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APPROXIMATION OF SOLUTIONS OF A DIFFERENCE-DIFFERENTIAL EQUATION

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Abstract. In the present paper we study the approximate solutions of a certain difference-differential equation under the given initial conditions. The well known Gronwall-Bellman integral inequality is used to establish the results. Applications to a Volterra type difference-integral equation are also given.

1. Introduction

Let R^n denote the real n-dimensional Euclidean space with the corresponding norm $|\cdot|$. Let $R_+ = [0,\infty)$ be a subset of real numbers. Consider the difference-differential equation

$$x'(t) = f\big(t, x(t), x(t-1)\big)\,, \tag{1.1}$$

for $t \in R_+$ under the initial conditions

$$x(t-1) = \phi(t) \quad (0 \le t < 1)\,, \quad x(0) = x_0\,, \tag{1.2}$$

where $f \in C(R_+ \times R^n \times R^n, R^n)$ and $\phi(t)$ is a continuous function for which $\lim\limits_{t\to 1-0} \phi(t)$ exists. Recently, in [6, 7] (see also [1]–[4], [9]–[10]) the problems of existence, uniqueness and continuous dependence of solutions of equation (1.1) are dealt under the conditions (1.2). In dealing with the equation (1.1) with (1.2), one of the basic question to be answered is: how can we find the solutions or closely approximate them? The study of this question is interesting and need a fresh outlook for handling the equation of the form (1.1), see [6, 7]. In the present paper, we continue our investigation in [6, 7] and offer the conditions for the error evoluation of approximate solutions of equation (1.1) with (1.2) by establishing some new bounds on solutions of approximate problems. The basic integral inequality with explicit estimate due to Gronwall-Bellman (see [5, p. 12]) is used to establish the results. The applications to study a Volterra type difference-integral equation are also given.

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2. Main Results

We need the following integral inequality, known in the literature as the Gronwall-Bellman inequality (see [5, p. 12]).

Lemma. *Let $u(t)$, $n(t)$, $e(t) \in C(R_+, R_+)$ and $n(t)$ be nondecreasing on R_+. If*

$$u(t) \le n(t) + \int_0^t e(s)\, u(s)\, ds\,, \tag{2.1}$$

for $t \in R_+$, then

$$u(t) \le n(t) \exp\Big(\int_0^t e(s)\, ds\Big)\,, \tag{2.2}$$

for $t \in R_+$.

Let $x_i(t)$ $(i = 1, 2)$ be functions which are continuous on R_+, differentiable in $0 < t < \infty$ and satisfy the inequalities

$$\big|x_i'(t) - f\big(t, x_i(t), x_i(t-1)\big)\big| \le \varepsilon_i\,, \tag{2.3}$$

for given constants $\varepsilon_i \ge 0$ $(i = 1, 2)$, where it is supposed that the initial conditions

$$x_i(t-1) = \phi_i(t) \quad (0 \le t < 1)\,,\ x_i(0) = x_i\,, \tag{2.4}$$

for $i = 1, 2$ are fulfilled and $\phi_i(t)$ are continuous functions for which $\lim\limits_{t \to 1-0} \phi_i(t)$ exist. Then we call $x_i(t)$ $(i = 1, 2)$ the ε_i-approximate solutions with respect to the equation (1.1) with the initial conditions (2.4).

The following theorem deals with the estimate on the difference between the two approximate solutions of equation (1.1) with (2.4).

