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GLOBAL EXISTENCE AND POLYNOMIAL DECAY FOR A PROBLEM WITH BALAKRISHNAN-TAYLOR DAMPING

Abderrahmane Zaraï and Nasser-eddine Tatar

Abstract. A viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping is considered. Using integral inequalities and multiplier techniques we establish polynomial decay estimates for the energy of the problem. The results obtained in this paper extend previous results by Tatar and Zaraï [25].

1. Introduction

The aim of this paper is to extend a previous work by Tatar and Zaraï [25] where an exponential decay result and a blow up result for solutions of the wave equation of Kirchhoff type with Balakrishnan-Taylor damping have been established. Here we study the case where the kernel h decays polynomially (or more precisely, of power type). Namely, we are concerned with the following initial-boundary value problem

$$(1)\qquad \begin{cases} u_{tt} - \big(\xi_0 + \xi_1 \|\nabla u(t)\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t))\big)\Delta u \\ \qquad + \int_0^t h(t-s)\Delta u(s)\, ds = |u|^p u \quad \text{in} \quad \Omega \times [0,+\infty) \\ u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega \\ u(x,t) = 0 \quad \text{in} \quad \Gamma \times [0,+\infty) \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ. Here h represents the kernel of the memory term. All the parameters ξ_0, ξ_1, and σ are assumed to be positive constants. When $\xi_1 = \sigma = h = 0$, the equation (1) reduces to a nonlinear wave equation which has been extensively studied and several results concerning existence and nonexistence have been established [4], [10]–[12] [15], [16], [19]. When ξ_0, $\xi_1 \neq 0$, $\sigma = h = 0$, the equation in (1) reduces to the well-known Kirchhoff equation which has been introduced in [13] in order to describe the nonlinear vibrations of an elastic string. More precisely, the first (one-dimensional) Kirchhoff

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equation was of the form

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left\{ p_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + f \,,$$

for $0 < x < L$, $t \geq 0$; where u is the lateral deflection, x the space coordinate, t the time, E the Young modulus, ρ the mass density, h the cross section area, L the length, p_0 the initial axial tension and f the external force. Kirchhoff [13] was the first one to study the oscillations of stretched strings and plates. The question of existence and nonexistence of solutions have been discussed by many authors (see [17], [20]–[23], [26]).

The model in hand, with Balakrishnan-Taylor damping ($\sigma > 0$) and $h = 0$, was initially proposed by Balakrishnan and Taylor in 1989 [3] and Bass and Zes [5]. It is related to the panel flutter equation and to the spillover problem. So far it has been studied by Y. You [27], H. R. Clark [9] and N-e. Tatar and A. Zaraï [25, 26]. In case $\sigma = 0$ the equation in (1) describes the motion of a deformable solid with an hereditary effect. This phenomena occurs in many practical situations such as in viscoelasticity. Again, one can find several papers in the literature especially on exponential and polynomial stability of the system (see [2], [6]–[8], [18], [24] and references therein, to cite but a few). The well-posedness is by now well established and can be found in the cited references (see also [3, 27]).

An important question about the asymptotic behavior of solutions has been raised by Clark in [9]. It has been proved there that solutions decay exponentially to the equilibrium state provided that we have a damping of the form Δu_t. This damping is known to be a *strong* damping. In [25], Tatar and Zaraï proved an exponential decay result of the energy provided that the kernel h decays exponentially. In this paper we improve this result by establishing sufficient conditions yielding polynomial stability of solutions under a weaker damping, namely the viscoelastic damping due to the material itself and in presence of a nonlinear source. This nonlinear source, of course, will compete with both kinds of damping. More precisely, we find a "stable" set of initial data where if we start there then the corresponding solutions will decay polynomially to the stationary state when the kernel h decays polynomially.

Our plan in this paper is as follows: in Section 2, we give some lemmas and assumptions which will be used later as well as a local existence theorem. In Section 3, we show that the energy of system (1) is global in time and decays polynomially when we start in a certain "stable" set.

2. Preliminaries

In this section, we present the following well-known lemmas which will be needed later.

Lemma 1 (See [1, 14])**.** *Let $E(t)$ be a non-increasing and nonnegative function defined on $[0, \infty)$. Assume there are positive constants λ, C and S_0 such that*

$$\int_S^\infty E^{1+\lambda}(t)\, dt = C E^\lambda(0) E(S) \,, \quad \forall \; S \geq S_0 \,,$$

then

$$E(t) \le E(0)\Big(\frac{(S_0+C)(1+\lambda)}{\lambda t + S_0 + C}\Big)^{\frac{1}{\lambda}}, \quad \forall\, t \ge 0\,.$$

Lemma 2. *Let p be a non-negative number $(n = 1, 2)$ or $0 \le p \le \frac{4}{n-2}$ $(n > 2)$ then, there exists a constant $C(p, \Omega)$ such that*

$$\|u\|_{p+2} \le C(p,\Omega)\|\nabla u\|_2\,, \quad \textit{for} \quad u \in H_0^1(\Omega)\,.$$

Now, we state the general hypotheses

(A1) $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1-function satisfying

$$\xi_0 - \int_0^\infty h(s)\,ds = \ell > 0\,,$$

(A2) There exist positive constants k and $\rho \in (2, \infty)$ such that

$$h'(t) \le -k h^{1+\frac{1}{\rho}}(t)\,, \quad t > 0\,.$$

It follows from **(A2)** that

$$h(t) \le \frac{K}{(1+t)^\rho}\,, \quad t \ge 0\,,$$

for some constant $K > 0$. Therefore, we have

$$h^\eta \in L^1(0,\infty) \quad \text{for any} \quad \eta > \frac{1}{\rho}\,.$$

Now, we state the local existence theorem which can be found for instance in [6].

Theorem 1 (Local existence)**.** *Let $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $0 < p < p^*$. Here $p^* = \frac{2}{n-2}$, if $n \ge 3$ (∞, if $n \le 2$). Assume further that* **(A1)** *and* **(A2)** *hold. Then, problem* (1) *admits a unique weak solution $u \in C\big([0,T]; H_0^1(\Omega)\big) \cap C^1\big([0,T]; L^2(\Omega)\big)$ for T small enough.*

3. Global existence

Our first result states that the solutions exist for all time $t \ge 0$ provided that we start in a "stable region" which we will define. We define the energy of problem (1) by

$$E(t) = \|u'\|^2 + \Big(\xi_0 + \frac{\xi_1}{2}\|\nabla u\|^2\Big)\|\nabla u\|^2 + (h\Box\nabla u)(t)$$

$$- \int_0^t h(s)\|\nabla u(t)\|^2\,ds - \frac{2}{p+2}\|u\|_{p+2}^{p+2}\,, \quad t > 0 \tag{2}$$

where

$$(h\Box\nabla u)(t) = \int_0^t h(t-s)\int_\Omega |\nabla u(s) - \nabla u(t)|^2\,dx\,ds\,.$$

