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ON NEW SUBCLASS OF ANALYTIC FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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Abstract. In the present paper, we obtain coefficient estimates for new subclass of analytic functions with respect to symmetric points. A sufficient condition for a function to belong to this class of function is also obtained.

1. Introduction

Let $\mathcal{A}$ be the class of functions f *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the open unit disc

$$D = \{z \in \mathbb{C} : |z| < 1\} .$$

As usual, we denote by S the subclass of $\mathcal{A}$, consisting of functions which are also *univalent* in D. We recall here the definitions of the well-known classes of starlike function and convex functions:

$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in D \right\} ,$$

$$C = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in D \right\} .$$

Let w be a fixed point in D and $\mathcal{A}(w) = \{f \in H(D) : f(w) = f'(w) - 1 = 0\}$. In [8], Kanas and Ronning introduced the following classes

$$S_w = \{f \in \mathcal{A}(w) : f \text{ is univalent in } D\}$$

$$ST_w = \left\{ f \in \mathcal{A}(w) : \operatorname{Re}\left(\frac{(z-w)\, f'(z)}{f(z)}\right) > 0, z \in D \right\} , \tag{1.2}$$

$$CV_w = \left\{ f \in \mathcal{A}(w) : 1 + \left(\operatorname{Re}\frac{(z-w)\, f''(z)}{f'(z)}\right) > 0, z \in D \right\} . \tag{1.3}$$

Later Acu and Owa [1] studied the classes extensively.

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The class ST_w is defined by geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class S_w^c is defined by the property that the image of any circular arc centered at w is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [7] and [6] for uniformly starlike and convex functions, except that in this case the point w is fixed.

Let Σ_w denoted the subclass of $\mathcal{A}(w)$ consisting of the function of the form

$$f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n \qquad (a_n \geq 0\,,\ z \neq w)\,. \tag{1.4}$$

The function f in Σ_w is said to be starlike function of order β if and only if

$$\operatorname{Re}\left\{-\frac{(z-w)f'(z)}{f(z)}\right\} > \beta \qquad (z \in D) \tag{1.5}$$

for some β $(0 \leq \beta < 1)$. We denote by $S_w^*(\beta)$ the class of all starlike functions of order β. Similarly , a function f in Σ_w is said to be convex of order β if and only if

$$\operatorname{Re}\Big(-1 - \frac{(z-w)f''(z)}{f'(z)}\Big) > \beta \qquad (z \in D) \tag{1.6}$$

for some β $(0 \leq \beta < 1)$. We denote by $C_w(\beta)$ the class of all convex functions of order β. The class Σ_w studied extensively by Ghanim and Darus ([4], [5]).

Let S_{ws}^* be the subclass of Σ_w consisting of functions given by (1.4) satisfying

$$\operatorname{Re}\{(z-w)f'(z)/(f(z)-f(-z))\} > 0\,, \tag{1.7}$$

for $z \in D$.

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [10]. Recently, El-Ashwah and Thomas [3] have introduced two function classes, namely the class of functions starlike with respect to conjugate points and the class of functions starlike with respect to symmetric conjugate points.

In this paper, we introduce the class $S_{ws}^*(\alpha,\beta)$ of functions f, regular and univalent in D given by (1.7) and satisfying the condition

$$\Big|\frac{(z-w)f'(z)}{f(z)-f(-z)} - 1\Big| < \beta\,\Big|\frac{\alpha(z-w)f'(z)}{f(z)-f(-z)} + 1\Big|$$

$z \in D$, $0 < \beta \leq 1$, $0 \leq \alpha \leq 1$ and we obtain coefficient estimates for functions in this class.We also obtain a sufficient condition for a function f to belong to the class $S_{ws}^*(\alpha,\beta)$.

Along with $S_{ws}^*(\alpha,\beta)$, we also consider the class $S_{wc}^*(\alpha,\beta)$ of functions f with respect to conjugate points, regular in D and satisfying

$$\Big|\frac{(z-w)f'(z)}{f(z)-\overline{f(\overline{z})}} - 1\Big| < \beta\,\Big|\frac{\alpha(z-w)f'(z)}{f(z)-\overline{f(\overline{z})}} + 1\Big|$$

with $z \in D$, $0 < \beta \leq 1$, and $0 \leq \alpha \leq 1$.

