ARCHIVUM MATHEMATICUM (BRNO)
Tomus 46 (2010), 39–46

TIMELIKE B_2-SLANT HELICES IN MINKOWSKI SPACE $\mathbf{E}_1^4$

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Abstract. We consider a unit speed timelike curve α in Minkowski 4-space $\mathbf{E}_1^4$ and denote the Frenet frame of α by $\{\mathbf{T},\mathbf{N},\mathbf{B}_1,\mathbf{B}_2\}$. We say that α is a generalized helix if one of the unit vector fields of the Frenet frame has constant scalar product with a fixed direction U of $\mathbf{E}_1^4$. In this work we study those helices where the function $\langle \mathbf{B}_2, U\rangle$ is constant and we give different characterizations of such curves.

1. Introduction and statement of results

A helix in Euclidean 3-space $\mathbf{E}^3$ is a curve where the tangent lines make a constant angle with a fixed direction. A helix curve is characterized by the fact that the ratio τ/κ is constant along the curve, where τ and κ denote the torsion and the curvature, respectively. Helices are well known curves in classical differential geometry of space curves [8] and we refer to the reader for recent works on this type of curves [4, 12]. Recently, Izumiya and Takeuchi have introduced the concept of slant helix by saying that the normal lines make a constant angle with a fixed direction [5]. They characterize a slant helix if and only if the function

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\Big(\frac{\tau}{\kappa}\Big)' \tag{1}$$

is constant. The article [5] motivated generalizations in a twofold sense: first, by considering arbitrary dimension of Euclidean space [7, 9]; second, by considering analogous problems in other ambient spaces, for example, in Minkowski space $\mathbf{E}_1^n$ [1, 3, 6, 11, 13].

In this work we consider the generalization of the concept of helix in Minkowski 4-space, when the helix is a timelike curve. We denote by $\mathbf{E}_1^4$ the Minkowski 4-space, that is, $\mathbf{E}_1^4$ is the real vector space $\mathbb{R}^4$ endowed with the standard Lorentzian metric

$$\langle\ ,\ \rangle = -dx_1^2+dx_2^2+dx_3^2+dx_4^2\,,$$

where (x_1,x_2,x_3,x_4) is a rectangular coordinate system of $\mathbb{R}^4$. An arbitrary vector $v\in\mathbf{E}_1^4$ is said spacelike (resp. timelike, lightlike) if $\langle v,v\rangle>0$ or $v=0$ (resp. $\langle v,v\rangle<0$, $\langle v,v\rangle=0$ and $v\neq 0$). Let $\alpha\colon I\subset\mathbb{R}\to\mathbf{E}_1^4$ be a (differentiable) curve

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2000 *Mathematics Subject Classification*: primary 53C50; secondary 53B30.

Key words and phrases: Minkowski space, timelike curve, Frenet equations, slant helix.

Partially supported by MEC-FEDER grant no. MTM2007-61775 and Junta de Andalucía grant no. P06-FQM-01642.

Received January 27, 2009, revised October 2009. Editor J. Slovák.

with $\alpha'(t) \neq 0$, where $\alpha'(t) = d\alpha/dt(t)$. The curve α is said timelike if all its velocity vectors $\alpha'(t)$ are timelike. Then it is possible to re-parametrize α by a new parameter s, in such way that $\langle \alpha'(s), \alpha'(s)\rangle = -1$, for any $s \in I$. We say then that α is a unit speed timelike curve.

Consider $\alpha = \alpha(s)$ a unit speed timelike curve in $\mathbf{E}_1^4$. Let $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}_1(s), \mathbf{B}_2(s)\}$ be the moving frame along α, where $\mathbf{T}, \mathbf{N}, \mathbf{B}_1$ and $\mathbf{B}_2$ denote the tangent, the principal normal, the first binormal and second binormal vector fields, respectively. Here $\mathbf{T}(s)$, $\mathbf{N}(s)$, $\mathbf{B}_1(s)$ and $\mathbf{B}_2(s)$ are mutually orthogonal vectors satisfying

$$\langle \mathbf{T}, \mathbf{T}\rangle = -1, \langle \mathbf{N}, \mathbf{N}\rangle = \langle \mathbf{B}_1, \mathbf{B}_1\rangle = \langle \mathbf{B}_2, \mathbf{B}_2\rangle = 1\,.$$

