = x \cdot \sigma(a) - \tau(a) \cdot x \qquad (a \in \mathcal{A}).$$

We use notations $Z_{(\sigma,\tau)}(\mathcal{A}, X)$ for the space of all continuous (σ, τ)-derivations $D\colon \mathcal{A} \to X$, and $B_{(\sigma,\tau)}(\mathcal{A}, X)$ for those which are (σ, τ)-inner derivation. Also we use notation $H_{(\sigma,\tau)}(\mathcal{A}, X)$ for the quotient space $Z_{(\sigma,\tau)}(\mathcal{A}, X)/B_{(\sigma,\tau)}(\mathcal{A}, X)$ which call the first (σ, τ)-cohomology group of $\mathcal{A}$ with coefficients in X. Derivations of these forms are studied in [8].

Throughout this paper, $\mathcal{A}$ and $\mathfrak{A}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \qquad (a, b \in \mathcal{A},\ \alpha \in \mathfrak{A}).$$

Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \qquad a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x,$$

$$(\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a) \qquad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X)$$

and the same for the right or two-sided actions. Then we say that X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If moreover

$$\alpha \cdot x = x \cdot \alpha \qquad (\alpha \in \mathfrak{A}, x \in X)$$

then X is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module. If X is a (commutative) Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is X^*, where the actions of $\mathcal{A}$ and $\mathfrak{A}$ on X^* are defined by

$$\langle \alpha \cdot f, x\rangle = \langle f, x \cdot \alpha\rangle\,, \quad \langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle \qquad (a \in \mathcal{A}, \alpha \in \mathfrak{A},\ x \in X, f \in X^*)$$

and the same for the right actions.

Note that when $\mathcal{A}$ acts on itself by algebra multiplication, it is not in general a Banach $\mathcal{A}$-$\mathfrak{A}$-module, as we have not assumed the compatibility condition

$$a \cdot (\alpha \cdot b) = (a \cdot \alpha) \cdot b \qquad (\alpha \in \mathfrak{A}, a, b \in \mathcal{A})\,.$$

If $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-module.

If $\mathcal{A}$ is a Banach $\mathfrak{A}$-module with compatible actions, then so is the dual space $\mathcal{A}^*$. If moreover $\mathcal{A}$ is a commutative $\mathfrak{A}$-module, then $\mathcal{A}^*$ is commutative $\mathcal{A}$-$\mathfrak{A}$-module.

Now let $\mathcal{A}$ and $\mathcal{B}$ be $\mathfrak{A}$-modules. Then a $\mathfrak{A}$-module morphism from $\mathcal{A}$ to $\mathcal{B}$ is a norm-continuous map $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ with $T(a \pm b) = T(a) \pm T(b)$ which is multiplicative, that is

$$T(\alpha{\cdot}a) = \alpha{\cdot}T(a)\,, \quad T(a{\cdot}\alpha) = T(a){\cdot}\alpha\,, \quad T(ab) = T(a)T(b)\,, \qquad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A})\,.$$

We denote by $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathcal{B})$, the space of all such morphisms and denote $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathcal{A})$ by $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$.

Let $\mathcal{A}$ and $\mathfrak{A}$ be as above and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. Suppose that φ and ψ in $\mathrm{Hom}_{\mathfrak{A}}(A)$. A bounded map $D\colon \mathcal{A} \longrightarrow X$ is called a *module* (φ, ψ)*-derivation* if

$$D(a \pm b) = D(a) \pm D(b)\,, \quad D(ab) = D(a) \cdot \varphi(b) + \psi(a) \cdot D(b) \qquad (a, b \in \mathcal{A})\,,$$

and

$$D(\alpha \cdot a) = \alpha \cdot D(a)\,, \quad D(a \cdot \alpha) = D(a) \cdot \alpha \qquad (a \in \mathcal{A}, \alpha \in \mathfrak{A})\,.$$

Note that $D\colon \mathcal{A} \longrightarrow X$ is bounded if there exist $M > 0$ such that $\|D(a)\| \leq M\|a\|$ for all $a \in \mathcal{A}$. Although D is not necessarily linear, but still its boundedness implies its norm continuity (since D preserves subtraction). There is a similar justification for $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathcal{B})$. When X is commutative $\mathcal{A}$-$\mathfrak{A}$-module, each $x \in X$ defines a module (φ, ψ)-derivation $D^x_{(\varphi,\psi)}(a) = x \cdot \varphi(a) - \psi(a) \cdot x$ on $\mathcal{A}$. These are called *module* (φ, ψ)*-inner derivations*.