Theorem 1. *Suppose that the function f in equation (1.1) satisfies the condition*

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \le h(t)[|x - \bar{x}| + |y - \bar{y}|]\,, \tag{2.5}$$

where $h \in C(R_+, R_+)$. Let $x_i(t)$ $(i = 1, 2)$ be respectively ε_i-approximate solutions of equation (1.1) on R_+ with (2.4) such that

$$|x_1 - x_2| \le \delta\,, \tag{2.6}$$

where $\delta \ge 0$ is a constant. Then

$$|x_1(t) - x_2(t)| \le c(t) \exp\Big(\int_0^t h(s)\, ds\Big)\,, \tag{2.7}$$

for $0 \le t < 1$ and

$$|x_1(t) - x_2(t)| \le c(t) \exp\Big(\int_0^t [h(s) + h(s+1)]\, ds\Big)\,, \tag{2.8}$$

for $1 \le t < \infty$, where

$$c(t) = \big((\varepsilon_1 + \varepsilon_2)t + \delta\big) + \int_0^1 h(s)|\phi_1(s) - \phi_2(s)|\, ds\,. \tag{2.9}$$

Proof. Since $x_i(t)$ $(i=1,2)$ for $t\in R_+$ are respectively ε_i-approximate solutions of equation (1.1) with (2.4), we have (2.3). By taking $t=s$ in (2.3) and integrating both sides over s from 0 to t, we have

$$\begin{aligned}
\varepsilon_i t &\geq \int_0^t \big|x_i'(s) - f\big(s, x_i(s), x_i(s-1)\big)\big|\, ds \\
&\geq \Big|\int_0^t \big\{x_i'(s) - f\big(s, x_i(s), x_i(s-1)\big)\big\}\, ds\Big| \\
&= \Big|\Big\{x_i(t) - x_i(0) - \int_0^t f\big(s, x_i(s), x_i(s-1)\big)\, ds\Big\}\Big|, \qquad (2.10)
\end{aligned}$$

for $i=1,2$. From (2.10) and using the elementary inequalities $|v-z|\leq|v|+|z|$, $|v|-|z|\leq|v-z|$, we observe that

$$\begin{aligned}
(\varepsilon_1+\varepsilon_2)t &\geq \Big|\Big\{x_1(t) - x_1(0) - \int_0^t f\big(s, x_1(s), x_1(s-1)\big)\, ds\Big\}\Big| \\
&\quad + \Big|\Big\{x_2(t) - x_2(0) - \int_0^t f\big(s, x_2(s), x_2(s-1)\big)\, ds\Big\}\Big| \\
&\geq \Big|\Big\{x_1(t) - x_1(0) - \int_0^t f\big(s, x_1(s), x_1(s-1)\big)\, ds\Big\} \\
&\quad - \Big\{x_2(t) - x_2(0) - \int_0^t f\big(s, x_2(s), x_2(s-1)\big)\, ds\Big\}\Big| \\
&\geq |x_1(t) - x_2(t)| - |x_1(0) - x_2(0)| \\
&\quad - \Big|\int_0^t f\big(s, x_1(s), x_1(s-1)\big)\, ds - \int_0^t f\big(s, x_2(s), x_2(s-1)\big)\, ds\Big|. \qquad (2.11)
\end{aligned}$$

Let $u(t)=|x_1(t)-x_2(t)|, t\in R_+$. From (2.11), we observe that

(2.12)
$$u(t) \leq (\varepsilon_1+\varepsilon_2)t + u(0) + \int_0^t \big|f\big(s, x_1(s), x_1(s-1)\big) - f\big(s, x_2(s), x_2(s-1)\big)\big|\, ds\,.$$

We consider the following two cases (see [6, 7]).

Case I: $0\leq t<1$. From (2.12) and hypotheses, we observe that

$$\begin{aligned}
u(t) &\leq \big((\varepsilon_1+\varepsilon_2)t + \delta\big) + \int_0^t \big|f\big(s, x_1(s), \phi_1(s)\big) - f\big(s, x_2(s), \phi_2(s)\big)\big|\, ds \\
&\leq \big((\varepsilon_1+\varepsilon_2)t + \delta\big) + \int_0^t h(s)\big[\big|x_1(s) - x_2(s)\big| + \big|\phi_1(s) - \phi_2(s)\big|\big]\, ds \\
&\leq c(t) + \int_0^t h(s)u(s)\, ds\,. \qquad (2.13)
\end{aligned}$$

Clearly $c(t)$ is nondecreasing in $t \in R_+$. Now an application of Lemma to (2.13) yields (2.7).