Then the Frenet equations for α are given by

$$\begin{bmatrix}\mathbf{T}'\\ \mathbf{N}'\\ \mathbf{B}_1'\\ \mathbf{B}_2'\end{bmatrix} = \begin{bmatrix}0 & \kappa_1 & 0 & 0\\ \kappa_1 & 0 & \kappa_2 & 0\\ 0 & -\kappa_2 & 0 & \kappa_3\\ 0 & 0 & -\kappa_3 & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}_1\\ \mathbf{B}_2\end{bmatrix}. \tag{2}$$

Recall the functions $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ are called respectively, the first, the second and the third curvatures of α. If $\kappa_3(s) = 0$ for any $s \in I$, then $\mathbf{B}_2(s)$ is a constant vector B and the curve α lies in a three-dimensional affine subspace orthogonal to B, which is isometric to the Minkowski 3-space $\mathbf{E}_1^3$.

We will assume throughout this work that all the three curvatures satisfy $\kappa_i(s) \neq 0$ for any $s \in I$, $1 \leq i \leq 3$.

Definition 1.1. A unit speed timelike curve $\alpha\colon I \to \mathbf{E}_1^4$ is said to be a generalized (timelike) helix if there exists a constant vector field U different from zero and a vector field $X \in \{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ such that the function

$$s \longmapsto \langle X(s), U\rangle\,, \quad s \in I$$

is constant.

In this work we are interested in generalized timelike helices in $\mathbf{E}_1^4$ where the function $\langle \mathbf{B}_2, U\rangle$ is constant. Motivated by the concept of slant helix in $\mathbf{E}^4$ [9], we give the following

Definition 1.2. A unit speed timelike curve α is called a B_2-slant helix if there exists a constant vector field U such that the function $\langle \mathbf{B}_2(s), U\rangle$ is constant.

Our main result in this work follows similar ideas as in [6] for timelike helices in $\mathbf{E}_1^4$. In this sense, we have the following characterization of B_2-slant helices in the spirit of the one given in equation (1) for a slant helix in $\mathbf{E}^3$:

A unit speed timelike curve in $\mathbf{E}_1^4$ is a B_2-slant helix if and only if the function

$$\frac{1}{\kappa_1^2}\Big(\frac{\kappa_3}{\kappa_2}\Big)'^2 - \Big(\frac{\kappa_3}{\kappa_2}\Big)^2$$

is constant.

When α is a lightlike curve, similar computations have been given by Erdogan and Yilmaz in [2].

2. Basic equations of timelike helices

Let α be a unit speed timelike curve in $\mathbf{E}_1^4$ and let U be a unit constant vector field in $\mathbf{E}_1^4$. For each $s \in I$, the vector U is expressed as linear combination of the orthonormal basis $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}_1(s), \mathbf{B}_2(s)\}$. Consider the differentiable functions a_i, $1 \leq i \leq 4$,

$$U = a_1(s)\,\mathbf{T}(s) + a_2(s)\mathbf{N}(s) + a_3(s)\,\mathbf{B}_1(s) + a_4(s)\,\mathbf{B}_2(s)\,, \quad s \in I\,, \tag{3}$$

that is,

$$a_1 = -\langle \mathbf{T}, U\rangle\,, \quad a_2 = \langle \mathbf{N}, U\rangle\,, \quad a_3 = \langle \mathbf{B}_1, U\rangle\,, \quad a_4 = \langle \mathbf{B}_2, U\rangle\,.$$

Because the vector field U is constant, a differentiation in (3) together (2) gives the following ordinary differential equation system

$$\begin{cases} a_1' + \kappa_1 a_2 = 0 \\ a_2' + \kappa_1 a_1 - \kappa_2 a_3 = 0 \\ a_3' + \kappa_2 a_2 - \kappa_3 a_4 = 0 \\ a_4' + \kappa_3 a_3 = 0 \end{cases} \tag{4}$$

In the case that U is spacelike (resp. timelike), we will assume that $\langle U, U\rangle = 1$ (resp. -1). This means that the constant M defined by

$$M := \langle U, U\rangle = -a_1^2 + a_2^2 + a_3^2 + a_4^2 \tag{5}$$

is 1, -1 or 0 depending if U is spacelike, timelike or lightlike, respectively.

We now suppose that α is a generalized helix. This means that there exists i, $1 \leq i \leq 4$, such that the function $a_i = a_i(s)$ is constant. Thus in the system (4) we have four differential equations and three derivatives of functions.

