ARCHIVUM MATHEMATICUM (BRNO)
Tomus 46 (2010), 79–86

SPACELIKE SUBMANIFOLDS IN INDEFINITE SPACE FORM $M_p^{n+p}(c)$

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Abstract. In this paper, we get an intrinsic inequality for spacelike submanifolds in indefinite space form $M_p^{n+p}(c)$, $(c > 0)$. We also get some rigidity theorems for such spacelike submanifolds.

1. Introduction

Let $M_p^{n+p}(c)$ be $n+p$-dimensional connected semi-Riemannian manifold of constant curvature c whose index is p. It is called indefinite space form of index p. Let M be an n-dimensional Riemannian manifold immersed in $M_p^{n+p}(c)$. The semi-Riemannian metric of $M_p^{n+p}(c)$ induces the Riemannian metric of M, M is called a spacelike submanifold. Spacelike submanifolds in indefinite space form $M_p^{n+p}(c)$ have been of increasing interesting in the recent years. There are many results about these submanifolds, for instance, Dong [3], Wu [6, 7], Liu[4]. In [5], the authors got an intrinsic inequality for spacelike hypersurfaces in de Sitter space form M_1^{n+1} whose index is 1. In this note, we generalize the intrinsic inequality for spacelike hypersurface of de Sitter space to spacelike submanifolds of indefinite space form $M_p^{n+p}(c)$ with index $p \geq 1$. From this inequality, we also get some rigidity theorems for such spacelike submanifolds.

2. Preliminaries

We choose a local field of semi-Riemannian orthonormal frames $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p}\}$ in $M_p^{n+p}(c)$ such that, restricted to M^n, $e_1, \dots, e_n$ are tangent to M^n. Let $\omega_1, \dots, \omega_n$ be its dual frame field such that the semi-Riemannian metric of $M_p^{n+p}(c)$ is given by $ds^2 = \sum_{A=1}^{n+p} \epsilon_A (\omega_A)^2$, where $\epsilon_i = 1$, $i = 1, \dots, n$ and $\epsilon_\alpha = -1$,

2000 *Mathematics Subject Classification*: primary 53C42; secondary 53B30.
Key words and phrases: spacelike submanifolds, indefinite space form.
Supported by the National Natural Science Foundation of China No. 10971029.
Received August 11, 2009, revised October 2009. Editor O. Kowalski.

$\alpha = n+1, \ldots, n+p$. Then the structure equations of $M_p^{n+p}(c)$ are given by

$$d\omega_A = -\sum_B \epsilon_B \omega_{AB} \wedge \omega_B\,, \quad \omega_{AB} + \omega_{BA} = 0\,, \tag{1}$$

$$d\omega_{AB} = -\sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{CD} K_{ABCD}\omega_C \wedge \omega_D\,, \tag{2}$$

$$K_{ABCD} = c\epsilon_A\epsilon_B(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})\,. \tag{3}$$

We restrict these forms to M^n, then

$$\omega_\alpha = 0\,, \quad \alpha = n+1, \ldots, n+p\,, \tag{4}$$

and the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. Since

$$0 = d\omega_\alpha = -\sum_i \omega_{\alpha,i} \wedge \omega_i\,, \tag{5}$$

by Cartan's lemma we may write

$$\omega_{\alpha,i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{6}$$

From these formulas, we obtain the structure equations of M^n:

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j\,, \quad \omega_{ij} + \omega_{ji} = 0\,, \tag{7}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l\,, \tag{8}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha)\,, \tag{9}$$

where R_{ijkl} are the components of curvature tensor of M^n. We call

$$h = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha \tag{10}$$

the second fundamental form of M^n. The mean curvature vector is $H = \sum_{i,\alpha} h_{ii}^\alpha e_\alpha = \sum_\alpha H^\alpha e_\alpha$, where $H^\alpha = \sum_i h_{ii}^\alpha$. We denote $|h|^2 = \sum_{i,j,\alpha}(h_{ij}^\alpha)^2$, and $|H|^2 = \sum_\alpha (H^\alpha)^2$. We call that M^n is maximal if its mean curvature field vanishes, i.e. $H = 0$.