Lemma 3. *$E(t)$ is a non-increasing function on $[0,\infty)$ and*

$$E'(t) = -2\sigma\Big(\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2\Big)^2 + (h'\square\nabla u)(t) - h(t)\|\nabla u\|^2 \le 0\,, \quad t>0\,. \tag{3}$$

Proof. Multiplying the equation in (1) by u_t and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\Big\{\|u'\|^2 + \Big(\xi_0 + \frac{\xi_1}{2}\|\nabla u\|^2\Big)\|\nabla u\|^2 - \frac{2}{p+2}\|u\|_{p+2}^{p+2}\Big\}$$
$$+\sigma\Big(\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2\Big)^2 - \int_\Omega \nabla u' \int_0^t h(t-s)\nabla u(s)\,ds\,dx = 0\,.$$

We remark that

$$-\int_\Omega \nabla u' \int_0^t h(t-s)\nabla u(s)\,ds\,dx = \frac{1}{2}\frac{d}{dt}\Big[(h\square\nabla u)(t) - \int_0^t h(s)\,ds\|\nabla u(t)\|^2\Big]$$
$$-\frac{1}{2}(h'\square\nabla u)(t) + \frac{1}{2}h(t)\|\nabla u\|^2$$

The relation (3) follows at once. □

Now let

$$F(t) = \Big(\xi_0 - \int_0^t h(s)\,ds\Big)\|\nabla u\|^2 + \frac{\xi_1}{2}\|\nabla u\|^4$$
$$+ (h\square\nabla u)(t) - \frac{2}{p+2}\|u\|_{p+2}^{p+2} \tag{4}$$

then

$$E(t) = \|u'\|^2 + F(t)\,. \tag{5}$$

We define the potential well by

$$\mathcal{W} = \big\{u\,/\,I(u(t)) := \ell\|\nabla u\|^2 - \|u\|_{p+2}^{p+2} > 0\big\} \cup \{0\}\,.$$

Lemma 4. *Let u be the local solution of (1). If $u_0 \in \mathcal{W}$ and*

$$\alpha = \frac{C(p,\Omega)^{p+2}}{\ell^{\frac{p+2}{2}}}\Big(\frac{p+2}{p}E(0)\Big)^{p/2} < 1\,, \tag{6}$$

then $u(t)\in\mathcal{W}$ for each $t\in[0,T]$. Here $C(p,\Omega)$ is the Sobolev-Poincaré constant.

Proof. Let $u_0\in\mathcal{W}$, then $I(u_0)>0$. By continuity, this implies the existence of $T_m \le T$ such that $I(u(t)) \ge 0$ for all $t\in[0,T_m]$. Therefore, from (4), (5) and Lemma 3 we have

$$\ell\|\nabla u\|^2 \le \Big(\xi_0 - \int_0^t h(s)\,ds\Big)\|\nabla u\|^2 \le \frac{p+2}{p}F(t)$$
$$\le \frac{p+2}{p}E(t) \le \frac{p+2}{p}E(0)\,. \tag{7}$$

This relation, together with Lemma 2, implies that for $t\in[0,T_m]$,

$$\|u\|_{p+2}^{p+2} \le C(p,\Omega)^{p+2}\|\nabla u\|^{p+2} \le \frac{C(p,\Omega)^{p+2}}{\ell}\Big(\frac{p+2}{p\ell}E(0)\Big)^{p/2}\ell\|\nabla u\|^2\,.$$

That is, by our assumption on α

$$\|u\|_{p+2}^{p+2} \leq \alpha \ell \|\nabla u\|^2 < \alpha\Big(\xi_0 - \int_0^t h(s)\,ds\Big)\|\nabla u\|^2$$

$$< \ell \|\nabla u\|^2\,, \quad \forall\, t \in [0, T_m]\,. \tag{8}$$

Hence

$$I(t) > 0\,, \quad \forall\, t \in [0, T_m]\,,$$

which means that $u(t) \in \mathcal{W}$, $\forall t \in [0, T_m]$. By repeating the procedure, T_m extends to T. □

Now we are in position to state and prove our first main result.

Theorem 2. *Suppose that $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ satisfy* (6)*, then the solution of problem* (1) *is global in time.*

Proof. It suffices to show that $\|u'\|^2 + \|\nabla u\|^2$ is bounded independently of t. By virtue of Lemma 3 and Lemma 4 we get

$$E(0) \geq E(t) \geq \|u'\|^2 + \ell \|\nabla u\|^2 - \frac{2}{p+2}\|u\|_{p+1}^{p+1}$$

$$\geq \|u'\|^2 + \frac{p\ell}{p+2}\|\nabla u\|^2 + \frac{2}{p+2} I(t)\,.$$

Therefore, as $I(t) > 0$, we see that

$$\|u_t\|^2 + \|\nabla u\|^2 \leq c\,E(0) \quad \forall\, t > 0$$

for some positive constant c. □

4. Polynomial decay

In this section we shall prove the polynomial decay of solutions of problem (1).

Proposition 1. *Suppose that $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ satisfy* (6)*, then we have for any $T \geq S \geq 0$*

$$\int_S^T E^{\frac{m}{\rho}}(t)\Big\{\Big(\xi_0 - \int_0^t h(s)\,ds\Big)\|\nabla u(t)\|^2 + \frac{\xi_1}{2}\|\nabla u\|^4 - \frac{2}{p+2}\|u\|_{p+2}^{p+2}\Big\}\,dt$$

$$\leq C_1 E^{\frac{m}{\rho}}(0)\,E(S)$$

for some positive constant C_1.

Proof. Let us multiply both sides of the equation in (1) by $E^{\frac{m}{\rho}}(t)u$ and integrate over $\Omega\times[S,T]$, we obtain

$$\int_S^T E^{\frac{m}{\rho}}(t)\Big\{\Big(\xi_0-\int_0^t h(s)\,ds\Big)\|\nabla u\|^2+\xi_1\|\nabla u\|^4-\|u\|_{p+2}^{p+2}\Big\}\,dt$$

$$=-\int_S^T\int_\Omega E^{\frac{m}{\rho}}(t)u''u\,dx\,dt-\sigma\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^2\int_\Omega \nabla u'\cdot\nabla u\,dx\,dt$$

$$+\int_S^T\int_\Omega E^{\frac{m}{\rho}}(t)\nabla u(t)\int_0^t h(t-s)[\nabla u(s)-\nabla u(t)]\,ds\,dx\,dt\,. \tag{9}$$