Case II: $1 \le t < \infty$. From (2.12) and hypotheses, we observe that

$$\begin{aligned} u(t) &\le \big((\varepsilon_1+\varepsilon_2)t+\delta\big) + \int_0^1 \big|f\big(s, x_1(s), \phi_1(s)\big) - f\big(s, x_2(s), \phi_2(s)\big)\big|\, ds \\ &\quad + \int_1^t \big|f\big(s, x_1(s), x_1(s-1)\big) - f\big(s, x_2(s), x_2(s-1)\big)\big|\, ds \\ &\le \big((\varepsilon_1+\varepsilon_2)t+\delta\big) + \int_0^1 h(s)\big[\big|x_1(s)-x_2(s)\big| + \big|\phi_1(s)-\phi_2(s)\big|\big]\, ds \\ &\quad + \int_1^t h(s)\big[\big|x_1(s)-x_2(s)\big| + \big|x_1(s-1)-x_2(s-1)\big|\big]\, ds \\ &= c(t) + \int_0^t h(s)u(s)\, ds + I_1\,, \end{aligned} \tag{2.14}$$

where

$$I_1 = \int_1^t h(s)\big|x_1(s-1)-x_2(s-1)\big|\, ds\,. \tag{2.15}$$

By making the change of variable, from (2.15), we obtain

$$I_1 \le \int_0^t h(s+1)u(s)\, ds\,. \tag{2.16}$$

Using (2.16) in (2.14), we get

$$u(t) \le c(t) + \int_0^t \big[h(s)+h(s+1)\big]u(s)\, ds\,. \tag{2.17}$$

Now an application of Lemma to (2.17) yields (2.8). □

Remark 1. In case if $x_1(t)$ is a solution of equation (1.1) with $x_1(t-1) = \phi_1(t)$ $(0 \le t < 1)$, $x_1(0) = x_1$, then we have $\varepsilon_1 = 0$ and from (2.7) and (2.8) we see that $x_2(t) \to x_1(t)$ as $\varepsilon_2 \to 0$, $\delta \to 0$ and $\phi_2(t) \to \phi_1(t)$ for $0 \le t < 1$. Furthermore, if we put $\varepsilon_1 = \varepsilon_2 = 0$, $\delta = 0$, $\phi_2(t) = \phi_1(t)$ $(0 \le t < 1)$ in (2.7) and (2.8), then the uniqueness of solutions of equation (1.1) with (1.2) is established.

As noted above, the estimate obtained in Theorem 1 by the approximation method is useful in studying the uniqueness of solutions of (1.1) with (1.2). In [9], it is pointed that if f in (1.1) is continuous, it does not always guarantee the uniqueness of solutions of (1.1)–(1.2). To illustrate this fact, we shall give the following example in [9].

Example 1. Consider the difference-differential equation

$$x'(t) = 2x(t-1)\sqrt{x(t)}\,, \tag{D}$$

for $t \in R_+$ under the initial conditions

$$x(t-1) = 1 \quad (0 \le t < 1)\,,\ \phi(1-0) = 1\,,\ x(0) = 0\,.$$

It is apparent that the function $y\sqrt{x}$ is continuous for $x \geq 0$, $y \geq 0$. Then, we define the following functions:

$$x_1(t) = \begin{cases} 1 & \text{for} \quad -1 \leq t < 1\,, \\ t^2 & \text{for} \quad 0 \leq t < 1\,, \end{cases} \qquad x_1(-0) = 1\,,$$

and

$$x_2(t) = \begin{cases} 1 & \text{for} \quad -1 \leq t < 1\,, \\ 0 & \text{for} \quad 0 \leq t < \infty\,, \end{cases} \qquad x_2(-0) = 1\,.$$

We can easily continue the function $x_1(t)$ so that it is continuous and satisfies (D) on $0 \leq t < \infty$. Hence, we obtain two solutions with the same initial conditions.