Let $h_{ij,k}^\alpha$ and $h_{ij,kl}^\alpha$ denote the covariant derivative and the second covariant derivative of h_{ij}^α. Then we have $h_{ij,k}^\alpha = h_{ik,j}^\alpha$ and

$$h_{ij,kl}^\alpha - h_{ij,lk}^\alpha = \sum_m h_{im}^\alpha R_{mjkl} + \sum_m h_{mj}^\alpha R_{mikl} + \sum_m h_{ij}^\beta R_{\alpha\beta kl}\,,$$

where $R_{\alpha\beta kl}$ are the components of the normal curvature tensor of M^n, that is

$$R_{\alpha\beta kl} = \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{ik}^\beta h_{il}^\alpha)\,.$$

If $R_{\alpha\beta kl} = 0$ at point x of M^n we say that the normal bundle connection of M^n is flat at x and it is well known [1] that $R_{\alpha\beta kl} = 0$ at point x if and only if the matrix (h_{ij}^α) are simultaneously diagonalizable at x.

3. Main results for space-like submanifolds

Lemma 3.1 (Cauchy-Swartz inequality)**.** *Let $a_1, \dots, a_n; b_1, \dots, b_n$ be real numbers, then*

$$\Big(\sum_i a_i b_i\Big)^2 \leq \Big(\sum_i a_i^2\Big)\Big(\sum_i b_i^2\Big)$$

and the equality holds if and only if there exists a constant λ such that $a_i = \lambda b_i$ or $b_i = \lambda a_i$, $i = 1, \dots, n$.

Theorem 3.2. *If M^n is a space-like submanifold of indefinite space form $M_p^{n+p}(c)$ $(c > 0)$, S and ρ are Ricci curvature tensor and the scalar curvature of M^n, respectively, then*

$$|S|^2 \geq 2c\rho(n-1) - c^2 n(n-1)^2 \,. \tag{11}$$

Moreover, $|S|^2 = 2c\rho(n-1) - c^2n(n-1)^2$ if and only if M^n is a spacelike Einstein submanifolds with $S = c(n-1)g$, where g is the Riemannian metric of M^n.

Proof. From the Gauss equation we get

$$\begin{aligned} S_{ij} &= \sum_k R_{kikj} = \sum_k \Big\{c(\delta_{kk}\delta_{ij} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{kk}^\alpha h_{ij}^\alpha - h_{ik}^\alpha h_{jk}^\alpha)\Big\} \\ &= c(n-1)\delta_{ij} - \sum_\alpha H^\alpha h_{ij}^\alpha + \sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \end{aligned} \tag{12}$$

So

$$\begin{aligned} |S|^2 &= \sum_{ij} S_{ij}^2 = \sum_{ij} \Big\{c(n-1)\delta_{ij} - \sum_\alpha H^\alpha h_{ij}^\alpha + \sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\Big\}^2 \\ &= \sum_{ij} \Big\{c^2(n-1)^2\delta_{ij} + \Big(\sum_\alpha H^\alpha h_{ij}^\alpha\Big)^2 + \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\Big)^2 \\ &\quad - 2c(n-1)\delta_{ij}\Big(\sum_\alpha H^\alpha h_{ij}^\alpha\Big) + 2c(n-1)\delta_{ij}\Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\Big) \\ &\quad - 2\Big(\sum_\alpha H^\alpha h_{ij}^\alpha\Big)\Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\Big)\Big\} \\ &= c^2 n(n-1)^2 + \sum_{ij}\Big(\sum_\alpha H^\alpha h_{ij}^\alpha\Big)^2 + \sum_{ij}\Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\Big)^2 \\ &\quad - 2c(n-1)|H|^2 + 2c(n-1)\Big(\sum_{i,k,\alpha} h_{ik}^\alpha h_{ik}^\alpha\Big) \\ &\quad - 2\sum_{i,j}\Big(\sum_\alpha H^\alpha h_{ij}^\alpha\Big)\Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\Big) \end{aligned}$$

and

$$\rho=\sum_i S_{ii}=\sum_i\Big\{c(n-1)-\sum_\alpha H^\alpha h^\alpha_{ii}+\sum_{k,\alpha}h^\alpha_{ik}h^\alpha_{ik}\Big\}$$
$$=cn(n-1)-|H|^2+\sum_{ij\alpha}(h^\alpha_{ij})^2$$
$$=cn(n-1)-|H|^2+|h|^2\,, \tag{13}$$