Definition 2.1. Let φ and ψ be in $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. The Banach algebra $\mathcal{A}$ is called module (φ, ψ)-amenable (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, each module (φ, ψ)-derivation $D : \mathcal{A} \longrightarrow X^*$ is (φ, ψ)-inner.

We use the notations $Z^{\mathfrak{A}}_{(\varphi,\psi)}(\mathcal{A}, X^*)$ for the space of all module (φ, ψ)-derivations $D\colon \mathcal{A} \longrightarrow X^*$, $B^{\mathfrak{A}}_{(\varphi,\psi)}(\mathcal{A}, X^*)$ for those which are inner (φ, ψ)-derivations, and $H^{\mathfrak{A}}_{(\varphi,\psi)}(\mathcal{A}, X^*)$ for the quotient space which we call the first relative (to $\mathfrak{A}$) (φ, ψ)-cohomology group of $\mathcal{A}$ with coefficients in X^*. Hence $\mathcal{A}$ is module (φ, ψ)-amenable

if and only if $H^{\mathfrak{A}}_{(\varphi,\psi)}(\mathcal{A}, X^*) = \{0\}$ for all commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X. We note that if φ and ψ are identity maps, then module (φ,ψ)-amenability is the same as module amenability (see [1]).

From [1, Proposition 2.1] we see that (φ,ψ)-amenability of $\mathcal{A}$ implies its module (φ,ψ)-amenability if $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$. Therefore (φ,ψ)-amenability is stronger than module (φ,ψ)-amenability.

Proposition 2.2. *Let $\mathcal{A}$ be a Banach algebra and $\psi, \varphi \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. If $\mathcal{A}$ is a module (φ,ψ)-amenable, then $\mathcal{A}$ is module $(\lambda\circ\varphi, \mu\circ\psi)$-amenable, for any λ and μ in $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$.*

Proof. Let X be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and $D\colon \mathcal{A} \to X^*$ be a module $(\lambda\circ\varphi, \mu\circ\psi)$-derivation. We consider another $\mathcal{A}$-module structure on X via

$$a \bullet x = \lambda(a)\cdot x\,, \quad x \bullet a = x\cdot\mu(a) \qquad (a \in \mathcal{A}, x \in X)\,.$$

It is easy to check that X with this product is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. We have

$$D(ab) = D(a)\cdot(\lambda\circ\varphi)(b)+(\mu\circ\psi)(a)\cdot D(b) = D(a)\bullet\varphi(b)+\psi(a)\bullet D(b)\,, \qquad (a, b \in \mathcal{A})\,.$$

Thus D is a module (φ,ψ)-derivation, and so, there exists $f \in X^*$ such that $D(a) = f \bullet \varphi(a) - \psi(a) \bullet f$. Therefore $D(a) = f\cdot(\lambda\circ\varphi)(a) + (\mu\circ\psi)(a)\cdot f$. □

Corollary 2.3. *If $\mathcal{A}$ is module amenable (as an $\mathfrak{A}$-module), then $\mathcal{A}$ is module (φ,ψ)-amenable, for each φ and ψ in $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$.*

In the following proposition we show that the converse of Corollary 2.3 in a special case.

Proposition 2.4. *Let $\mathcal{A}$ be an Banach $\mathfrak{A}$-module and $\varphi \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. If φ is an epimorphism and $\mathcal{A}$ is module (φ,φ)-amenable, then $\mathcal{A}$ is module amenable.*

Proof. Assume that X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and $D\colon \mathcal{A} \to X^*$ is a module derivation. Obviously $d = D\circ\varphi$ is a module (φ,φ)-derivation and so, by module (φ,φ)-amenability of $\mathcal{A}$ there exists $f \in X^*$ such that $d(a) = f\cdot\varphi(a)-\varphi(a)\cdot f$ for all $a \in \mathcal{A}$. Let $b \in \mathcal{A}$, then there exist $a \in \mathcal{A}$ such that $\varphi(a) = b$. Hence $D(b) = D(\varphi(a)) = d(a) = f\cdot\varphi(a) - \varphi(a)\cdot f = f\cdot b - b\cdot f$. Therefore D is a module inner derivation. □