We analogously obtain coefficient estimates for functions in the class $S^*_{wc}(\alpha,\beta)$.

2. Coefficient estimates

We need a lemma of Lakshminarasimhan [9].

Lemma 2.1. *Let $H(z)$ be analytic in D and satisfy the condition*

$$\Big|\frac{1-H(z)}{1+\alpha H(z)}\Big| < \beta \tag{2.1}$$

$z\in D$, $0<\beta\le 1$, $0\le\alpha\le 1$, with $H(0)=1$. Then we have

$$H(z)=\frac{1-z\phi(z)}{1+\alpha z\phi(z)}\,, \tag{2.2}$$

where $\phi(z)$ is analytic in D and $\big|\phi(z)\big|\le\beta$ for $z\in D$. Conversely, any function $H(z)$ given by (2.2) *above is analytic in D and satisfies* (2.1).

Next, we shall prove a lemma, in order to obtain the coefficient estimates for functions f in the classes $S^*_{ws}(\alpha,\beta)$ and $S^*_{wc}(\alpha,\beta)$.

Lemma 2.2. *Let f and g belong to Σ_w and satisfy*

$$\Big|\frac{(z-w)f'(z)}{g(z)}-1\Big| < \beta\,\Big|\frac{\alpha(z-w)f'(z)}{g(z)}+1\Big|\,, \tag{2.3}$$

$0<\beta\le 1$, $0\le\alpha\le 1$ and $z\in D$, with f given by (1.4), *and $g(z)=\frac{1}{z-w}+\sum\limits_{n=1}^{\infty} b_n(z-w)^n$. Then for $n\ge 1$*

$$\Big|na_n-b_n\Big|^2 \le 2(\alpha\beta^2+1)\Big(1+\sum_{j=0}^{n-1} j\,|a_j|\,|b_j|\Big)\,. \tag{2.4}$$

Proof. We use the method of Clunie and Keogh [2] and Thomas [11]. By Lemma 2.1 we have

$$\frac{(z-w)f'(z)}{g(z)}=\frac{1-(z-w)\phi(z)}{1+\alpha(z-w)\phi(z)}\,,$$

$\phi(z)$ is analytic in D and $|\phi(z)|\le\beta$ for $(z-w)\in D$. Then

$$(z-w)f'(z)=g(z)\Big[\frac{1-(z-w)\phi(z)}{1+\alpha(z-w)\phi(z)}\Big]$$

or

$$\big[\alpha(z-w)(f'(z)+g(z)\big](z-w)\phi(z)=g(z)-(z-w)f'(z)\,.$$

Now if

$$\psi(z)=(z-w)\phi(z)=\sum_{n=0}^{\infty} t_n(z-w)^n$$

then

$$|\psi(z)|\le\beta|z-w| \quad \text{for} \quad (z-w)\in D\,.$$

Therefore

$$\left[\frac{-(\alpha+1)}{z-w}+\alpha\sum_{n=1}^{\infty}na_n(z-w)^n+\sum_{n=1}^{\infty}b_n(z-w)^n\right]\left[\sum_{n=0}^{\infty}t_n(z-w)^n\right]$$
$$(2.5)\qquad =\sum_{n=1}^{\infty}b_n\,(z-w)^n-\sum_{n=1}^{\infty}na_n\,(z-w)^n\,.$$

Comparing the coefficient of $(z-w)^n$ in (2.5), we have

$$(b_n-na_n)=(\alpha+1)t_{n-1}$$
$$+(\alpha a_1+b_1)t_{n-2}+(\alpha 2a_2+b_2)t_{n-3}+\cdots+(\alpha(n-1)a_{n-1}+b_{n-1})t_1\,.$$