The first case that appears is that the function a_1 is constant, that is, the function $\langle \mathbf{T}(s), U\rangle$ is constant. If U is timelike, that is, the tangent lines of α make a constant (hyperbolic) angle with a fixed timelike direction, the curve α is called a timelike cylindrical helix [6]. Then it is known that α is timelike cylindrical helix if and only if the function

$$\frac{1}{\kappa_3^2}\Big(\frac{\kappa_1}{\kappa_2}\Big)'^2 + \Big(\frac{\kappa_1}{\kappa_2}\Big)^2$$

is constant [6].

However the hypothesis that U is timelike can be dropped and we can assume that U has any causal character, as for example, spacelike or lightlike. We explain this situation. In Euclidean space one speaks on the angle that makes a fixed direction with the tangent lines (cylindrical helices) or the normal lines (slant helices). In Minkowski space, one can only speak about the angle between two vectors $\{u, v\}$ if both are timelike and the belong to the same timecone (hyperbolic angle). See [10, page 144]. This is the reason to avoid any reference about 'angles' in Definition 1.1.

Suppose now that the function $\langle \mathbf{T}(s), U\rangle$ is constant, independent on the causal character of U. From the expression of U in (3), we know that $a_1' = 0$ and by using (4), we obtain $a_2 = 0$ and

$$a_3 = \frac{\kappa_1}{\kappa_2} a_1\,, \quad a_3' = \kappa_3 a_4\,, \quad a_4' + \kappa_3 a_3 = 0\,.$$

Consider the change of variable $t(s) = \int_0^s \kappa_3(x)\,dx$. Then $\frac{dt}{ds}(s) = \kappa_3(s)$ and the last two above equations write as $a_3''(t) + a_3(t) = a_4''(t) + a_4(t) = 0$. Then one obtains that there exist constants A and B such that

$$a_3(s) = A\cos\int_0^s \kappa_3(s)ds + B\sin\int_0^s \kappa_3(s)\,ds$$

$$a_4(s) = -A\sin\int_0^s \kappa_3(s)ds + B\cos\int_0^s \kappa_3(s)ds\,.$$

Since $a_3^2 + a_4^2 = \langle U, U\rangle + a_1^2$ is constant, and

$$a_4 = \frac{1}{\kappa_3} a_3' = \frac{1}{\kappa_3}\Big(\frac{\kappa_1}{\kappa_2}\Big)' a_1\,,$$

it follows that

$$\frac{1}{\kappa_3^2}\Big(\frac{\kappa_1}{\kappa_2}\Big)'^2 + \Big(\frac{\kappa_1}{\kappa_2}\Big)^2 = \text{constant.}$$

Thus we have proved the following theorem.

Theorem 2.1. *Let α be a unit speed timelike curve in $\mathbf{E}_1^4$. Then the function $\langle \mathbf{T}(s), U\rangle$ is constant for a fixed constant vector field U if and only if the the function*

$$\frac{1}{\kappa_3^2}\Big(\frac{\kappa_1}{\kappa_2}\Big)'^2 + \Big(\frac{\kappa_1}{\kappa_2}\Big)^2$$

is constant.

When U is a timelike constant vector field, we re-discover the result given in [6].

3. Timelike B_2-slant helices

Let α be a B_2-slant helix, that is, a unit speed timelike curve in $\mathbf{E}_1^4$ such that the function $\langle \mathbf{B}_2(s), U\rangle$, $s \in I$, is constant for a fixed constant vector field U. We point out that U can be of any causal character. In the particular case that U is spacelike, and since $\mathbf{B}_2$ is too, we can say that a B_2-slant helix is a timelike curve whose second binormal lines make a constant angle with a fixed (spacelike) direction.

Using the system (3), the fact that α is a B_2-slant helix means that the function a_4 is constant. Then (4) gives $a_3 = 0$ and (3) writes as

$$U = a_1(s)\,\mathbf{T}(s) + a_2(s)\mathbf{N}(s) + a_4\,\mathbf{B}_2(s)\,, \quad a_4 \in \mathbb{R} \tag{6}$$

where

$$a_2 = \frac{\kappa_3}{\kappa_2} a_4 = -\frac{1}{\kappa_1} a_1'\,, \qquad a_2' + \kappa_1 a_1 = 0\,. \tag{7}$$

We remark that $a_4 \neq 0$: on the contrary, and from (4), we conclude $a_i = 0$, $1 \leq i \leq 4$, that is, $U = 0$: contradiction.