By an integration by parts we see that

$$-\int_S^T\int_\Omega E^{\frac{m}{\rho}}(t)u''u\,dx\,dt=\int_S^T E^{\frac{m}{\rho}}(t)\|u'\|^2\,dt-\int_\Omega E^{\frac{m}{\rho}}(t)u'(t)\,u(t)\Big|_S^T\,dx$$
$$+\int_S^T \big(E^{\frac{m}{\rho}}(t)\big)'\int_\Omega u'(t)\,u(t)\,dx\,dt$$

and (9) becomes

$$\int_S^T E^{\frac{m}{\rho}}(t)\big(\xi_0-\int_0^t h(s)\,ds\big)\|\nabla u(t)\|^2dt=\int_S^T E^{\frac{m}{\rho}}(t)\|u'\|^2dt$$

$$-\sigma\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^2\int_\Omega\nabla u'\nabla u\,dx\,dt-\int_\Omega E^{\frac{m}{\rho}}(t)u'(t)u(t)\Big|_S^T\,dx$$

$$+\int_S^T\int_\Omega E^{\frac{m}{\rho}}(t)\nabla u(t)\int_0^t h(t-s)\big[\nabla u(s)-\nabla u(t)\big]\,ds\,dx\,dt$$

$$-\xi_1\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^4dt+\int_S^T E^{\frac{m}{\rho}}(t)\|u\|_{p+2}^{p+2}\,dt$$

$$+\int_S^T\big(E^{\frac{m}{\rho}}(t)\big)'\int_\Omega u'(t)u(t)\,dx\,dt\,. \tag{10}$$

We start with the memory term. By using Cauchy-Schwarz inequality and the ε-Young inequality, we obtain

$$\int_S^T\int_\Omega E^{\frac{m}{\rho}}(t)\nabla u(t)\int_0^t h(t-s)\big[\nabla u(s)-\nabla u(t)\big]\,ds\,dx\,dt$$

$$\le\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u(t)\|\Big[\int_\Omega\Big(\int_0^t h(t-s)|\nabla u(s)-\nabla u(t)|\,ds\Big)^2dx\Big]^{\frac12}\,dt$$

$$\le\frac{1}{2\varepsilon_0}\int_S^T E^{\frac{m}{\rho}}(t)\Big(\int_0^t h(t-s)\|\nabla u(s)-\nabla u(t)\|\,ds\Big)^2\,dt$$

$$+\frac{\varepsilon_0}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u(t)\|^2\,dt \tag{11}$$

for some $\varepsilon_0>0$.

Recalling that $h'(t) \leq -kh^{1+\frac{1}{\rho}}(t)$ and using (3) it appears that

$$\begin{aligned}
&\int_S^T E^{\frac{m}{\rho}}(t)\Big[\int_\Omega\Big(\int_0^t h(t-s)|\nabla u(s)-\nabla u(t)|\,ds\Big)^2 dx\Big]^{\frac{1}{2}}\,dt\\
&\quad=\int_S^T E^{\frac{m}{\rho}}(t)\Big(\int_0^t h^{\frac{1}{2}(1-\frac{1}{\rho})}(t-s)h^{\frac{1}{2}(1+\frac{1}{\rho})}(t-s)\|\nabla u(s)-\nabla u(t)\|\,ds\Big)^2\,dt\\
&\quad\leq\int_S^T E^{\frac{m}{\rho}}(t)\Big(\int_0^t h^{1-\frac{1}{\rho}}(s)\,ds\Big)\Big(\int_0^t h^{1+\frac{1}{\rho}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\Big)\,dt\\
&\quad\leq\Big(\int_0^\infty h^{1-\frac{1}{\rho}}(s)\,ds\Big)\int_S^T E^{\frac{m}{\rho}}(t)\Big(\int_0^t h^{1+\frac{1}{\rho}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\Big)\,dt\\
&\quad\leq-\frac{\bar{h}_\rho}{k}\int_S^T E^{\frac{m}{\rho}}(t)\Big(\int_0^t h'(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\Big)\,dt\\
&\quad\leq-\frac{1}{k}\bar{h}_\rho\int_S^T E^{\frac{m}{\rho}}(t)E'(t)\,dt\leq\frac{\bar{h}_\rho}{k}E^{\frac{m}{\rho}}(0)\,E(S)
\end{aligned}\tag{12}$$

where

$$\int_0^\infty h^{1-\frac{1}{\rho}}(s)\,ds=\bar{h}_\rho\,.$$

Therefore, (11) and (12) yield

$$\begin{aligned}
&\int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega\int_0^t h(t-s)\nabla u(t)\big[\nabla u(s)-\nabla u(t)\big]\,ds\,dx\,dt\\
&\qquad\leq\frac{\varepsilon_0}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u(t)\|^2\,dt+\frac{\bar{h}_\rho}{2\varepsilon_0 k}E^{\frac{m}{\rho}}(0)\,E(S)\,.
\end{aligned}\tag{13}$$

Next, we note that from (4) we have

$$F(t)\geq\frac{p}{p+2}\Big\{\Big(\xi_0-\int_0^t h(s)\,ds\Big)\|\nabla u\|^2+(h\Box\nabla u)(t)\Big\}+\frac{\xi_1}{2}\|\nabla u\|^4\tag{14}$$

from which we entail that

$$\|\nabla u\|^2\leq\frac{p+2}{p(\xi_0-\int_0^t h(s)\,ds)}\,E(t)\,.\tag{15}$$

Hence, using Poincaré inequality

$$\|u\|^2\leq B\|\nabla u\|^2\leq\frac{(p+2)B}{p\ell}E(t)\tag{16}$$

where B is the Poincaré constant.

Applying the above inequality (16) and having in mind that $F(t)\geq 0$; we find

$$\Big|\int_\Omega u'(t)u(t)\,dx\Big|\leq\frac{1}{2}\|u'\|^2+\frac{(p+2)B}{2p\ell}E(t)\leq\frac{1}{2}\Big(1+\frac{(p+2)B}{p\ell}\Big)\,E(t)\,.\tag{17}$$

Then, from (17) and the fact that $E(t)$ is non-increasing we infer that

$$-\int_\Omega E^{\frac{m}{\rho}}(t)u'(t)u(t)\Big|_S^T\,dx \le \Big(1+\frac{(p+2)B}{p\ell}\Big)E^{\frac{m}{\rho}}(0)\,E(S) \tag{18}$$

and

$$\begin{aligned}\int_S^T \big(E^{\frac{m}{\rho}}(t)\big)'\int_\Omega u'(t)u(t)\,dx\,dt &\le -\int_S^T (E^{\frac{m}{\rho}}(t))'|\int_\Omega u'(t)u(t)\,dx|\,dt\\ &\le -\frac{1}{2}\Big(1+\frac{(p+2)B}{p\ell}\Big)\int_S^T \big(E^{\frac{m}{\rho}}(t)\big)'E(t)\,dt\\ &\le \frac{1}{2}\Big(1+\frac{(p+2)B}{p\ell}\Big)E^{\frac{m}{\rho}}(0)\,E(S)\,. \end{aligned}\tag{19}$$