So

$$\begin{aligned}|S|^2&=c^2n(n-1)^2+\sum_{ij}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2+\sum_{ij}\Big(\sum_{k,\alpha}h^\alpha_{ik}h^\alpha_{jk}\Big)^2\\&\quad-2c(n-1)|H|^2+2c(n-1)\big(\rho+|H|^2-cn(n-1)\big)\\&\quad-2\sum_{i,j}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)\Big(\sum_{k,\alpha}h^\alpha_{ik}h^\alpha_{jk}\Big)\\&=2c\rho(n-1)-c^2n(n-1)^2+\sum_{ij}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2+\sum_{ij}\Big(\sum_{k,\alpha}h^\alpha_{ik}h^\alpha_{jk}\Big)^2\\&\quad-2\sum_{i,j}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)\Big(\sum_{k,\alpha}h^\alpha_{ik}h^\alpha_{jk}\Big)\\&\geq 2c\rho(n-1)-c^2n(n-1)^2+\sum_{ij}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2+\sum_{ij}\Big(\sum_{k,\alpha}h^\alpha_{ik}h^\alpha_{jk}\Big)^2\\&\quad-2\Big(\sum_{ij}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2\Big)^{1/2}\Big(\sum_{ij}\Big(\sum_{k,\alpha}h^\alpha_{ik}h^\alpha_{jk}\Big)^2\Big)^{1/2}\\&=2c\rho(n-1)-c^2n(n-1)^2+\Big\{\Big(\sum_{ij}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2\Big)^{1/2}\\&\quad-\Big(\sum_{ij}\Big(\sum_{k,\alpha}h^\alpha_{ik}h^\alpha_{jk}\Big)^2\Big)^{1/2}\Big\}^2\geq 2c\rho(n-1)-c^2n(n-1)^2\,.\end{aligned}\tag{14}$$

The first inequality has used Lemma 3.1.

So we have

$$|S|^2\geq 2c\rho(n-1)-c^2n(n-1)^2\,.$$

Now we will prove the second part of this theorem.

If M^n is a spacelike Einstein submanifold with $S=c(n-1)g$, then we have the following equations:

$$|S|^2=c^2n(n-1)^2\,,\quad\text{and}\quad\rho=cn(n-1)\,,$$

i.e.

$$|S|^2=2c\rho(n-1)-c^2n(n-1)^2\,.$$

Conversely, if the Eq. (14) becomes an equality, then all the inequality of Eq. (14) will become equality. From the Lemma 3.1, there exist a constant λ such that

$$\sum_{\alpha} H^\alpha h_{ij}^\alpha = \lambda \sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha$$

or

$$\lambda \sum_{\alpha} H^\alpha h_{ij}^\alpha = \sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \quad \text{for all} \quad i,j \in \{1,\dots,n\} \tag{15}$$

and

$$\sum_{ij} \Big(\sum_{\alpha} H^\alpha h_{ij}^\alpha \Big)^2 = \sum_{ij} \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \Big)^2 . \tag{16}$$

(I) If $\lambda = 0$, we know that

$$\sum_{\alpha} H^\alpha h_{ij}^\alpha = 0 \quad \text{or} \quad \sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha = 0 \quad \text{for all} \quad i,j \in \{1,\dots,n\} . \tag{17}$$

then

$$H = 0 \quad \text{or} \quad \sum_{i,k,\alpha} [h_{ik}^\alpha]^2 = 0 \tag{18}$$

If $H = 0$, then M^n is maximal. From the Eq. (14), we have the following equations:

$$\begin{aligned} |S|^2 &= 2c\rho(n-1) - c^2 n(n-1)^2 + \sum_{ij} \Big(\sum_{\alpha} H^\alpha h_{ij}^\alpha \Big)^2 + \sum_{ij} \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \Big)^2 \\ &\quad - 2 \Big(\sum_{ij} \Big(\sum_{\alpha} H^\alpha h_{ij}^\alpha \Big)^2 \Big)^{1/2} \Big(\sum_{ij} \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \Big)^2 \Big)^{1/2} \\ &= 2c\rho(n-1) - c^2 n(n-1)^2 + \sum_{ij} \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \Big)^2 . \end{aligned} \tag{19}$$

We have that

$$\sum_{ij} \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \Big)^2 = 0 , \tag{20}$$

for $i,j \in \{1,\dots,n\}$. From this equation, we get

$$\sum_{k,\alpha} h_{ik}^\alpha h_{ik}^\alpha = 0 \quad \text{for} \quad i = 1,\dots,n . \tag{21}$$

So $h_{ij}^\alpha = 0$, for $i,j \in \{1,\dots,n\}$ and $\alpha \in \{n+1,\dots,n+p\}$, i.e. M^n is totally geodesic.