Thus the coefficient combination on the right-hand side of (2.5) depends only on the coefficients combination $(\alpha a_1+b_1),\ldots,(\alpha(n-1)a_{n-1}+b_{n-1})$ on the left-hand side. Hence, for $n\geq 1$, we can write

$$\left[\frac{-(\alpha+1)}{z-w}+\sum_{j=0}^{n-1}(\alpha ja_j+b_j)(z-w)^j\right]\psi(z)$$
$$(2.6)\qquad =\sum_{j=1}^{n}(b_j-ja_j)\,(z-w)^j+\sum_{j=n+1}^{\infty}c_j(z-w)^j\,.$$

Squaring the modulus of the both sides of (2.6) and integrating along $|z-w|=r<1$, and using the fact that $|\psi(z)|\leq\beta|z-w|$, we obtain

$$\sum_{j=1}^{n}|ja_j-b_jt|^2r^{2(j+1)}+\sum_{j=n+1}^{\infty}|c_j|r^{2(j+1)}<\beta^2\left[(\alpha+1)^2+\sum_{j=0}^{n-1}|\alpha ja_j+b_j|^2r^{2(j+1)}\right]$$

Letting $r\to 1$ on the left-hand side of this inequality, we obtain

$$\sum_{j=1}^{n}|ja_j-b_j|^2<\beta^2(\alpha+1)^2+\beta^2\sum_{j=0}^{n-1}|\alpha ja_j+b_j|^2\,.$$

This implies that

$$|na_n-b_n|^2\leq\beta^2(\alpha+1)^2+\beta^2\sum_{j=0}^{n-1}|\alpha ja_j+b_j|^2-\sum_{j=0}^{n-1}|ja_j-b_j|^2$$
$$(2.7)\qquad \leq\beta^2(\alpha+1)^2+(\alpha^2\beta^2-1)\sum_{j=0}^{n-1}j^2|a_j|^2+(\beta^2-1)\sum_{j=0}^{n-1}|b_j|^2$$
$$+2\alpha\beta^2\sum_{j=0}^{n-1}j|a_jb_jt|+2\sum_{j=0}^{n-1}j|a_jb_j|$$

or

$$|na_n-b_n|^2\leq 2(\alpha\beta^2+1)+2\alpha\beta^2\sum_{j=0}^{n-1}j|a_j|\,|b_j|+2\sum_{j=0}^{n-1}j|a_j|\,|b_j|\,.$$

Thus

$$|na_n - b_n|^2 \le 2(\alpha\beta^2+1)\Big(1+\sum_{j=0}^{n-1} j|a_j|\,|b_j|\Big)\,,$$

since $0 \le \beta < 1$, $0 \le \alpha \le 1$. □

Theorem 2.1. *Let f and g belong to Σ_w and be given as in Lemma 2.2. Then for $n \ge 1$*

$$|na_n - b_n|^2 \le 2(\alpha\beta^2+1)\big(1 + CA(1-1/n, f)^{\frac{1}{2}} A(1-1/n, g)^{\frac{1}{2}}\big)\,,$$

where $A(r,f)$ denotes the area enclosed by $f(|z-w| = r)$ and where C is a constant.