It follows from (7) that the function a_1 satisfies the following second order differential equation:

$$\frac{1}{\kappa_1}\frac{d}{ds}\Big(\frac{1}{\kappa_1}a_1'\Big) - a_1 = 0\,.$$

If we change variables in the above equation as $\dfrac{1}{\kappa_1}\dfrac{d}{ds} = \dfrac{d}{dt}$, that is, $t = \int_0^s \kappa_1(s)\,ds$, then we get

$$\frac{d^2 a_1}{dt^2} - a_1 = 0\,.$$

The general solution of this equation is

$$a_1(s) = A\cosh\int_0^s \kappa_1(s)\,ds + B\sinh\int_0^s \kappa_1(s)\,ds\,, \tag{8}$$

where A and B are arbitrary constants. From (7) and (8) we have

$$a_2(s) = -A\sinh\int_0^s \kappa_1(s)\,ds - B\cosh\int_0^s \kappa_1(s)\,ds\,. \tag{9}$$

The above expressions of a_1 and a_2 give

$$\begin{aligned} A &= -\Big[\frac{\kappa_3}{\kappa_2}\sinh\int_0^s \kappa_1(s)\,ds + \frac{1}{\kappa_1}\Big(\frac{\kappa_3}{\kappa_2}\Big)'\cosh\int_0^s \kappa_1(s)\,ds\Big]a_4\,,\\ B &= -\Big[\frac{1}{\kappa_1}\Big(\frac{\kappa_3}{\kappa_2}\Big)'\sinh\int_0^s \kappa_1(s)\,ds + \frac{\kappa_3}{\kappa_2}\cosh\int_0^s \kappa_1(s)\,ds\Big]a_4\,. \end{aligned} \tag{10}$$

From (10),

$$A^2 - B^2 = \Big[\frac{1}{\kappa_1^2}\Big(\frac{\kappa_3}{\kappa_2}\Big)'^2 - \frac{\kappa_3^2}{\kappa_2^2}\Big]a_4^2\,.$$

Therefore

$$\frac{1}{\kappa_1^2}\Big(\frac{\kappa_3}{\kappa_2}\Big)'^2 - \frac{\kappa_3^2}{\kappa_2^2} = \text{constant} := m\,. \tag{11}$$

Conversely, if the condition (11) is satisfied for a timelike curve, then we can always find a constant vector field U such that the function $\langle \mathbf{B}_2(s), U\rangle$ is constant: it suffices if we define

$$U = \Big[-\frac{1}{\kappa_1}\Big(\frac{\kappa_3}{\kappa_2}\Big)'\mathbf{T} + \frac{\kappa_3}{\kappa_2}\mathbf{N} + \mathbf{B}_2\Big]\,.$$

By taking account of the differentiation of (11) and the Frenet equations (2), we have that $\frac{dU}{ds} = 0$ and this means that U is a constant vector. On the other hand, $\langle \mathbf{B}_2(s), U\rangle = 1$. The above computations can be summarized as follows:

Theorem 3.1. *Let α be a unit speed timelike curve in $\mathbf{E}_1^4$. Then α is a B_2-slant helix if and only if the function*

$$\frac{1}{\kappa_1^2}\Big(\frac{\kappa_3}{\kappa_2}\Big)'^2 - \Big(\frac{\kappa_3}{\kappa_2}\Big)^2$$

is constant.

From (5), (8) and (9) we get

$$A^2 - B^2 = a_4^2 - M = a_4^2\, m\,.$$

Thus, the sign of the constant m agrees with the one $A^2 - B^2$. So, if U is timelike or lightlike, m is positive. If U is spacelike, then the sign of m depends on $a_4^2 - 1$. For example, $m = 0$ if and only if $a_4^2 = 1$. With similar computations as above, we have

Corollary 3.2. *Let α be a unit speed timelike curve in $\mathbf{E}_1^4$ and let U be a unit spacelike constant vector field. Then $\langle \mathbf{B}_2(s), U\rangle^2 = 1$ for any $s \in I$ if and only if there exists a constant A such that*

$$\frac{\kappa_3}{\kappa_2}(s) = A\ \exp\Big(\int_0^s \kappa_1(t)\, dt\Big)\,.$$

As a consequence of Theorem 3.1, we obtain other characterization of B_2-slant helices. The first one is the following

Corollary 3.3. *Let α be a unit speed timelike curve in $\mathbf{E}_1^4$. Then α is a B_2-slant helix if and only if there exists real numbers C and D such that*

$$\frac{\kappa_3}{\kappa_2}(s) = C \sinh \int_0^s \kappa_1(s)\, ds + D \cosh \int_0^s \kappa_1(s)\, ds\,, \tag{12}$$

Proof. Assume that α is a B_2-slant helix. From (7) and (9), the choice $C = -A/a_4$ and $D = -B/a_4$ yields (12).