The continuity of derivatives of solutions can not always be guaranteed at $t = 1$. It will easily be expected that discontinuity of derivatives at $t = 1$ is caused by the discontinuity of the initial conditions, that is $\phi(-0) \neq x(0)$. The next example shows that the solutions and their derivatives are continuous for $0 \leq t < \infty$ (see [9]).

Example 2. Consider the difference-differential equation

$$x'(t) = 2\big[x(t-1) + 1\big]\sqrt{x(t)}\,,$$

for $t \in R_+$, under the initial conditions

$$x(t-1) = 0 \quad (-1 \leq t < 0)\,,\ x(-0) = 0\,,\ x(0) = 0\,.$$

Next, we consider the equation (1.1) with (1.2) together with the following difference- differential equation

$$y'(t) = \bar{f}\big(t, y(t), y(t-1)\big)\,, \tag{2.18}$$

for $t \in R_+$ under the initial conditions

$$y(t-1) = \psi(t) \quad (0 \leq t < 1)\,,\ y(0) = y_0\,, \tag{2.19}$$

where $\bar{f} \in C(R_+ \times R^n \times R^n, R^n)$ and $\psi(t)$ is a continuous function for which $\lim\limits_{t\to 1-0} \psi(t)$ exists.

The next theorem deals with the closeness of the solutions of equations (1.1)–(1.2) and (2.18)–(2.19).

Theorem 2. *Suppose that the function f in equation* (1.1) *satisfies the condition* (2.5) *and there exist constants $\bar{\varepsilon} \geq 0$, $\bar{\delta} \geq 0$ such that*

$$\big|f(t,x,y) - \bar{f}(t,x,y)\big| \leq \bar{\varepsilon}\,, \tag{2.20}$$

$$|x_0 - y_0| \leq \bar{\delta}\,, \tag{2.21}$$

where f, x_0 and $\bar{f}$, y_0 are as in equations (1.1)–(1.2) *and* (2.18)–(2.19). *Let $x(t)$ and $y(t)$ be respectively, solutions of equations* (1.1)–(1.2) *and* (2.18)–(2.19) *on R_+. Then*

$$\big|x(t) - y(t)\big| \leq d(t) \exp\Big(\int_0^t h(s)\, ds\Big)\,, \tag{2.22}$$

for $0 \le t < 1$ *and*

$$|x(t) - y(t)| \le d(t) \exp\Big(\int_0^t \big[h(s) + h(s+1)\big]\, ds\Big), \tag{2.23}$$

for $1 \le t < \infty$*, where*

$$d(t) = \bar{\varepsilon} t + \bar{\delta} + \int_0^1 h(s)\big|\phi(s) - \psi(s)\big|\, ds\,. \tag{2.24}$$

Proof. Let $v(t) = |x(t) - y(t)|$, $t \in R_+$. Using the facts that $x(t)$ and $y(t)$ are the solutions of equations (1.1)–(1.2) and (2.18)–(2.19), we observe the following two cases.

Case I: $0 \le t < 1$. From the hypotheses, we have

$$\begin{aligned} v(t) &\le |x_0 - y_0| + \int_0^t \big| f\big(s, x(s), \phi(s)\big) - f\big(s, y(s), \psi(s)\big)\big|\, ds \\ &\quad + \int_0^t \big| f\big(s, y(s), \psi(s)\big) - \bar{f}\big(s, y(s), \psi(s)\big)\big|\, ds \\ &\le \bar{\delta} + \bar{\varepsilon} t + \int_0^t h(s)\big[\big|x(s) - y(s)\big| + \big|\phi(s) - \psi(s)\big|\big]\, ds \\ &\le d(t) + \int_0^t h(s) v(s)\, ds\,. \end{aligned} \tag{2.25}$$

Clearly $d(t)$ is nondecreasing in $t \in R_+$. Now an application of Lemma to (2.25) yields (2.22).