Proof. We have by (2.4) of Lemma 2.2

$$|na_n - b_n|^2 \le 2(\alpha\beta^2+1)(1+\sum_{j=0}^{n-1} j|a_j|\,|b_j|)\,.$$

The Cauchy-Schwarz inequality gives for $0 < r < 1$

$$\begin{aligned}
|na_n - b_n|^2 &\le 2(\alpha\beta^2+1) \\
&\quad + 2\alpha\beta^2\Big(\sum_{j=0}^{n-1} j|a_j|^2\Big)^{\frac{1}{2}}\Big(\sum_{j=0}^{n-1} j|b_j|^2\Big)^{\frac{1}{2}} + 2\Big(\sum_{j=0}^{n-1} j|a_j|^2\Big)^{\frac{1}{2}}\Big(\sum_{j=0}^{n-1} j|b_j|^2\Big)^{\frac{1}{2}} \\
&\le 2(\alpha\beta^2) + 1 + \frac{2\alpha\beta^2}{r^{2n}}\Big(\sum_{j=0}^{n-1} j|a_j|^2 r^{2j}\Big)^{\frac{1}{2}}\Big(\sum_{j=0}^{n-1} j|b_j|^2 r^{2j}\Big)^{\frac{1}{2}} \\
&\quad + \frac{2}{r^{2n}}\Big(\sum_{j=0}^{n-1} j|a_j|^2 r^{2j}\Big)^{\frac{1}{2}}\Big(\sum_{j=0}^{n-1} j|b_j|^2 r^{2j}\Big)^{\frac{1}{2}} \\
&\le 2(\alpha\beta^2+1) + \frac{2\alpha\beta^2}{\pi r^{2n}} A(r,f)^{\frac{1}{2}} A(r,g)^{\frac{1}{2}} + \frac{2}{\pi r^{2n}} A(r,f)^{\frac{1}{2}} A(r,g)^{\frac{1}{2}}
\end{aligned}$$

Since $A(r,f) = \pi \sum\limits_{n=0}^{\infty} n|a_n|^2 r^{2n}$.

Choosing $r = 1 - \frac{1}{n}$ for $n \ge 1$, the result follows. □

Theorem 2.2. *Let $f \in ST_{ws}(\alpha,\beta)$ and be given by* (1.4). *Then*

(1) $(m+1)^2|a_{2m+1}|^2 \le \frac{1}{2}(\alpha\beta^2+1)\Big(\sum\limits_{j=0}^{m}(2j+1)|a_{2j}|^2\Big)$ *$m \ge 0$, $|a_0| = 1$;*

(2) $m^2|a_{2m}|^2 \le \frac{1}{2}(\alpha\beta^2+1)\Big(\sum\limits_{j=0}^{m-1}(2j+1)|a_{2j}|^2\Big)$, *$m \ge 1$. Further, if $\alpha\beta < 1$;*

(3) $(m+1)^2|a_{2m+1}|^2 \le \frac{\beta^2-1}{4}\Big(\sum\limits_{j=0}^{m}|a_{2j}|^2\Big) + \frac{\beta+1}{2}\Big(\sum\limits_{j=0}^{m}(2j+1)|a_{2j}|^2\Big)$ *for $m \ge 0$, $|a_0| = 1$ and*

(4) $m^2|a_{2m}|^2 \le \frac{\beta^2-1}{4}\Big(\sum\limits_{j=0}^{m-1}|a_{2j}|^2\Big) + \frac{\beta+1}{2}\Big(\sum\limits_{j=0}^{m-1}(2j+1)|a_{2j}|^2\Big)$ *$m \ge 1$, $|a_0| = 1$.*

The inequalities (1) *and* (2) *are sharp.*

Proof. Since $f \in ST_{ws}(\alpha, \beta)$, by Lemma 2.1 we have $\frac{(z-w)f'(z)}{g(z)} = h(z)$, where g is an odd starlike function with $g(z) = \frac{f(z)-f(-z)}{2}$ and $h(z) = \frac{1-(z-w)\phi(z)}{1+\alpha(z-w)\phi(z)}$ $\phi(z)$ analytic in D and $|\phi(z)| \leq \beta$ for $z \in D$. Thus, with $g(z) = \frac{1}{(z-w)} + \sum\limits_{n=1}^{\infty} a_{2n}(z-w)^{2n}$ for $z \in D$, using (2.4) of Lemma 2.2 with b_n suitably chosen, the inequalities (1) and (2) in the theorem follow. Indeed, when $\alpha\beta < 1$ using (2.7) of Lemma 2.2

$$|na_n - b_n|^2 \leq 2(\beta+1)\Big(1 + \sum_{j=0}^{n-1} j|a_j|^2|b_j|^2\Big) + (\beta^2 - 1)\sum_{j=0}^{n-1}|b_j|^2$$

and the inequalities (3) and (4) follow. □

Theorem 2.3. *If $f \in S^*_{ws}(\alpha, \beta)$ with $\alpha\beta < 1$, then $a_n = O\big(\frac{1}{n}\big)$ as $n \to \infty$.*