Proposition 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules and $\varphi \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$, $\psi \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{B})$. If there is λ in $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A},\mathcal{B})$ such that $\lambda\circ\varphi = \psi\circ\lambda$ and range of λ is a dense subset of $\mathcal{B}$, then module (φ,φ)-amenability of $\mathcal{A}$ implies module (ψ,ψ)-amenability of $\mathcal{B}$.*

Proof. Let X be a commutative Banach $\mathcal{B}$-$\mathfrak{A}$-module and $D\colon \mathcal{B} \to X^*$ be a module (ψ,ψ)-derivation. X can be considered as Banach $\mathcal{A}$-module by the following actions

$$a * x = \lambda(a)\cdot x\,, \quad x * a = x\cdot\lambda(a) \qquad (a \in \mathcal{A}, x \in X)\,.$$

Then X with this product is a Banach $\mathcal{A}$-$\mathfrak{A}$-module, so $\overline{D} = D \circ \lambda \colon \mathcal{A} \to X^*$ is (φ,φ)-derivation because

$$\begin{aligned}\overline{D}(ab) = D\big(\lambda(a)\lambda(b)\big) &= D(a)\cdot\psi\big(\lambda(b)\big) + \psi\big(\lambda(a)\big)\cdot D\big(\lambda(b)\big)\\ &= D\big(\lambda(a)\big)\cdot\lambda\big(\varphi(b)\big) + \lambda\big(\varphi(a)\big)\cdot D\big(\lambda(b)\big)\\ &= \overline{D}(a) * \varphi(b) + \varphi(a) * \overline{D}(b)\,,\end{aligned}$$

for all a, b in $\mathcal{A}$. Due to module (φ,φ)-amenability of $\mathcal{A}$, there exist $f \in X^*$ such that $\overline{D}(a) = f * \varphi(a) - \varphi(a) * f$. Thus $D(\lambda(a)) = f\cdot\psi(\lambda(a)) + \psi(\lambda(a))\cdot f$. By density of range of λ and continuity of D, D is inner. □

By using the Proposition 2.5, if φ and ψ are identity map and $\lambda(\mathcal{A})$ is dense in $\mathcal{B}$, then module amenability of $\mathcal{A}$ implies module amenability $\mathcal{B}$. Therefore Proposition 2.5 could be considered as a generalization of [1, Proposition 2.5].

Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with compatible actions and J be the closed ideal of $\mathcal{A}$ generated by elements $(\alpha\cdot a)b - a(b\cdot\alpha)$ for all $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$, then the quotient Banach algebra $\mathcal{A}/J$ is Banach $\mathfrak{A}$-module with compatible actions. Suppose that $\varphi, \psi \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$ such that $\varphi(J) \subseteq J, \psi(J) \subseteq J$. Then one can define maps $\widehat{\varphi}$, $\widehat{\psi}\colon \mathcal{A}/J \to \mathcal{A}/J$ by $\widehat{\varphi}(a+J) = \varphi(a) + J$ and $\widehat{\psi}(a+J) = \psi(a) + J$.

We say that $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$ if there is a bounded net $\{\zeta_j\}$ in $\mathfrak{A}$ such that $\|\zeta_j\cdot a - a\| \to 0$ and $\|a\cdot\zeta_j - a\| \to 0$, for each $a \in \mathcal{A}$.

Proposition 2.6. *Let φ, ψ be in $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. If $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$, then $(\widehat{\varphi},\widehat{\psi})$-amenability of $\mathcal{A}/J$ implies module (φ,ψ)-amenability $\mathcal{A}$.*

Proof. Let X be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and $D : \mathcal{A} \to X^*$ be a module (φ,ψ)-derivation. We can show that $J\cdot X = X\cdot J = 0$, so X is a Banach $\mathcal{A}/J$-module with module actions

$$(a+J)\cdot x := a\cdot x\,, \quad x\cdot(a+J) := x\cdot a \qquad (x \in X, a \in \mathcal{A})\,.$$