Now, using (8) it is easy to see that

$$\begin{aligned}\int_S^T E^{\frac{m}{\rho}}(t)\|u\|_{p+2}^{p+2}\,dt &\le \alpha\int_S^T \ell E^{\frac{m}{\rho}}(t)\|\nabla u\|^2\,dt\\ &< \alpha\int_S^T E^{\frac{m}{\rho}}(t)\Big(\xi_0-\int_0^t h(s)\,ds\Big)\|\nabla u\|^2\,dt\end{aligned}\tag{20}$$

and for $\varepsilon_1>0$

$$\begin{aligned}-\sigma\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^2\int_\Omega \nabla u'\nabla u\,dx\,dt &\le \frac{\sigma\varepsilon_1}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^4dt\\ &\quad+\frac{\sigma}{2\varepsilon_1}\int_S^T E^{\frac{m}{\rho}}(t)(\nabla u',\nabla u)^2\,dt\,.\end{aligned}$$

Thanks also to formula (3) which implies that

$$\begin{aligned}-\sigma\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^2\int_\Omega \nabla u'\nabla u\,dx\,dt &\le \frac{\sigma\varepsilon_1}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^4dt\\ &\quad+\frac{1}{4\varepsilon_1}E^{\frac{m}{\rho}}(0)\,E(S)\,.\end{aligned}\tag{21}$$

Taking into account the estimates (13) and (18)–(21) in relation (10), we end up with

$$\begin{aligned}\int_S^T E^{\frac{m}{\rho}}(t)\Big(\xi_0-\int_0^t h(s)\,ds\Big)\|\nabla u(t)\|^2\,dt &\le \int_S^T E^{\frac{m}{\rho}}(t)\|u'\|^2\,dt\\ &+\frac{\varepsilon_0}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u(t)\|^2\,dt+\frac{\sigma\varepsilon_1}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^4\,dt\\ &+\alpha\int_S^T E^{\frac{m}{\rho}}(t)\Big(\xi_0-\int_0^t h(s)\,ds\Big)\|\nabla u\|^2\,dt+\frac{3}{2}\Big(1+\frac{(p+2)B}{p\ell}\Big)E^{\frac{m}{\rho}}(0)\,E(S)\\ &+\frac{1}{4\varepsilon_1}E^{\frac{m}{\rho}}(0)\,E(S)-\xi_1\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^4\,dt+\frac{\bar{h}_\rho}{2\varepsilon_0 k}E^{\frac{m}{\rho}}(0)\,E(S)\,.\end{aligned}$$

If we put $\varepsilon_0 = (1-\alpha)\big(\xi_0 - \int_0^\infty h(s)\,ds\big)$, then we obtain

$$\begin{aligned}\int_S^T E^{\frac{m}{\rho}}(t)\Big(\xi_0 - \int_0^t h(s)\,ds\Big)\|\nabla u(t)\|^2\,dt \\ \le \frac{2}{1-\alpha}\int_S^T E^{\frac{m}{\rho}}(t)\|u'\|^2\,dt + \frac{2}{1-\alpha}\Big(\frac{\sigma\varepsilon_1}{2} - \xi_1\Big)\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^4\,dt \\ + \frac{2}{1-\alpha}\Big(\frac{3}{2} + \frac{1}{4\varepsilon_1} + \frac{3(p+2)B}{2p\ell} + \frac{\bar{h}_\rho}{2\varepsilon_0 k}\Big)E^{\frac{m}{\rho}}(0)\,E(S)\,. \end{aligned} \tag{22}$$

Now, multiplying both sides of the equation in (1) by the expression

$$E^{\frac{m}{\rho}}(t)\int_0^t h(t-s)\big[u(s)-u(t)\big]\,ds\,,$$

integrating over $\Omega\times[S,T]$ and setting

$$\begin{aligned}(h\diamond u)(t) &= \int_0^t h(t-s)\big[u(s)-u(t)\big]\,ds\,,\\ (h\diamond \nabla u)(t) &= \int_0^t h(t-s)\big[\nabla u(s)-\nabla u(t)\big]\,ds\end{aligned}$$

we find

$$\begin{aligned}&\int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega u''(h\diamond u)(t)\,dx\,dt\\ &\quad -\int_S^T E^{\frac{m}{\rho}}(t)(\xi_0+\xi_1\|\nabla u(t)\|_2^2+\sigma\big(\nabla u(t),\nabla u_t(t)\big))\int_\Omega \Delta u(h\diamond u)(t)\,dx\,dt\\ &\quad +\int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega\Big(\int_0^t h(t-s)\Delta u(s)\,ds\Big)(h\diamond u)(t)\,dx\,dt\\ &= \int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega |u|^p u(h\diamond u)(t)\,dx\,dt\,.\end{aligned}\tag{23}$$

An integration by parts yields

$$\begin{aligned}\int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega u''(h\diamond u)(t)\,dx\,dt &= E^{\frac{m}{\rho}}(t)\int_\Omega u'(t)(h\diamond u)(t)\,dx\Big|_S^T\\ &\quad -\int_S^T \big(E^{\frac{m}{\rho}}(t)\big)'\int_\Omega u'(t)(h\diamond u)(t)\,dx\,dt\\ &\quad -\int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega u'(t)(h'\diamond u)(t)\,dx\,dt\\ &\quad +\int_S^T E^{\frac{m}{\rho}}(t)\Big(\int_0^t h(s)\,ds\Big)\|u'(t)\|^2\,dt\,,\end{aligned}$$

and a substitution in (23) gives

$$\int_S^T E^{\frac{m}{\rho}}(t)\Big(\int_0^t h(s)\,ds\Big)\|u'(t)\|^2\,dt = \int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega |u|^p u(h\diamond u)(t)\,dx\,dt$$

$$- E^{\frac{m}{\rho}}(t)\int_\Omega u'(t)(h\diamond u)(t)\,dx\Big|_S^T + \int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega u'(t)(h'\diamond u)(t)\,dx\,dt$$

$$+ \int_S^T \big(E^{\frac{m}{\rho}}(t)\big)'\int_\Omega u'(t)(h\diamond u)(t)\,dx\,dt$$

$$+ \int_S^T E^{\frac{m}{\rho}}(t)\big(\xi_0+\xi_1\|\nabla u(t)\|_2^2+\sigma(\nabla u(t),\nabla u_t(t))\big)\int_\Omega \Delta u(h\diamond u)(t)\,dx\,dt$$

$$- \int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega \Big(\int_0^t h(t-s)\Delta u(s)\,ds\Big)(h\diamond u)(t)\,dx\,dt\,. \tag{24}$$