Case II: $1 \le t < \infty$. From the hypotheses, we have

$$\begin{aligned} v(t) &\le |x_0 - y_0| + \int_0^1 \big| f\big(s, x(s), \phi(s)\big) - f\big(s, y(s), \psi(s)\big)\big|\, ds \\ &\quad + \int_0^1 \big| f\big(s, y(s), \psi(s)\big) - \bar{f}\big(s, y(s), \psi(s)\big)\big|\, ds \\ &\quad + \int_1^t \big| f\big(s, x(s), x(s-1)\big) - f\big(s, y(s), y(s-1)\big)\big|\, ds \\ &\quad + \int_1^t \big| f\big(s, y(s), y(s-1)\big) - \bar{f}\big(s, y(s), y(s-1)\big)\big|\, ds \\ &\le \bar{\delta} + \bar{\varepsilon} t + \int_0^1 h(s)\big|\phi(s) - \psi(s)\big|\, ds + \int_0^t h(s)\big|x(s) - y(s)\big|\, ds + I_2\,, \end{aligned} \tag{2.26}$$

where

$$I_2 = \int_1^t h(s)\big|x(s-1) - y(s-1)\big|\, ds\,. \tag{2.27}$$

From (2.26) and (2.27), it is easy to observe that (see [6, 7])

$$v(t) \le d(t) + \int_0^t \big[h(s) + h(s+1)\big]\, v(s)\, ds\,. \tag{2.28}$$

Now an application of Lemma to (2.28) yields (2.23). □

A slight variant of Theorem 2 is embodied in the following theorem.

Theorem 3. *Suppose that the functions f and $\bar f$ in equations* (1.1) *and* (2.18) *satisfies the condition*

$$\big|f(t,x,y) - \bar f(t,\bar x,\bar y)\big| \le p(t)\big[|x-\bar x| + |y-\bar y|\big]\,, \tag{2.29}$$

where $p \in C(R_+, R_+)$ and the condition (2.21) *holds. Let $x(t)$ and $y(t)$ be respectively, solutions of equations* (1.1)–(1.2) *and* (2.18)–(2.19) *on R_+. Then*

$$\big|x(t) - y(t)\big| \le d_0 \exp\Big(\int_0^t p(s)\, ds\Big)\,, \tag{2.30}$$

for $0 \le t < 1$ and

$$\big|x(t) - y(t)\big| \le d_0 \exp\Big(\int_0^t \big[p(s) + p(s+1)\big]\, ds\Big)\,, \tag{2.31}$$

for $1 \le t < \infty$, where

$$d_0 = \bar\delta + \int_0^1 p(s)\big|\phi(s) - \psi(s)\big|\, ds\,. \tag{2.32}$$

Proof. Let $w(t) = |x(t) - y(t)|$, $t \in R_+$. Using the facts that $x(t)$ and $y(t)$ are respectively, solutions of equations (1.1)–(1.2) and (2.18)–(2.19), we have

$$w(t) \le |x_0 - y_0| + \int_0^t \big|f\big(s, x(s), x(s-1)\big) - \bar f\big(s, y(s), y(s-1)\big)\big|\, ds\,. \tag{2.33}$$

The rest of the proof can be complited by closely looking at the proofs of Theorems 1 and 2 given above with suitable modifications. We omit the details. □

Remark 2. We note that the result given in Theorem 2 relates the solutions of equations (1.1)–(1.2) and (2.18)–(2.19) in the sense that if f is close to $\bar f$, x_0 is close to y_0 and $\phi(t)$ is close to $\psi(t)$ for $0 \le t < 1$, then the solutions of equations (1.1)–(1.2) and (2.18)–(2.19) are also close together. The result obtained in Theorem 3 provide conditions that yields the estimate on the difference between the solutions of equations (1.1)-(1.2) and (2.18)–(2.19).