Consider $\widehat{D}\colon \mathcal{A}/J \to X$, defined by $\widehat{D}(a+J) = D(a)$ for all $a \in \mathcal{A}$. $\widehat{D}$ is well defined because

$$\begin{aligned}D(\alpha\cdot ab - ab\cdot\alpha) &= \alpha\cdot D(ab) - D(ab)\cdot\alpha\\ &= \alpha\cdot\big(D(a)\cdot b + a\cdot D(b)\big) - \big(D(a)\cdot b + a\cdot D(b)\big)\cdot\alpha\\ &= \alpha\cdot\big(D(a)\cdot b\big) - \big(D(a)\cdot b\big)\cdot\alpha\\ &\quad + \alpha\cdot\big(a\cdot D(b)\big) - \big(a\cdot D(b)\big)\cdot\alpha = 0\,.\end{aligned}$$

Now for each a, b in $\mathcal{A}$ we have

$$\begin{aligned}\widehat{D}(ab+J) = D(ab) &= D(a)\cdot\varphi(b) + \psi(a)\cdot D(b)\\ &= \widehat{D}(a+J)\cdot\big(\varphi(b)+J\big) + \big(\psi(a)+J\big)\cdot\widehat{D}(b+J)\\ &= \widehat{D}(a+J)\cdot\widehat{\varphi}(b+J) + \widehat{\psi}(a+J)\cdot\widehat{D}(b+J)\,.\end{aligned}$$

Since $\mathfrak{A}$ has the bounded approximate identity (ζ_i) for $\mathcal{A}$, then for each $\rho \in \mathbb{C}$ and $a \in \mathcal{A}$ we have

$$\begin{aligned}\widehat{\varphi}(\rho a + J) = \varphi(\rho a) + J &= \lim_i \varphi(\rho a \cdot \zeta_i) + J = \lim_i \varphi(a \cdot \rho\zeta_i) + J \\ &= \lim_i \rho\varphi(a \cdot \zeta_i) + J = \rho\varphi(a) + J = \rho\widehat{\varphi}(a + J)\,.\end{aligned}$$

Now, it follows from the proof of [1, Proposition 2.1] that $\widehat{D}$ is $\mathbb{C}$-linear, and so it is $(\widehat{\varphi}, \widehat{\psi})$-inner. Hence there exist $f \in X^*$ such that

$$D(a) = \widehat{D}(a + J) = f \cdot \widehat{\varphi}(a + J) - \widehat{\psi}(a + J) \cdot f = f \cdot \varphi(a) - \psi(a) \cdot f\,.$$

Therefore D is a module (φ, ψ)-inner. □

We say the Banach algebra $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left if for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$, $\alpha \cdot a = f(\alpha)a$, where f is a continuous linear functional on $\mathfrak{A}$.

Proposition 2.7. *Let φ, ψ be as above and $\mathcal{A}$ be module (φ, ψ)-amenable as an $\mathfrak{A}$-module with trivial left action. If $\mathcal{A}/J$ has an identity, then $\mathcal{A}/J$ is $(\widehat{\varphi}, \widehat{\psi})$-amenable.*

Proof. Without loss of generality we assume that X is an unital $\mathcal{A}/J$-module and $\widehat{D}\colon \mathcal{A}/J \to X^*$ be an $(\widehat{\varphi}, \widehat{\psi})$-derivation. Then X is a $\mathcal{A}$-module via

$$a \cdot x := (a + J) \cdot x, \quad x \cdot a := x \cdot (a + J) \qquad (x \in X, a \in \mathcal{A})\,,$$

and also X is $\mathfrak{A}$-module with trivial actions, that is $\alpha \cdot x = x \cdot \alpha = f(\alpha)x$ for all $x \in \mathcal{X}$ and $\alpha \in \mathfrak{A}$. Since $f(\alpha)a - a \cdot \alpha \in J$ [2, Lemma 3.1], we have $f(\alpha)a + J = a \cdot \alpha + J$ for all $\alpha \in \mathfrak{A}$. Hence the actions of $\mathfrak{A}$ and $\mathcal{A}$ on X are compatible. Therefore X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module. Consider $D = \widehat{D} \circ \pi\colon \mathcal{A} \to X^*$ where $\pi\colon \mathcal{A} \to \mathcal{A}/J$ is the natural $\mathfrak{A}$-module. Obviously $D(a \pm b) = D(a) \pm D(b)$ for all $a, b \in \mathcal{A}$, and $\mathcal{A}/J$ has an identity, so by [2, Theorem 3.2], $D(a \cdot \alpha) = D(a) \cdot \alpha, D(\alpha \cdot a) = D(a) = \alpha \cdot D(a)$ for all $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. Also we have