Moreover, in virtue of (5) and (14), we have

$$\begin{aligned}\Big|\int_\Omega u'(t)(h\diamond u)(t)\,dx\Big| &\le \frac{\varepsilon_2}{2}\|u'(t)\|^2 + \frac{1}{2\varepsilon_2}\int_\Omega \big((h\diamond u)(t)\big)^2\,dx\\ &\le \frac{\varepsilon_2}{2}E(t) + \frac{B^2}{2\varepsilon_2}\Big(\int_0^t h(s)\,ds\Big)(h\Box\nabla u)(t)\\ &\le \frac{1}{2}\Big(\varepsilon_2+\frac{B^2(\xi_0-\ell)}{\varepsilon_2}\Big)E(t) \le \frac{1}{2}\Big(\varepsilon_2+\frac{B^2(\xi_0-\ell)}{k\varepsilon_2}\Big)E(S)\end{aligned}$$

for some $\varepsilon_2>0$ and $t\ge S$. So,

$$\begin{aligned}&\int_\Omega E^{\frac{m}{\rho}}(t)u'(t)\int_0^t h(t-s)\big[u(s)-u(t)\big]\,ds\,dx\Big|_S^T\\ &\qquad\le \Big(\varepsilon_2+\frac{B^2(\xi_0-\ell)}{\varepsilon_2}\Big)E^{\frac{m}{\rho}}(0)\,E(S)\,.\end{aligned} \tag{25}$$

On the other hand, it is clear that

$$\begin{aligned}\int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega u'(t)(h'\diamond u)(t)\,dx\,dt &\le \frac{\varepsilon_3}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|u'(t)\|^2 dt\\ &+ \frac{1}{2\varepsilon_3}\int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega\Big(\int_0^t |h'(t-s)||u(s)-u(t)|\,ds\Big)^2\,dx\,dt\end{aligned}$$

and for some $\varepsilon_3>0$

$$\begin{aligned}\int_S^T \big(E^{\frac{m}{\rho}}(t)\big)'\int_\Omega u'(t)(h\diamond u)(t)\,dx\,dt &\le -\Big(\varepsilon_2+\frac{B^2(\xi_0-\ell)}{\varepsilon_2}\Big)\int_S^T \big(E^{\frac{m}{\rho}}(t)\big)'E(t)\,dt\\ &\le -\Big(\varepsilon_2+\frac{B^2(\xi_0-\ell)}{\varepsilon_2}\Big)E^{\frac{m}{\rho}}(0)\,E(S)\,.\end{aligned}$$

Since $h'(t) \le 0$, the relation (3) implies that

$$\begin{aligned}
&\int_\Omega \Big(\int_0^t |h'(t-s)||u(s)-u(t)|\,ds \Big)^2\,dx \\
&\quad \le -\int_0^t h'(s)\,ds \int_0^t |h'(t-s)|\,\|u(s)-u(t)\|^2\,ds \\
&\quad \le -h(0)B(h'\Box\nabla u)(t) \le -h(0)BE'(t)\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_S^T E^{\frac{m}{\rho}}(t) \int_\Omega u'(t) \int_0^t h'(t-s)\big[u(s)-u(t)\big]\,ds\,dx\,dt \\
&\qquad \le \frac{\varepsilon_3}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|u'(t)\|^2 dt + \frac{h(0)B}{2\varepsilon_3} E^{\frac{m}{\rho}}(0)\,E(S)\,. \qquad (26)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int_S^T E^{\frac{m}{\rho}}(t)\big(\xi_0 + \xi_1\|\nabla u(t)\|_2^2 + \sigma(\nabla u(t),\nabla u_t(t))\big)\int_\Omega \Delta u(h\diamond u)(t)\,dx\,dt \\
&= -\,\xi_0 \int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega \nabla u(h\diamond\nabla u)(t)\,dx\,dt \\
&\quad -\int_S^T \xi_1 E^{\frac{m}{\rho}}(t)\|\nabla u(t)\|_2^2 \int_\Omega \nabla u(h\diamond\nabla u)(t)\,dx\,dt \\
&\quad -\int_S^T \sigma E^{\frac{m}{\rho}}(t)\big(\nabla u(t),\nabla u_t(t)\big)\int_\Omega \nabla u(h\diamond\nabla u)(t)\,dx\,dt
\end{aligned}$$

and the application of ε-Young inequality and the relation (7) and (3) yield

$$\begin{aligned}
&\int_S^T E^{\frac{m}{\rho}}(t)\big(\xi_0 + \xi_1\|\nabla u(t)\|_2^2 + \sigma(\nabla u(t),\nabla u'(t))\big)\int_\Omega \Delta u(h\diamond u)(t)\,dx\,dt \\
&\le -\,\xi_0 \int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega \nabla u(h\diamond\nabla u)(t)\,dx\,dt + \frac{\varepsilon_4}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u(t)\|^2\,dt \\
&\quad + \sigma^2\frac{\varepsilon_5}{2}\int_S^T E^{\frac{m}{\rho}}(t)(\nabla u,\nabla u')^2\|\nabla u(t)\|^2\,dt \\
&\quad + \frac{1}{2\varepsilon_5}\int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega \big((h\diamond\nabla u)(t)\big)^2\,dx\,dt \\
&\quad + \frac{\xi_1^2 E(0)^2(p+2)^2\bar{h}_\rho}{2(p\ell)^2\varepsilon_4 k} E^{\frac{m}{\rho}}(0)\,E(S) \qquad (27)
\end{aligned}$$

for ε_4, $\varepsilon_5 > 0$. Thus, (27) and (3) imply that

$$\int_S^T E^{\frac{m}{\rho}}(t)\big(\xi_0 + \xi_1\|\nabla u(t)\|_2^2 + \sigma\big(\nabla u(t), \nabla u'(t)\big)\big) \int_\Omega \Delta u(h \diamond u)(t)\,dx\,dt$$

$$\leq -\xi_0 \int_S^T E^{\frac{m}{\rho}}(t) \int_\Omega \nabla u(h \diamond \nabla u)(t)\,dx\,dt$$

$$+ \frac{\varepsilon_4}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u(t)\|^2\,dt + \frac{\xi_1^2 E(0)^2(p+2)^2\bar{h}_\rho}{2(p\ell)^2\varepsilon_4 k} E^{\frac{m}{\rho}}(0)\,E(S)$$

$$+ \Big(\frac{\varepsilon_5\sigma(p+2)E(0)}{4p\ell} + \frac{\bar{h}_\rho}{2\varepsilon_5 k}\Big) E^{\frac{m}{\rho}}(0)\,E(S)\,. \tag{28}$$

Moreover, we have

$$-\int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega \Big(\int_0^t h(t-s)\Delta u\,ds\Big)(h \diamond u)(t)\,dx\,dt$$

$$= \int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega \Big(\int_0^t h(t-s)\nabla u\,ds\Big)(h \diamond \nabla u)(t)\,dx\,dt$$

$$= \int_S^T E^{\frac{m}{\rho}}(t)\Big(\int_0^t h(s)\,ds\Big)\int_\Omega \nabla u(h \diamond \nabla u)(t)\,dx\,dt$$

$$+ \int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega \big((h \diamond \nabla u)(t)\big)^2\,dx\,dt\,. \tag{29}$$