Proof. We observe that when $\alpha\beta < 1$, for $f \in ST_{ws}(\alpha, \beta)$, $\frac{(z-w)f'(z)}{f(z)-f(-z)}$ is bounded, we first prove that

$$\big(n - (1 - (-1)^n)\big)^2 |a_n|^2 \leq 2(\beta + 1)\Big(1 + \sum_{j=0}^{n-1} j|a_j|^2\Big).$$

If $f \in ST_{ws}(\alpha, \beta)$ is given by (1.4), we have using Lemma 2.1

$$\frac{(z-w)f'(z)}{f(z)-f(-z)} = \frac{1-(z-w)\phi(z)}{1+\alpha(z-w)\phi(z)},$$

$\phi(z)$ is analytic in D and $|\phi(z)| \leq \beta$ for $(z-w) \in D$. Then

$$\begin{aligned}\big[\alpha(z-w)f'(z) + f(z) - f(-z)\big]&(z-w)\phi(z) \\ &= \big[f(z) - f(-z)\big] - (z-w)f'(z)\end{aligned}$$

Now if

$$\psi(z) = (z-w)\phi(z) = \sum_{n=0}^{\infty} t_n (z-w)^n,$$

then

$$|f(z)| \leq \beta|z-w| \quad \text{for} \quad (z-w) \in D$$

therefore

$$\begin{aligned}&\Big[\frac{\alpha}{z-w} + \alpha\sum_{n=1}^{\infty} na_n(z-w)^n + \frac{2}{z-w} + \sum_{n=1}^{\infty}\big(1 - (-1)^n\big)a_n(z-w)^n\Big] \\ &\times\Big(\sum_{n=0}^{\infty} t_n(z-w)^n\Big) = \Big[\frac{1}{z-w} + \sum_{n=1}^{\infty}\big((1-(-1)^n) - n\big)a_n(z-w)^n\Big].\end{aligned} \tag{2.8}$$

Equating coefficients of $(z-w)^n$ in (2.8), we have

$$\big((1-(-1)^n)-n\big) = (\alpha+1)t_{n-1} + \big(\alpha a_1 + (1-(-1)^1)\big)t_{n-2} + \big(\alpha 2a_2 + (1-(-1)^2)\big)t_{n-3} + \cdots + \big(\alpha(n-1)a_{n-1} + (1-(-1)^{n-1})\big)t_1\,.$$

Thus the coefficient combination on the right side of (2.8) depends only on the coefficient combination

$$\big(\alpha a_1 + (1-(-1)^1)\big), \ldots, \big(\alpha(n-1)a_{n-1} + (1-(-1)^{n-1})\big)$$

on the left-hand side. Hence, for $n \geq 1$, we can write

$$\Big[\frac{\alpha+1}{z-w} + \sum_{j=0}^{n-1} \big(\alpha j + (1-(-1)^j)\big)a_j(z-w)^j\Big]\psi(z) = \sum_{j=1}^{n} \big((1-(-1)^j)-j\big)a_j(z-w)^j + \sum_{j=n+1}^{\infty} c_j(z-w)^j\,. \tag{2.9}$$

Squaring the modules of both sides of (2.9) and integrating along $|z-w| = r < 1$, we obtain (using the fact that $|\psi(z)| \leq \beta|z-w|$)

$$\sum_{j=1}^{n} \big(j-(1-(-1)^j)\big)^2|a_j|^2 r^{2j} + \sum_{j=n+1}^{\infty} |c_j|^2 r^{2j} < \beta^2\Big[\frac{(\alpha+1)^2}{r^2} + \sum_{j=0}^{n-1} \big(\alpha j + (1-(-1)^j)\big)^2|a_j|^2 r^{2j}\Big]\,.$$

Letting $r \to 1$ on the left-hand side of the last inequality we obtain

$$\sum_{j=1}^{n} \big(j-(1-(-1)^j)\big)^2|a_j|^2 < \beta^2\Big[(\alpha+1)^2 + \sum_{0=1}^{n-1} \big(\alpha j + (1-(-1)^j)\big)^2|a_j|^2\Big]\,.$$