We now suppose that (12) is satisfied. A straightforward computation gives

$$\frac{1}{\kappa_1^2}\Big(\frac{\kappa_3}{\kappa_2}\Big)'^2 - \Big(\frac{\kappa_3}{\kappa_2}\Big)^2 = C^2 - D^2\,.$$

We now use Theorem 3.1. □

We end this section with a new characterization for B_2-slant helices. Let now assume that α is a B_2-slant helix in $\mathbf{E}_1^4$. By differentiation (11) with respect to s we get

$$\frac{1}{\kappa_1}\Big(\frac{\kappa_3}{\kappa_2}\Big)'\Big[\frac{1}{\kappa_1}\Big(\frac{\kappa_3}{\kappa_2}\Big)'\Big]' - \Big(\frac{\kappa_3}{\kappa_2}\Big)\Big(\frac{\kappa_3}{\kappa_2}\Big)' = 0\,, \tag{13}$$

and hence

$$\frac{1}{\kappa_1}\Big(\frac{\kappa_3}{\kappa_2}\Big)' = \frac{\Big(\frac{\kappa_3}{\kappa_2}\Big)\Big(\frac{\kappa_3}{\kappa_2}\Big)'}{\Big[\frac{1}{\kappa_1}\Big(\frac{\kappa_3}{\kappa_2}\Big)'\Big]'}\,,$$

If we define a function $f(s)$ as

$$f(s) = \frac{\Big(\frac{\kappa_3}{\kappa_2}\Big)\Big(\frac{\kappa_3}{\kappa_2}\Big)'}{\Big[\frac{1}{\kappa_1}\Big(\frac{\kappa_3}{\kappa_2}\Big)'\Big]'}\,,$$

then

$$f(s)\kappa_1(s) = \Big(\frac{\kappa_3}{\kappa_2}\Big)'\,. \tag{14}$$

By using (13) and (14), we have

$$f'(s) = \frac{\kappa_1 \kappa_3}{\kappa_2}\,.$$

Conversely, consider the function $f(s) = \frac{1}{\kappa_1}\left(\frac{\kappa_3}{\kappa_2}\right)'$ and assume that $f'(s) = \frac{\kappa_1\kappa_3}{\kappa_2}$. We compute

$$\frac{d}{ds}\left[\frac{1}{\kappa_1^2}\left(\frac{\kappa_3}{\kappa_2}\right)'^2 - \frac{\kappa_3^2}{\kappa_2^2}\right] = \frac{d}{ds}\left[f(s)^2 - \frac{f'(s)^2}{\kappa_1^2}\right] := \varphi(s)\,. \tag{15}$$

As $f(s)f'(s) = \left(\frac{\kappa_3}{\kappa_2}\right)\left(\frac{\kappa_3}{\kappa_2}\right)'$ and $f''(s) = \kappa_1'\left(\frac{\kappa_3}{\kappa_2}\right) + \kappa_1\left(\frac{\kappa_3}{\kappa_2}\right)'$ we obtain

$$f'(s)f''(s) = \kappa_1\kappa_1'\left(\frac{\kappa_3}{\kappa_2}\right)^2 + \kappa_1^2\left(\frac{\kappa_3}{\kappa_2}\right)\left(\frac{\kappa_3}{\kappa_2}\right)'.$$

As consequence of above computations

$$\varphi(s) = 2\Big(f(s)f'(s) - \frac{f'(s)f''(s)}{\kappa_1^2} + \frac{\kappa_1' f'(s)^2}{\kappa_1^3}\Big) = 0\,,$$

that is, the function $\frac{1}{\kappa_1^2}\left(\frac{\kappa_3}{\kappa_2}\right)'^2 - \left(\frac{\kappa_3}{\kappa_2}\right)^2$ is constant. Therefore we have proved the following

Theorem 3.4. *Let α be a unit speed timelike curve in $\mathbf{E}_1^4$. Then α is a B_2-slant helix if and only if the function $f(s) = \frac{1}{\kappa_1}\left(\frac{\kappa_3}{\kappa_2}\right)'$ satisfies $f'(s) = \frac{\kappa_1\kappa_3}{\kappa_2}$.*

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