$$\begin{aligned}D(ab) = \widehat{D}\big(\pi(ab)\big) &= \widehat{D}(a + J) \cdot \widehat{\varphi}(b + J) + \widehat{\psi}(a + J) \cdot \widehat{D}(b + J) \\ &= D(a) \cdot (\varphi(b) + J) + (\psi(a) + J) \cdot D(b) \\ &= D(a) \cdot \varphi(b) + \psi(a) \cdot D(b)\,,\end{aligned}$$

for all $a, b \in \mathcal{A}$, that is $D \in Z^{\mathfrak{A}}_{(\varphi,\psi)}(\mathcal{A}, X^*)$. Hence there exist $f \in X^*$ such that $D(a) = f \cdot \varphi(a) - \psi(a) \cdot f$ for all $a \in \mathcal{A}$. Therefore $\widehat{D}$ is $(\widehat{\varphi}, \widehat{\psi})$-inner. □

3. Module (φ, ψ)-amenability of semigroup algebras

Recall that a discrete semigroup S is called an *inverse semigroup* if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotents of S is denoted by E. We start this section with a definition of inverse semigroups.

Throughout this section, S is an inverse semigroup with set of idempotents E. There is a natural order on E, defined by

$$e \leq d \Longleftrightarrow ed = e \qquad (e, d \in E)\,.$$

It is easy to see that E is indeed a commutative subsemigroup of S (see [6, Theorem V.1.2]). In particular $\ell^1(E)$ could be regarded as a subalgebra of $\ell^1(S)$, and thereby $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module with compatible canonical actions [1]. However, for technical reasons, here we let $\ell^1(E)$ act on $\ell^1(S)$ by multiplication from right and trivially from left, that is

$$\delta_e \cdot \delta_s = \delta_s\,, \quad \delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e \qquad (s \in S, e \in E)\,.$$

In this case, J is the closed linear span of $\{\delta_{\mathrm{set}} - \delta_{st}\ s, t \in S, e \in E\}$. We consider the following equivalence relation on S

$$s \approx t \Longleftrightarrow \delta_s - \delta_t \in J \qquad (s, t \in S)\,.$$

Recall that E is called *directed upward* if for every e, $f \in E$ there exist $g \in E$ such that $eg = e$ and $fg = f$. This is precisely the assertion that S satisfies the condition D_1 of Duncan and Namioka [5]. It is shown in [2] that if E is directed upward, then the quotient $S/\approx$ is a discrete group. Unital inverse semigroups have a directed upward set of idempotents. Also if E is totally ordered, it is clearly directed upward. The examples of the latter include the bicyclic semigroup and the semigroup of natural numbers with *max* operation. On the other hand, the set of idempotents of the free inverse semigroup on two generators is not directed upward. Indeed, if the generators are a and b, there is no idempotent which is bigger than both aa^* and bb^*.

Consider the quotient map $\pi\colon S \to S/\approx$, $s \mapsto [s]$. As in [11, Theorem 3.3], we may observe that $\ell^1(S)/J \cong \ell^1(S/\approx)$. Now, if $\varphi \in \mathrm{Hom}_{\ell^1(E)}(\ell^1(S))$, by using the discussion before Proposition 2.6 we define $\widehat{\varphi}$ in $\mathrm{Hom}(\ell^1(S/\approx))$ by $\widehat{\varphi}(\delta_s + J) = \varphi(\delta_s) + J$ and extended by linearity (see also the proof of Proposition 2.6). The following result is the main aim of this section. In fact we show that there exists a one-to-one correspondence between $H^{\ell^1(E)}_{(\varphi,\psi)}(\ell^1(S), X^*)$ and $H_{(\widehat{\varphi},\widehat{\psi})}(\ell^1(S/\approx), X^*)$, where φ and ψ are in $\mathrm{Hom}_{\ell^1(E)}(\ell^1(S))$ and X is a commutative Banach $\ell^1(S)$-$\ell^1(E)$-module.