This implies

$$\begin{aligned}\big(n-(1-(-1)^n)\big)^2|a_n|^2 &< \beta^2(\alpha+1)^2 + \beta^2\sum_{j=0}^{n-1} \big(\alpha j + (1-(-1)^j)\big)^2|a_j|^2 \\ &\quad - \sum_{j=1}^{n} \big(j-(1-(-1)^j)\big)^2|a_j|^2 \leq \beta^2(\alpha+1)^2 + (\alpha^2\beta^2-1) \\ &\quad \times \sum_{j=0}^{n-1} j^2|a_j|^2 + (\beta^2-1)\sum_{j=0}^{n-1} 1-(-1)^j)^2|a_j|^2 \\ &\quad + 2\alpha\beta^2\sum_{j=0}^{n-1} j(1-(-1)^j)^2|a_j|^2 + 2\sum_{j=0}^{n-1} j(1-(-1)^j)^2|a_j|^2\end{aligned} \tag{2.10}$$

or

$$\big(n-(1-(-1)^n)\big)^2|a_n|^2 \le 2(\beta+1)+2\beta\sum_{j=0}^{n-1} j|a_j|^2 + 2\sum_{j=0}^{n-1} j|a_j|^2$$

$$\le 2(\beta+1)\Big(1+\sum_{j=0}^{n-1} j|a_j|^2\Big) \tag{2.11}$$

since $\alpha\beta < 1$.
It remains to show that $a_n = O\big(\frac{1}{n}\big)$ as $n \to \infty$. From (2.11) we have

$$\big(n-(1-(-1)^n)\big)^2|a_n|^2 \le 2(\beta+1)\left(1+\sum_{j=0}^{n-1} j|a_j|^2\right). \tag{2.12}$$

Since $\frac{(z-w)f'(z)}{f(z)-f(-z)}$ is bounded, it follows that $f(z)$ is bounded. Now following Clunie and Keogh [2] we conclude that $\triangle$, the area of the image of $f(z)$, is given by

$$\triangle = \pi\Big(1+\sum_{j=1}^{\infty} j|a_j|^2\Big), \tag{2.13}$$

and consequently, $\sum\limits_{j=1}^{\infty} j|a_j|^2 < \infty$ and hence $r_n = \sum\limits_{j=1}^{\infty} j|a_j|^2 \to 0$ as $n \to \infty$.
Thus we have

$$\sum_{j=0}^{n-1} j|a_j|^2 = \sum_{j=0}^{n-1}(r_j - r_{j+1}) = r_1 - r_n = O(1) \quad \text{as} \quad n \to \infty. \tag{2.14}$$

Using (2.12) and (2.14), we have $a_n = O\big(\frac{1}{n}\big)$ as $n \to \infty$. □

3. Sufficient condition

We obtain a sufficient condition for functions to belong to the class $S^*_{ws}(\alpha,\beta)$.

Theorem 3.1. *Let* $f(z) = \frac{1}{z-w} + \sum\limits_{n=1}^{\infty} a_n(z-w)^n$ *be analytic in the unit disc* D. *If for* $0 \le \alpha \le 1$, $\frac{1}{2} < \beta \le 1$.

$$\sum_{n=1}^{\infty}\Big[\frac{(1+\alpha\beta)n}{(\beta(\alpha+2)-1)} + \frac{(\beta(1-(-1)^n)) - (1-(-1)^n)}{(\beta(\alpha+2)-1)}\Big]|a_n| \le 1$$

or equivalently,

$$\Big[\sum_{m=0}^{\infty}\frac{(1+\alpha\beta)(2m+1)|a_{2m+1}|}{(\beta(\alpha+2)-1)} + \sum_{m=0}^{\infty}\frac{\{(1+\alpha\beta)(2m+2)|a_{2m+2}| + 2(\beta-1)|a_{2m+2}|\}}{(\beta(\alpha+2)-1)}\Big] \le 1, \tag{3.1}$$

then f *belongs to the class* $ST_{ws}(\alpha,\beta)$.