So, thanks to (12) we obtain

$$-\int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega \Big(\int_0^t h(t-s)\Delta u\,ds\Big)(h \diamond u)(t)\,dx\,dt$$

$$\leq \int_S^T E^{\frac{m}{\rho}}(t)\Big(\int_0^t h(s)\,ds\Big)\int_\Omega \nabla u(t)$$

$$\times (h \diamond \nabla u)(t)\,dx\,dt + \frac{\bar{h}_\rho}{k}E^{\frac{m}{\rho}}(0)\,E(S)\,. \tag{30}$$

The first term in the right-hand side of (30) together with the first term in the right-hand side of (28), can be estimated as follows

$$\int_S^T E^{\frac{m}{\rho}}(t)\Big(\int_0^t h(s)\,ds - \xi_0\Big)\int_\Omega \nabla u(t)\,(h \diamond \nabla u)(t)\,dx\,dt$$

$$\leq \int_S^T E^{\frac{m}{\rho}}(t)\Big(\xi_0 - \int_0^t h(s)\,ds\Big)\Big|\int_\Omega \nabla u(t)\,(h \diamond \nabla u)(t)\,dx\Big|\,dt$$

$$\leq \xi_0 \int_S^T E^{\frac{m}{\rho}}(t)\Big|\int_\Omega \nabla u(t)\,(h \diamond \nabla u)(t)\,dx\Big|\,dt$$

and by (13) we get

$$\begin{aligned}
&\int_S^T E^{\frac{m}{\rho}}(t)\Big(\int_0^t h(s)\,ds-\xi_0\Big)\int_\Omega \nabla u(t)\,(h\diamond\nabla u)(t)\,dx\,dt\\
&\qquad\le \frac{\xi_0\varepsilon_4}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u(t)\|^2\,dt+\frac{\xi_0\bar h_\rho}{2\varepsilon_4 k}\,E^{\frac{m}{\rho}}(0)\,E(S)\,. \tag{31}
\end{aligned}$$

Now, the relations (28)–(31) lead to

$$\begin{aligned}
&\int_S^T E^{\frac{m}{\rho}}(t)\big(\xi_0+\xi_1\|\nabla u(t)\|_2^2+\sigma(\nabla u(t),\nabla u_t(t))\big)\int_\Omega \Delta u(h\diamond u)(t)\,dx\,dt\\
&\quad-\int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega\Big(\int_0^t h(t-s)\Delta u\,ds\Big)(h\diamond u)(t)\,dx\,dt\\
&\quad\le \bar h_\rho\Big(\frac{1}{k}+\frac{\xi_1^2E(0)^2(p+2)^2}{2(p\ell)^2\varepsilon_4 k}+\frac{\varepsilon_5\sigma(p+2)E(0)}{4p\ell\bar h_\rho}+\frac{1}{2\varepsilon_5 k}+\frac{\xi_0}{2\varepsilon_4 k}\Big)\\
&\quad\times E^{\frac{m}{\rho}}(0)\,E(S)+\varepsilon_4\Big(\frac12+\frac{\xi_0}{2}\Big)\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u(t)\|^2\,dt\,. \tag{32}
\end{aligned}$$

In addition to that, it is easy to see that the relation (12) implies that

$$\begin{aligned}
&\int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega |u|^p u\,(h\diamond u)(t)\,dx\,dt\\
&\quad\le \int_S^T \Big(E^{\frac{m}{\rho}}(t)\int_\Omega |u|^{2p+2}\,dx\Big)^{\frac12}\Big(E^{\frac{m}{\rho}}(t)\int_\Omega ((h\diamond u)(t))^2\,dx\Big)^{\frac12}\,dt\\
&\quad\le \frac{\varepsilon_4}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|u\|_{2p+2}^{2p+2}\,dt+\frac{1}{2\varepsilon_4}\int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega\Big(\int_0^t h^{1-\frac{1}{\rho}}(s)ds\Big)\\
&\qquad\times\Big(\int_0^t h^{1+\frac{1}{\rho}}(t-s)\big[u(s)-u(t)\big]^2ds\Big)\,dx\,dt\\
&\quad\le \frac{\varepsilon_4}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|u\|_{2p+2}^{2p+2}\,dt+\frac{B\bar h_\rho}{2\varepsilon_4}\,E^{\frac{m}{\rho}}(0)\,E(S)\,. \tag{33}
\end{aligned}$$

To estimate the term $\int_S^T E^{\frac{m}{\rho}}(t)\|u\|_{2p+2}^{2p+2}\,dt$ we use Sobolev-Poincaré inequality

$$\begin{aligned}
\|u\|_{2p+2}^{2p+2}&\le C_*(p,\Omega)^{2p+2}\|\nabla u\|_2^{2p+2}\\
&\le C(p,\Omega)^{p+2}\Big(\frac{p+2}{p\ell}\,E(0)\Big)^p\|\nabla u\|^2=:\beta\|\nabla u\|^2\,. \tag{34}
\end{aligned}$$

Here $C_*(p,\Omega)$ is the Sobolev-Poincaré constant, with $0 \le p < +\infty$ $(n=1,2)$ or $0 \le p \le \frac{2}{n-2}$ $(n>2)$. Therefore,

$$\int_S^T E^{\frac{m}{\rho}}(t)\int_\Omega |u|^p u(h\diamond u)(t)\,dx\,dt$$

$$\le \frac{\beta\varepsilon_4}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^2\,dt + \frac{B\bar{h}_\rho}{2\varepsilon_4 k}E^{\frac{m}{\rho}}(0)\,E(S)\,. \tag{35}$$

Now, combining the relations (25), (26), (32) and (35) with (24), we obtain

$$\int_S^T E^{\frac{m}{\rho}}(t)\Big(\int_0^t h(s)\,ds\Big)\|u'(t)\|^2\,dt$$

$$\le CE^{\frac{m}{\rho}}(0)\,E(S) + \frac{\varepsilon_3}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|u'(t)\|^2\,dt$$

$$+\,\varepsilon_4\Big(\frac{1+\xi_0+\beta}{2}\Big)\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^2\,dt\,,$$

where

$$C = \bar{h}_\rho\Big[\frac{1}{k} + \frac{\xi_1^2E(0)^2(p+2)^2}{2(p\ell)^2\varepsilon_4 k} + \frac{\varepsilon_5\sigma(p+2)E(0)}{4p\ell\bar{h}_\rho} + \frac{\xi_0}{2\varepsilon_4 k}$$

$$+\frac{1}{2\varepsilon_5 k} + \frac{B}{2\varepsilon_4 k} + \frac{2\varepsilon_2}{\bar{h}_\rho} + \frac{2B^2(\xi_0-\ell)}{\varepsilon_2\bar{h}_\rho} + \frac{h(0)B}{2\varepsilon_3\bar{h}_\rho}\Big]$$

or

$$\int_S^T E^{\frac{m}{\rho}}(t)\Big(\int_0^t h(s)\,ds - \frac{\varepsilon_3}{2}\Big)\|u'(t)\|^2\,dt$$

$$\le CE^{\frac{m}{\rho}}(0\,E(S) + \varepsilon_4\Big(\frac{1+\xi_0+\beta}{2}\Big)\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^2\,dt\,. \tag{36}$$