Theorem 3.1. *Let S be an inverse semigroup with a directed upward set of idempotents E. If $\ell^1(S)$ is an $\ell^1(E)$-module with trivial left action and φ, ψ are in $\mathrm{Hom}_{\ell^1(E)}(\ell^1(S))$, then*

$$H^{\ell^1(E)}_{(\varphi,\psi)}\big(\ell^1(S), X^*\big) \cong H_{(\widehat{\varphi},\widehat{\psi})}\big(\ell^1(S/\approx), X^*\big)\,.$$

Proof. Since $S/\approx$ is a discrete group, $\ell^1(S)/J \cong \ell^1(S/\approx)$ has an identity. Also S is an inverse semigroup with a directed upward set of idempotents E, so E satisfies condition D_1 of Duncan and Namioka. Hence $\ell^1(E)$ has a bounded approximate identity [5]. Now, if (ζ_j) is a bounded approximate identity of $\ell^1(E)$, then $\zeta_j * \delta_s = \zeta_j * \delta_{ss^*s} = (\zeta_j \cdot \delta_{ss^*}) * \delta_s \to \delta_s$, and similarly for the right side multiplication. Therefore $\ell^1(E)$ has a bounded approximate identity for $\ell^1(S)$. It follows from Proposition 2.6 and Proposition 2.7 that the map

$$Z^{\ell^1(E)}_{(\varphi,\psi)}\big(\ell^1(S), X^*\big) \to Z_{(\widehat{\varphi},\widehat{\psi})}\big(\ell^1(S/\approx), X^*\big)\,, \quad (D \mapsto \widehat{D})\,.$$

induce an isomorphism between the quotient spaces $H^{\ell^1(E)}_{(\varphi,\psi)}(\ell^1(S), X^*)$ and $H_{(\widehat{\varphi},\widehat{\psi})}(\ell^1(S/\approx), X^*)$. □

The following corollary is a result of Theorem 3.1 in which φ and ψ are identity maps.

Corollary 3.2. *With the hypothesis of above theorem, S is amenable if and only if $S/\approx$ is amenable.*

Proof. The discrete group $S/\approx$ is amenable if and only if $\ell^1(S/\approx)$ is amenable by Johnson's theorem [7]. We conclude from Theorem 3.2, $\ell^1(S/\approx)$ is amenable if and only if $\ell^1(S)$ is module amenable (as an $\ell^1(E)$-module). Now the result follows from [1, Theorem 3.1]. □

Let $\sim$ be the congruence relation on S where $s \sim t$ if and only if there exist $e \in E$ such that $se = te$. The quotient semigroup $G(S) \equiv S/\sim$ is then a group. It is indeed the maximal group homomorphic image of S [9]. Also the inverse semigroup S is amenable if and only if the discrete group $G(S)$ is amenable [5, Theorem 1]. Now consider epimorphisms $P\colon S \to S/\approx;\ s \mapsto [s]$ and $Q\colon S \to G(S);\ s \mapsto [[s]]$, then maximality of $G(S)$ implies that there is a group homomorphism $R\colon S/\approx \to G(S)$ such that $R \circ P = Q$. Clearly R is onto. To see it is one-to-one, let $[[s]] = [[t]]$, then there is $e \in E$ such that $se = te$. Since $\delta_{sa} - \delta_{sea} \in J$ for all $s,\ a \in S$ and $e \in E$, $\delta_s - \delta_{se} = \delta_{ss^*s} - \delta_{ss^*se} = \delta_{ss^*s} - \delta_{ses^*s} \in J$. Hence $\delta_s - \delta_t = \delta_s - \delta_{se} + \delta_{se} - \delta_t = (\delta_s - \delta_{se}) - (\delta_t - \delta_{te}) \in J$, so $[s] = [t]$. Therefore R is a group isomorphism. Now if $S/\approx$ is amenable, then by Corollary 3.2 and [1, Theorem 3.1], $\ell^1(S)$ is $\ell^1(E)$-module amenable. Therefore $\ell^1(S)$ is module (φ,ψ)-amenable, for each $\varphi, \psi \in \mathrm{Hom}_{\ell^1(E)}(\ell^1(S))$ by Corollary 2.3. We close this section by some examples.

Example 3.3. (i) Let $(\mathbb{N}, \vee)$ be the commutative semigroup of positive integers with maximum operation $m \vee n = \max(m, n)$, then each element of $\mathbb{N}$ is an idempotent, hence $\mathbb{N}/\approx$ is the trivial group with one element. So $\mathbb{N}/\approx$ is amenable. Therefore $(\mathbb{N}, \vee)$ is amenable by the above corollary.