Proof. We use the method of Clunic and Keogh [2]. Suppose that $f(z) = \frac{1}{z-w} + \sum\limits_{n=1}^{\infty} a_n(z-w)^n$, then for $|z-w|<1$

$$\begin{aligned}
&\big|(z-w)f'(z) - f(z) - f(-z)\big| - \beta\big|\alpha(z-w)f'(z) + f(z) - f(-z)\big|\\
&\quad \times \Big|\frac{-1}{(z-w)} + \sum_{n=1}^{\infty} na_n(z-w)^n - \sum_{n=1}^{\infty}(1-(-)^n)a_n(z-w)^n\Big|\\
&\quad - \beta\Big|\frac{-\alpha}{(z-w)} + \alpha\sum_{n=1}^{\infty} na_n(z-w)^n - \frac{2}{(z-w)} + \sum_{n=1}^{\infty}(1-(-1)^n)a_n(z-w)^n\Big|\\
&= \Big|\frac{-1}{(z-w)} + \sum_{n=1}^{\infty} na_n(z-w)^n - \sum_{n=1}^{\infty}(1-(-1)^n)a_n(z-w)^n\Big|\\
&\quad - \beta\Big|\frac{-(\alpha+2)}{(z-w)} + \alpha\sum_{n=1}^{\infty} na_n(z-w)^n + \sum_{n=1}^{\infty}(1-(-1)^n)a_n(z-w)^n\Big|\\
&= \Big|\frac{-1}{(z-w)} + \sum_{n=1}^{\infty}\big(n-(1-(-1)^n)\big)a_n(z-w)^n\Big|\\
&\quad - \beta\Big|\frac{-(\alpha+2)}{(z-w)} + \sum_{n=1}^{\infty}\big(n\alpha+(1-(-1)^n)\big)a_n(z-w)^n\Big|\\
&\le \frac{1}{r} + \sum_{n=1}^{\infty}\big(n-(1-(-1)^n)\big)|a_n|r^n\\
&\quad - \beta\Big[\frac{\alpha+2}{r} - \sum_{n=1}^{\infty}\big(n\alpha+(1-(-1)^n)\big)|a_n|r^n\Big]\\
&< \Big[\sum_{m=0}^{\infty}(1+\alpha\beta)(2m+1)|a_{2m+1}| + \sum_{m=0}^{\infty}\{(1+\alpha\beta)(2m+2)|a_{2m+2}|\\
&\quad + 2(\beta-1)|a_{2m+2}|\}\Big]r - \frac{(\beta(\alpha+2)-1)}{r}\\
&< \Big[\sum_{m=0}^{\infty}(1+\alpha\beta)(2m+1)|a_{2m+1}| + \sum_{m=0}^{\infty}\big\{(1+\alpha\beta)(2m+2)|a_{2m+2}|\\
&\quad + 2(\beta-1)|a_{2m+2}|\big\} - (\beta(\alpha+2)-1)\Big]r \le 0
\end{aligned}$$

by (3.1). Hence, it follows that in $|(z-w)|<1$

$$\Big|\Big(\frac{(z-w)f'(z)}{f(z)-f(-z)} - 1\Big)\Big/\Big(\frac{\alpha(z-w)f'(z)}{f(z)-f(-z)} + 1\Big)\Big| < \beta$$

so that $f(z) \in ST_{ws}(\alpha,\beta)$. We note that

$$f(z) = \frac{1}{z-w} + \frac{(\beta(\alpha+2)-1)}{(1+\alpha\beta)n + (\beta(1-(-1)^n)) - (1-(-1)^n)}(z-w)^n$$

is an extremal function with respect to the theorem since

$$\Big|\Big(\frac{(z-w)f'(z)}{f(z)-f(-z)}-1\Big)\Big/\Big(\frac{\alpha(z-w)f'(z)}{f(z)-f(-z)}+1\Big)\Big|=\beta$$

for $z=1$, $\frac{1}{2}<\beta\le 1$, $0\le\alpha\le 1$, $n=1,2,3,\dots$. □

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