If $\sum_{i,k,\alpha} [h_{ik}^\alpha]^2 = 0$,so $h_{ij}^\alpha = 0$, for $i,j \in \{1,\dots,n\}$ and $\alpha \in \{n+1,\dots,n+p\}$, i.e. M^n is totally geodesic.

From the Eq. (12), we know that

$$S_{ij} = c(n-1)\delta_{ij} . \tag{22}$$

(II) If $\lambda \neq 0$, from the equation $\sum_\alpha H^\alpha h_{ij}^\alpha = \lambda \sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha$, and equation (16), we have the following equation:

$$(\lambda^2 - 1)\Big[\sum_{ij}\Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\Big)^2\Big] = 0\,, \tag{23}$$

then $\sum_{ij}\big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\big)^2 = 0$ or $\lambda^2 = 1$.

If $\sum_{ij}\big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\big)^2 = 0$, then $\big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\big)^2 = 0$ for all i, j. So $h_{ij}^\alpha = 0$, for $i, j \in \{1, \dots, n\}$ and $\alpha \in \{n+1, \dots, n+p\}$, i.e. M^n is totally geodesic.

If $\lambda^2 = 1$, then $\lambda = 1$ or $\lambda = -1$. If $\lambda = -1$, then $\sum_\alpha H^\alpha h_{ij}^\alpha = -\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha$, so we have that $H^2 + |h|^2 = 0$, i.e. $h = 0$. If $\lambda = 1$, then $\sum_\alpha H^\alpha h_{ij}^\alpha = \sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha$.

From equation (12), we have the following equation:

$$S_{ij} = c(n-1)\delta_{ij}\,. \tag{24}$$

□

Remark 3.3. When $p = 1$, i.e. M^n is a space-like hypersurface, the inequality given in [5].

Corollary 3.4. *If M^n is a maximal space-like submanifold of indefinite space form $M_p^{n+p}(c)(c > 0)$, S and ρ are Ricci curvature tensor and the scalar curvature of M^n, respectively, then*

$$|S|^2 = 2c\rho(n-1) - c^2 n(n-1)^2 \tag{25}$$

if and only if M^n is totally geodesic.

Proof. If M^n is totally geodesic, then from equations (12) and (13),

$$|S|^2 = c^2 n(n-1)^2\,, \quad \text{and} \quad \rho = cn(n-1)\,,$$

i.e.

$$|S|^2 = 2c\rho(n-1) - c^2 n(n-1)^2\,.$$

Conversely, from equations $H = 0$, (19), (20) and (21), we know that M^n is totally geodesic. □

Theorem 3.5. *If M^n is a complete spacelike submanifold with flat normal bundle and with positive sectional curvature immersed in indefinite space form $M_p^{n+p}(c)$, ($c > 0$, $p \geq 2$, $n \geq 2$), S and ρ are Ricci curvature tensor and the scalar curvature of M^n, respectively, then*

$$|S|^2 = 2c\rho(n-1) - c^2 n(n-1)^2 \tag{26}$$

if and only if M^n is totally geodesic.

Proof. If M^n is totally geodesic, then from equations (12) and (13),

$$|S|^2 = c^2 n(n-1)^2\,, \quad \text{and} \quad \rho = cn(n-1)\,,$$

i.e.

$$|S|^2 = 2c\rho(n-1) - c^2 n(n-1)^2\,.$$

Conversely, from case (I) and case (II) in the proof of Theorem 3.2, we will prove that M^n must be geodesic under the conditions: $\lambda = 1$ and

$$\sum_{\alpha} H^\alpha h_{ij}^\alpha = \sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \,, \tag{27}$$

for $i, j \in \{1, \ldots, n\}$.