It is clear that

$$\int_0^S h(s)\,ds \ge \int_0^{S_0} h(s)\,ds > 0\,, \quad S \ge S_0$$

and choosing

$$\varepsilon_3 < \int_0^{S_0} h(s)\,ds =: h_0\,,$$

we find

$$\frac{1}{2}\Big(\int_0^{S_0} h(s)\,ds\Big)\int_S^T E^{\frac{m}{\rho}}(t)\|u'(t)\|^2 dt \le CE^{\frac{m}{\rho}}(0)E(S)$$

$$+\,(1+\xi_0+\beta)\frac{\varepsilon_4}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^2\,dt\,.$$

So

$$\int_S^T E^{\frac{m}{\rho}}(t)\|u'(t)\|^2\,dt \le \frac{2C}{\int_0^{S_0} h(s)\,ds}\,E^{\frac{m}{\rho}}(0)\,E(S)$$

$$+\,\varepsilon_4\Big(\frac{1+\xi_0+\beta}{h_0}\Big)\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^2\,dt\,. \tag{37}$$

Plugging estimate (37) into (22) we obtain

$$\int_S^T E^{\frac{m}{\rho}}(t)\Big(\xi_0-\int_0^t h(s)\,ds\Big)\|\nabla u(t)\|^2\,dt$$

$$\le \frac{2}{1-\alpha}\Big(\frac{3}{2}+\frac{1}{4\varepsilon_1}+\frac{3(p+2)B}{2p\ell}+\frac{\bar{h}_\rho}{2\varepsilon_0 k}\Big)E^{\frac{m}{\rho}}(0)\,E(S)$$

$$+\,\frac{2}{1-\alpha}\Big(\frac{\sigma\varepsilon_1}{2}-\xi_1\Big)\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^4\,dt+\frac{4C}{(1-\alpha)h_0}\,E^{\frac{m}{\rho}}(0)\,E(S)$$

$$+\,\frac{2\varepsilon_4(1+\xi_0+\beta)}{(1-\alpha)h_0}\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^2\,dt\,. \tag{38}$$

The choice $\varepsilon_4=\frac{(1-\alpha)h_0\ell}{4(1+\xi_0+\beta)}$ allows us to write

$$\int_S^T E^{\frac{m}{\rho}}(t)\Big(\xi_0-\int_0^t h(s)\,ds\Big)\|\nabla u(t)\|^2\,dt$$

$$\le \frac{4}{1-\alpha}\Big\{\Big[\frac{2C}{h_0}+\frac{3}{2}+\frac{1}{4\varepsilon_1}+\frac{3(p+2)B}{2p\ell}+\frac{\bar{h}_\rho}{2\varepsilon_0 k}\Big]$$

$$\times\,E^{\frac{m}{\rho}}(0)\,E(S)+\Big(\frac{\sigma\varepsilon_1}{2}-\xi_1\Big)\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^4\,dt\Big\}$$

and if we choose $\varepsilon_1=\frac{7+\alpha}{4\sigma}\xi_1$, since $\alpha<1$, the last relation reduces to

$$\int_S^T E^{\frac{m}{\rho}}(t)\Big(\xi_0-\int_0^t h(s)\,ds\Big)\|\nabla u(t)\|^2\,dt$$

$$+\,\frac{\xi_1}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^4\,dt \le \frac{4\tilde{C}}{1-\alpha}\,E^{\frac{m}{\rho}}(0)\,E(S) \tag{39}$$

where $\tilde{C}=\frac{2C}{h_0}+\frac{3}{2}+\frac{1}{4\varepsilon_1}+\frac{3(p+2)B}{2p\ell}+\frac{\bar{h}_\rho}{2\varepsilon_0 k}$.

Next, the relations (37) and (39) imply

$$\int_S^T E^{\frac{m}{\rho}}(t)\|u'(t)\|^2\,dt \le \frac{2C}{h_0}\,E^{\frac{m}{\rho}}(0)\,E(S)$$

$$+\,\frac{1-\alpha}{4}\int_S^T E^{\frac{m}{\rho}}(t)\Big(\xi_0-\int_0^t h(s)\,ds\Big)\|\nabla u\|^2\,dt \le 2\tilde{C}E^{\frac{m}{\rho}}(0)\,E(S)\,. \tag{40}$$

Finally, in virtue of (8) and (39) we get

$$\begin{aligned}\frac{2}{p+2}\int_S^T E^{\frac{m}{\rho}}(t)\|u\|_{p+2}^{p+2}\,dt &\le \frac{2\alpha}{p+2}\int_S^T E^{\frac{m}{\rho}}(t)\Big(\xi_0-\int_0^t h(s)\,ds\Big)\|\nabla u\|^2\,dt\\ &\le \frac{2\alpha\tilde{C}}{p+2}E^{\frac{m}{\rho}}(0)\,E(S)\le \tilde{C}E^{\frac{m}{\rho}}(0)\,E(S)\,. \qquad (41)\end{aligned}$$

So, combining (39)–(41) we obtain

$$\begin{aligned}&\int_S^T E^{\frac{m}{\rho}}(t)\|u'(t)\|^2\,dt+\int_S^T E^{\frac{m}{\rho}}(t)\Big(\xi_0-\int_0^t h(s)\,ds\Big)\|\nabla u(t)\|^2\,dt\\ &\qquad+\frac{\xi_1}{2}\int_S^T E^{\frac{m}{\rho}}(t)\|\nabla u\|^4\,dt-\frac{2}{p+2}\int_S^T E^{\frac{m}{\rho}}(t)\|u\|_{p+2}^{p+2}\,dt\\ &\qquad\le\Big(3+\frac{4}{1-\alpha}\Big)\tilde{C}E^{\frac{m}{\rho}}(0)\,E(S)\,. \qquad (42)\end{aligned}$$

$\square$

Theorem 3. *Suppose that $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ satisfy (6), then we have the following decay estimate*

$$E(t)\le E(0)\Big(\frac{c(1+\rho)}{t+c\rho}\Big)^{\rho},\qquad \forall\ t\ge 0$$

for some positive constant c.