(ii) Let $\mathcal{C}$ be the bicyclic inverse semigroup generated by a and b, that is

$$\mathcal{C} = \{a^m b^n : m, n \geq 0\}, \quad (a^m b^n)^* = a^n b^m .$$

The set of idempotents of $\mathcal{C}$ is $E_{\mathcal{C}} = \{a^n b^n : n = 0, 1, \ldots\}$ which is totally ordered with the following order

$$a^n b^n \leq a^m b^m \to m \leq n .$$

It have been showed in [2] that $\mathcal{C}/\approx$ is isomorphic to integer numbers $\mathbb{Z}$, hence it is amenable. Therefore $\ell^1(\mathcal{C})$ is module (φ,ψ)-amenable for all $\varphi, \psi \in \mathrm{Hom}_{\ell^1(E_{\mathcal{C}})}\big(\ell^1(\mathcal{C})\big)$.

(iii) Let S be an amenable E-unitary inverse semigroup with infinite number of idempotents (see [6] and [10]). Then $\ell^1(S)$ is module amenable [1]. So if φ and ψ are in $\mathrm{Hom}_{\ell^1(E)}(\ell^1(S))$, then $\ell^1(S)$ is module (φ,ψ)-amenable, but not (φ,ψ)-amenable even when φ and ψ are identity maps [5].

(iv) If S is a *Brandt semigroup* of an amenable group over an infinite index set (see [5] and [10]), then $\ell^1(S)$ is module amenable [1]. Therefore we conclude by

Corollary 2.3 that $\ell^1(S)$ is module (φ, ψ)-amenable for all $\varphi, \psi \in \mathrm{Hom}_{\ell^1(E)}(\ell^1(S))$ without having a bounded approximate identity [5].

Acknowledgement. The author would like to thank Dr. Massoud Amini, for his valuable discussions and useful comments. He also would like to thank the referee for careful reading of the paper and giving some useful suggestions.

References

[1] Amini, M., *Module amenability for semigroup algebras*, Semigroup Forum **69** (2004), 243–254.

[2] Amini, M., Bodaghi, A., Bagha, D. Ebrahimi, *Module amenability of the second dual and module topological center of semigroup algebras*, Semigroup Forum **80** (2010), 302–312.

[3] Amioni, M., *Corrigendum, Module amenability for semigroup algebras*, Semigroup Forum **72** (2006), 493.

[4] Dale, H. G., *Banach Algebra and Automatic Continuity*, Oxford university Press, 2000.

[5] Duncan, J., Namioka, I., *Amenability of inverse semigroups and their semigroup algebra*, Proc. Roy. Soc. Edinburgh Sect. A **80** (3–4) (1978), 309–321.

[6] Howie, J. M., *An Introduction to Semigroup Theory*, London Academic Press, 1976.

[7] Johnson, B. E., *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. **127** (1972), iii+96 pp.

[8] Moslehian, M. S., Motlagh, A. N., *Some notes on (σ, τ)-amenability of Banach algebras*, Stud. Univ. Babeş-Bolyai Math. **53** (3) (2008), 57–68.

[9] Munn, W. D., *A class of irreducible matrix representations of an arbitrary inverse semigroup*, Proc. Glasgow Math. Assoc. **5** (1961), 41–48.

[10] Paterson, A. L. T., *Groupoids, Inverse Semigroups, and Their Operator Algebras*, Birkhäuser, Boston, 1999.

[11] Rezavand, R., Amini, M., Sattari, M. H., Bagh, D. Ebrahimi, *Module Arens regularity for semigroup algebras*, Semigroup Forum **77** (2008), 300–305.

[12] Wilde, C., Argabright, L., *Invariant means and factor semigroup*, Proc. Amer. Math. Soc. **18** (1967), 226–228.

Department of Mathematics, Islamic AZAD University,
Garmsar Branch, Garmsar, Iran
E-mail: abasalt@putra.upm.edu.my abasalt_bodaghi@yahoo.com

Current address:
Laboratory of Theoretical Studies,
Institute for Mathematical Research, University Putra Malaysia,
43400 UPM Serdang, Selangor Darul Ehsan, Malaysia