If $H = 0$, from Corollary 3.4, we know that M^n is totally geodesic. Now we suppose $H \neq 0$, and choose $e_{n+1} = \frac{H}{|H|}$. Then, it follows that

$$H = \sum_i h_{ii}^{n+1} e_{n+1} \,, \quad \text{and} \quad H^\alpha = \sum_i h_{ii}^\alpha = 0 \,, \quad \alpha > n+1 \,. \tag{28}$$

Since the normal bundle of M^n is flat, we choose $e_1, \cdots, e_n$ such that

$$h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}, \quad \text{for} \quad \alpha = n+1, \ldots, n+p \,.$$

From equation (27), we have the following equations:

$$|H|^2 = |H^{n+1}|^2 = |h|^2 \,. \tag{29}$$

Taking the covariant derivative of (29), we obtain

$$H^{n+1} H_k^{n+1} = \sum_{ij\alpha} h_{ij}^\alpha h_{ij,k}^\alpha \tag{30}$$

and by Lemma 3.1, we have

$$|H|^2 |\nabla H|^2 \leq |h|^2 |\nabla h|^2 \,. \tag{31}$$

Then the Laplacian of $|h|^2$ is given by:

$$\begin{aligned} \frac{1}{2}\triangle |h|^2 = \frac{1}{2}\triangle |H|^2 &= |\nabla h|^2 + \sum_{ij\alpha} h_{ij}^\alpha \triangle h_{ij}^\alpha \\ &= |\nabla h|^2 + \sum_i \lambda^{n+1}(H^{n+1}) + \frac{1}{2} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2 \end{aligned} \tag{32}$$

We define an operator $\Box$ acting on any function f by:

$$\Box f = \sum_{ij} (H^{n+1}\delta_{ij} - h_{ij}^{n+1}) f_{,ij}$$

Since $(H^{n+1}\delta_{ij} - h_{ij}^{n+1})$ is trace free, it follows from [2] that $\Box$ is self-adjoint relative to L^2-inner product of M^n, i.e.,

$$\int_{M^n} f \Box g = \int_{M^n} g \Box f \,.$$

Thus we have

$$\begin{aligned} \Box H^{n+1} &= \sum_{ij} (H^{n+1}\delta_{ij} - h_{ij}^{n+1}) H_{ij}^{n+1} \\ &= \frac{1}{2}\triangle |H|^2 - |\nabla H|^2 - \sum_i \lambda^{n+1}(H^{n+1}) \end{aligned} \tag{33}$$

From equations (30),(31),(32),(33),

$$\Box H^{n+1} \geq \frac{1}{2} R_{ijij} (\lambda_i^\alpha - \lambda_j^\alpha)^2 \,. \tag{34}$$

Because $S_{ij} = c(n-1)\delta_{ij}$, we see by the Bonnet-Myers theorem that M^n is bounded and hence compact.

Since $\Box$ is self-adjoint, we have

$$0 \geq \int_{M^n} \frac{1}{2} R_{ijij} (\lambda_i^\alpha - \lambda_j^\alpha)^2 \,. \tag{35}$$

Then, by hypothesis $R_{ijij} > 0$, so $\lambda_i^\alpha = \lambda_j^\alpha$ for $\alpha \in \{n+1, \dots, n+p\}$ and $i, j \in \{1, \dots, n\}$.

From equation (27), we have

$$(n-1)(\lambda_1^{n+1})^2 = (\lambda_1^{n+2})^2 + \cdots + (\lambda_1^{n+p})^2. \tag{36}$$

From equation (28), we have

$$n\lambda_1^{n+2} = \cdots = n\lambda_1^{n+p} = 0 \,, \tag{37}$$

then we have

$$(n-1)(\lambda_1^{n+1})^2 = 0, \tag{38}$$

so $\lambda_1^{n+1} = \lambda_1^{n+2} = \cdots = \lambda_1^{n+p} = 0$, i.e. M^n is a totally geodesic submanifold. $\Box$

Acknowledgement. The author would like to thank Professors Y. X. Dong and X. L. Chao for their valuable suggestions and useful discussions. Also, he would like to thank the referee whose valuable suggestions make this paper more perfect.

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