Proof. First, applying Hölder's inequality we see that

$$\begin{aligned}(h\Box\nabla u)(t)&=\int_0^t h^{\frac{m-1}{\rho+m}}(t-s)\|\nabla u(s)-\nabla u(t)\|^{\frac{2m}{\rho+m}}\\ &\quad\times h^{1-\frac{m-1}{\rho+m}}(t-s)\|\nabla u(s)-\nabla u(t)\|^{\frac{2\rho}{\rho+m}}\,ds\\ &\le\Big(\int_0^t h^{1-\frac{1}{m}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\Big)^{\frac{m}{\rho+m}}\\ &\quad\times\Big(\int_0^t h^{1+\frac{1}{\rho}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\Big)^{\frac{\rho}{\rho+m}}\,.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_S^T E^{\frac{m}{\rho}}(t)(h\Box\nabla u)(t)\,dt\\ &\qquad\le\int_S^T E^{\frac{m}{\rho}}(t)\Big(\int_0^t h^{1-\frac{1}{m}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\Big)^{\frac{m}{\rho+m}}\\ &\qquad\quad\times\Big(\int_0^t h^{1+\frac{1}{\rho}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\Big)^{\frac{\rho}{\rho+m}}\,dt\,.\end{aligned}$$

Applying Hölder's inequality again it appears that

$$\begin{aligned}
&\int_S^T E^{\frac{m}{\rho}}(t)(h\Box\nabla u)(t)\,dt \\
&\qquad \le \Big(\int_S^T E^{1+\frac{m}{\rho}}(t)\int_0^t h^{1-\frac{1}{m}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\,dt\Big)^{\frac{m}{\rho+m}} \\
&\qquad \times \Big(\int_S^T\int_0^t h^{1+\frac{1}{\rho}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\,dt\Big)^{\frac{\rho}{\rho+m}}. \qquad (43)
\end{aligned}$$

By virtue of the condition $h'(t) \le -kh^{1+\frac{1}{\rho}}(t)$, we see that

$$\begin{aligned}
&\int_S^T E^{\frac{m}{\rho}}(t)(h\Box\nabla u)(t)\,dt \\
&\qquad \le \Big(\int_S^T E^{1+\frac{m}{\rho}}(t)\int_0^t h^{1-\frac{1}{m}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\,dt\Big)^{\frac{m}{\rho+m}} \\
&\qquad \times \Big(\frac{1}{k}\Big)^{\frac{\rho}{\rho+m}}\Big(-\int_S^T\int_0^t h'(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\,dt\Big)^{\frac{\rho}{\rho+m}} \\
&\qquad \le \Big(\int_S^T E^{1+\frac{m}{\rho}}(t)\int_0^t h^{1-\frac{1}{m}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\,dt\Big)^{\frac{m}{\rho+m}} \\
&\qquad \times \Big(\frac{1}{k}\Big)^{\frac{\rho}{\rho+m}}\Big(-\int_S^T E'(t)\,dt\Big)^{\frac{\rho}{\rho+m}} \\
&\qquad \le \Big(\int_S^T E^{1+\frac{m}{\rho}}(t)\int_0^t h^{1-\frac{1}{m}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\,dt\Big)^{\frac{m}{\rho+m}} \\
&\qquad \times \Big(\frac{1}{k}\Big)^{\frac{\rho}{\rho+m}} E^{\frac{\rho}{\rho+m}}(S).
\end{aligned}$$

Further, by Young inequality, we have for $\varepsilon_6 > 0$

$$\begin{aligned}
\int_S^T E^{\frac{m}{\rho}}(t)(h\Box\nabla u)(t)\,dt &\le C\Big(\int_S^T E^{1+\frac{m}{\rho}}(t)\,dt\Big)^{\frac{m}{\rho+m}} \\
&\quad \times \Big\|\int_0^t h^{1-\frac{1}{m}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\Big\|_\infty^{\frac{m}{\rho+m}} E^{\frac{\rho}{\rho+m}}(S) \\
&\le C(\varepsilon_6)\Big\|\int_0^t h^{1-\frac{1}{m}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\Big\|_\infty^{\frac{m}{\rho}} E(S) \\
&\quad + \varepsilon_6\int_S^T E^{1+\frac{m}{\rho}}(t)\,dt \qquad (44)
\end{aligned}$$

for some constant $C(\varepsilon_6) > 0$.

So, combining (42) and (44) we obtain

$$\int_S^T E^{1+\frac{m}{\rho}}(t)\,dt \le \Big(3+\frac{4}{1-\alpha}\Big)\tilde{C}E^{\frac{m}{\rho}}(0)\,E(S)+\varepsilon_6\int_S^T E^{1+\frac{m}{\rho}}(t)\,dt$$

$$+\,C(\varepsilon_6)\|\int_0^t h^{1-\frac{1}{m}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\|_\infty^{\frac{m}{\rho}}\,E(S) \tag{45}$$

or

$$\int_S^T E^{1+\frac{m}{\rho}}(t)\,dt$$

$$\le C_2\Big(E^{\frac{m}{\rho}}(0)+\Big\|\int_0^t h^{1-\frac{1}{m}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\Big\|_\infty^{\frac{m}{\rho}}\Big)\,E(S) \tag{46}$$

for some positive constant C_2.

On the other hand we have when $m=2$

$$\int_0^t h^{\frac12}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds \le \frac{2(p+2)}{p\ell}\int_0^t h^{\frac12}(t-s)\big(E(s)+E(t)\big)\,ds$$

$$\le \frac{4(p+2)}{p\ell}\int_0^t h^{\frac12}(s)\,ds\,E(0)\,, \qquad \forall\ t\ge 0\,.$$

Therefore

$$\int_S^T E^{1+\frac{2}{\rho}}(t)dt \le C_3E^{\frac{2}{\rho}}(0)\,E(S)\,, \qquad \forall\ S\ge S_0\,. \tag{47}$$

Hence, applying Lemma 1 with $\lambda=\frac{2}{\rho}$, we find

$$E(t)\le E(0)\Big(\frac{(S_0+C_3)(1+\lambda)}{\lambda t+S_0+C_3}\Big)^{\frac{\rho}{2}}\,,\quad \forall\ t\ge 0\,.$$

Now if $m=1$, from

$$\int_0^t \|\nabla u(s)-\nabla u(t)\|^2\,ds \le 2\frac{p+2}{p\ell}\Big(\int_0^t E(s)\,ds+\sup_{t\ge 0} t\,E(t)\Big)$$

$$\le C_3E(0)\,,\quad \forall\ t\ge 0$$

and (46), we can write

$$\int_S^T E^{1+\frac{1}{\rho}}(t)\,dt \le C_4E^{\frac{1}{\rho}}(0)\,E(S)\,,\quad \forall\ S\ge S_0\,.$$

Hence, applying Lemma 1 with $\lambda=\frac{1}{\rho}$ we deduce that

$$E(t)\le E(0)\Big(\frac{(S_0+C_4)(1+\rho)}{t+\rho(S_0+C_4)}\Big)^{\rho}\,,\quad \forall\ t\ge 0\,.$$

□

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Cheikh El Arbi Tébessi University,
12002 Tébessa, Algeria
E-mail: zaraiabdoo@yahoo.fr

King Fahd University of Petroleum and Minerals,
Department of Mathematics and Statistics,
Dhahran 31261, Saudi Arabia
E-mail: tatarn@kfupm.edu.sa