3. Applications

In this section we use the idea of approximation of solutions to study the Volterra type difference-integral equation of the form

$$y(t) = g(t) + \int_0^t F\big(t, s, y(s), y(s-1)\big)\, ds\,, \tag{3.1}$$

for $t \in R_+$ with the given condition

$$y(t-1) = \phi(t) \quad (0 \le t < 1)\,, \tag{3.2}$$

where $g \in C(R_+, R^n)$, for $s \le t$, $F \in C(R_+^2 \times R^n \times R^n, R^n)$ and $\phi(t)$ is as in (1.2). The special version of equation (3.1) with (3.2) occur in a natural way while studying the perturbed difference-differential equations (see [1, 7, 8]). Here, we note that the problem of existence and uniqueness for the solutions of equation (3.1) with (3.2) can be dealt with, by modifying the ideas employed in [6, 7]. Below, we formulate in brief the results similar to those of given in Theorems 1–3 related to the equation (3.1) with (3.2).

We call a function $y \in C(R_+, R^n)$ an ε-approximate solution of equation (3.1) with (3.2), if there exists a constant $\varepsilon \ge 0$ such that

$$\Big| y(t) - g(t) - \int_0^t F\big(t, s, y(s), y(s-1)\big)\, ds \Big| \le \varepsilon\,, \tag{3.3}$$

for $t \in R_+$.

The following result deals with the estimate on the difference between the two approximate solutions of equation (3.1) with given initial conditions.

Theorem 4. *Suppose that the function F in equation (3.1) satisfies the condition*

$$\big|F(t, s, u, v) - F(t, s, \bar{u}, \bar{v})\big| \le Lq(s)\big[|u - \bar{u}| + |v - \bar{v}|\big]\,, \tag{3.4}$$

where $L \ge 0$ is a constant and $q \in C(R_+, R_+)$. For $i = 1, 2$ let $y_i(t)$ be respectively, ε_i-approximate solutions of equation (3.1) with

$$y_i(t-1) = \phi_i(t) \quad (0 \le t < 1)\,, \tag{3.5}$$

on R_+, where $\phi_i(t)$ be as in (2.4). Then

$$\big|y_1(t) - y_2(t)\big| \le \alpha \exp\Big(\int_0^t Lq(s)\, ds\Big)\,, \tag{3.6}$$

for $0 \le t < 1$ and

$$\big|y_1(t) - y_2(t)\big| \le \alpha \exp\Big(\int_0^t L\big[q(s) + q(s+1)\big]\, ds\Big)\,, \tag{3.7}$$

for $1 \le t < \infty$, where

$$\alpha = \varepsilon_1 + \varepsilon_2 + \int_0^1 Lq(s)\big|\phi_1(s) - \phi_2(s)\big|\, ds\,. \tag{3.8}$$

Proof. Since $y_i(t)$ $(i = 1, 2)$ are respectively ε_i-approximate solutions of equation (3.1) with (3.5), we have

$$\Big| y_i(t) - g(t) - \int_0^t F\big(t, s, y_i(s), y_i(s-1)\big)\, ds \Big| \le \varepsilon_i\,. \tag{3.9}$$

From (3.9) and using the elementary inequalities $|v-z| \le |v|+|z|$, $|v|-|z| \le |v-z|$, we observe that

$$
\begin{aligned}
\varepsilon_1+\varepsilon_2 &\ge \Big|y_1(t)-g(t)-\int_0^t F\big(t,s,y_1(s),y_1(s-1)\big)\,ds\Big| \\
&\quad + \Big|y_2(t)-g(t)-\int_0^t F\big(t,s,y_2(s),y_2(s-1)\big)\,ds\Big| \\
&\ge \Big|\Big\{y_1(t)-g(t)-\int_0^t F\big(t,s,y_1(s),y_1(s-1)\big)\,ds\Big\} \\
&\quad - \Big\{y_2(t)-g(t)-\int_0^t F\big(t,s,y_2(s),y_2(s-1)\big)\,ds\Big\}\Big| \\
&\ge \big|y_1(t)-y_2(t)\big| - \Big|\int_0^t F\big(t,s,y_1(s),y_1(s-1)\big)\,ds \\
&\quad - \int_0^t F\big(t,s,y_2(s),y_2(s-1)\big)\,ds\Big|\,.
\end{aligned}
\tag{3.10}
$$

The rest of the proof can be completed by following the proof of Theorem 1 with suitable modifications. We omit the details. ☐

Consider the equation (3.1) with (3.2) together with the corresponding Volterra type difference-integral equation

$$
z(t)=\bar g(t)+\int_0^t \bar F\big(t,s,z(s),z(s-1)\big)\,ds\,, \tag{3.11}
$$

for $t\in R_+$, with the given condition

$$
z(t-1)=\psi(t) \quad (0\le t<1)\,, \tag{3.12}
$$

where $\bar g\in C(R_+,R^n)$, for $s\le t$, $\bar F\in C(R_+^2\times R^n\times R^n,R^n)$ and $\psi(t)$ is as in (2.19).

The following theorems that relates the solutions of equations (3.1)–(3.2) and (3.11)–(3.12) holds.

Theorem 5. *Suppose that the function F in equation* (3.1) *satisfies the condition* (3.4) *and there exist constants $\varepsilon\ge 0$, $\delta\ge 0$ such that*

$$
\big|F(t,s,u,v)-\bar F(t,s,u,v)\big|\le\varepsilon\,, \tag{3.13}
$$

$$
\big|g(t)-\bar g(t)\big|\le\delta\,, \tag{3.14}
$$

where g, F and $\bar g$, $\bar F$ are as in equations (3.1) *and* (3.11). *Let $x(t)$ and $y(t)$ be respectively, solutions of equations* (3.1)–(3.2) *and* (3.11)–(3.12) *on R_+. Then*

$$
\big|x(t)-y(t)\big|\le m(t)\exp\Big(\int_0^t Lq(s)\,ds\Big)\,, \tag{3.15}
$$

for $0 \le t < 1$ *and*

$$\big|x(t) - y(t)\big| \le m(t) \exp\Big(\int_0^t L\big[q(s) + q(s+1)\big]\, ds\Big)\,, \tag{3.16}$$

for $1 \le t < \infty$, *where*

$$m(t) = \varepsilon t + \delta + \int_0^1 Lq(s)\big|\phi(s) - \psi(s)\big|\, ds\,. \tag{3.17}$$

Theorem 6. *Suppose that* F *and* $\bar{F}$ *in equations* (3.1) *and* (3.11) *satisfies the condition*

$$\big|F(t,s,u,v) - \bar{F}(t,s,\bar{u},\bar{v})\big| \le Mr(s)\big[|u-\bar{u}| + |v-\bar{v}|\big]\,, \tag{3.18}$$

where $M \ge 0$ *is a constant and* $r \in C(R_+, R_+)$ *and the condition* (3.14) *holds. Let* $x(t)$ *and* $y(t)$ *be respectively, solutions of equations* (3.1)–(3.2) *and* (3.11)–(3.12) *on* R_+. *Then*

$$\big|x(t) - y(t)\big| \le \beta \exp\Big(\int_0^t Mr(s)\, ds\Big)\,, \tag{3.19}$$

for $0 \le t < 1$ *and*

$$\big|x(t) - y(t)\big| \le \beta \exp\Big(\int_0^t M\big[r(s) + r(s+1)\big] ds\Big)\,, \tag{3.20}$$

for $1 \le t < \infty$, *where*

$$\beta = \delta + \int_0^1 Mr(s)\big|\phi(s) - \psi(s)\big|\, ds\,. \tag{3.21}$$

The proofs of Theorems 5, 6 are straightforward in view of the results given above. Here, we omit the details.

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