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LOWER BOUNDS FOR EXPRESSIONS OF LARGE SIEVE TYPE

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Abstract. We show that the large sieve is optimal for almost all exponential sums.

Let a_n, $1 \leq n \leq N$ be complex numbers, and set $S(\alpha) = \sum_{n\leq N} a_n e(n\alpha)$, where $e(\alpha) = \exp(2\pi i\alpha)$. Large Sieve inequalities aim at bounding the number of places where this sum can be extraordinarily large, the basic one being the bound

$$\sum_{q\leq Q} \sum_{\substack{1\leq a\leq q\\(a,q)=1}} \left|S\left(\frac{a}{q}\right)\right|^2 \leq (N+Q^2) \sum_{n\leq N} |a_n|^2$$

(see e.g. [3] for variations and applications). P. Erdős and A. Rényi [1] considered lower bounds of the same type, in particular they showed that the bound

$$\sum_{q\leq Q} \sum_{(a,q)=1} \left|S\left(\frac{a}{q}\right)\right|^2 \ll N \sum_{n\leq N} |a_n|^2\,, \tag{1}$$

valid for $Q \ll \sqrt{N}$, is wrong for almost all choices of coefficients $a_n \in \{1,-1\}$, provided that $Q > C\sqrt{N}\log N$, and that the standard probabilistic argument fails to decide whether (1) is true in the range $\sqrt{N} < Q < \sqrt{N}\log N$. In this note, we show that (1) indeed fails throughout this range.

Theorem 1. *Let $S(\alpha)$ be as above. Then*

$$\sum_{q\leq Q} \sum_{(a,q)=1} \left|S\left(\frac{a}{q}\right)\right|^2 \geq \varepsilon Q^2 \sum_{n\leq N} |a_n|^2 \tag{2}$$

holds true with probability tending to 1 provided ε tends to 0, and Q^2/N tends to infinity.

Our approach differs from [1] in so far as we first prove an unconditional lower bound, which involves an awkward expression, and show then that almost always this expression is small. We show the following.

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Lemma 1. *Let $S(\alpha)$ be as above, and define*

$$M(x) = \sup_{\mathfrak{m}} \frac{\int\limits_{\mathfrak{m}} |S(u)|^2\, du}{\int\limits_0^1 |S(u)|^2\, du},$$

where $\mathfrak{m}$ ranges over all measurable subsets of $[0,1]$ of measure x. Then for any real parameter $A > 1$ we have the estimate

$$\sum_{q \le Q} \sum_{(a,q)=1} \left|S\Big(\frac{a}{q}\Big)\right|^2 \ge \Big(\frac{Q^2}{A}\Big(1 - M\Big(\frac{1}{A}\Big)\Big) - 6\pi N A\Big) \sum_{n \le N} |a_n|^2 . \tag{3}$$

Proof. Our proof adapts Gallagher's proof of an upper bound large sieve [2]. For every $f \in C^1([0,1])$, we have

$$f(1/2) = \int\limits_0^1 f(u)\, du + \int\limits_0^{1/2} u f'(u)\, du - \int\limits_{1/2}^1 (1-u) f'(u)\, du .$$

Putting $f(u) = |S(u)|^2$, and using the linear substitution $u \mapsto (\alpha - \delta/2) + \delta u$, we obtain for every $\delta > 0$ and any $\alpha \in [0,1]$

$$\begin{aligned} |S(\alpha)|^2 = \frac{1}{\delta} \int\limits_{\alpha-\delta/2}^{\alpha+\delta/2} |S(u)|^2\, du &+ \frac{1}{\delta} \int\limits_{\alpha-\delta/2}^{\alpha} \big(\delta/2 - |u-\alpha|\big)\big(S'(u)S(-u) - S(u)S'(-u)\big)\, du \\ &- \frac{1}{\delta} \int\limits_{\alpha}^{\alpha+\delta/2} \big(\delta/2 - |u-\alpha|\big)\big(S'(u)S(-u) - S(u)S'(-u)\big)\, du . \end{aligned}$$

We have $|S(u)| = |S(-u)|$ and $|S'(-u)| = |S'(u)|$, thus $|S'(u)S(-u) - S(u)S'(-u)| \le 2|S(u)S'(u)|$, and we obtain

$$\begin{aligned} |S(\alpha)|^2 &\ge \frac{1}{\delta} \int\limits_{\alpha-\delta/2}^{\alpha+\delta/2} |S(u)|^2\, du - \frac{1}{\delta} \int\limits_{\alpha-\delta/2}^{\alpha+\delta/2} 2\Big(\frac{1}{2} - \frac{|u-\alpha|}{\delta}\Big) |S(u)S'(u)|\, du \\ &\ge \frac{1}{\delta} \int\limits_{\alpha-\delta/2}^{\alpha+\delta/2} |S(u)|^2\, du - \int\limits_{\alpha-\delta/2}^{\alpha+\delta/2} |S(u)S'(u)|\, du . \end{aligned}$$

We now set $\delta = A/Q^2$. We can safely assume that $\delta < \frac{1}{2}$, since our claim would be trivial otherwise. Summing over all fractions $\alpha = \frac{a}{q}$ with $q \le Q$, $(a,q) = 1$, we get

$$(4)\quad \sum_{q \le Q} \sum_{(a,q)=1} \left| S\Big(\frac{a}{q}\Big) \right|^2 \ge \frac{Q^2}{A} \int_0^1 |S(u)|^2 du - \frac{Q^2}{A} \int_{m(Q,A)} |S(u)|^2 \, du - \int_0^1 R(u) |S(u) S'(u)| \, du \,,$$

where

$$R(u) = \#\Big\{ a, q : (a,q) = 1, q \le Q, \Big| u - \frac{a}{q} \Big| \le \frac{A}{Q^2} \Big\} \,,$$

and

$$m(Q,A) = \big\{ u \in [0,1] : R(u) = 0 \big\} \,.$$

To bound $R(u)$, let $\frac{a_1}{q_1} < \frac{a_2}{q_2} < \cdots < \frac{a_k}{q_k}$ be the list of all fractions with $q_i \le Q$, $\big|u - \frac{a_i}{q_i}\big| \le \frac{A}{Q^2}$. We have for $i \ne j$ the bound

$$\Big| \frac{a_i}{q_i} - \frac{a_j}{q_j} \Big| \ge \frac{1}{q_i q_j} \ge \frac{1}{Q^2} \,,$$

that is, the fractions $\frac{a_1}{q_1}, \dots, \frac{a_k}{q_k}$ form a set of points with distance $> \frac{1}{Q^2}$ in an interval of length $\frac{2A}{Q^2}$. There can be at most $2A + 1$ such points, hence, $R(u) \le 3A$.

Next, we bound $|m(Q,A)|$. By Dirichlet's theorem, we have that for each real number $\alpha \in [0,1]$ there exists some $q \le Q$ and some a, such that $|\alpha - \frac{a}{q}| \le \frac{1}{qQ}$. If $\alpha \in m(Q,A)$, we must have $\frac{1}{qQ} > \frac{A}{Q^2}$, that is, $q < Q/A$. Hence, we obtain

$$\begin{aligned} |m(Q,A)| &\le \Big| \bigcup_{q < Q/A} \bigcup_{(a,q)=1} \Big[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ} \Big] \setminus \Big[\frac{a}{q} - \frac{A}{Q^2}, \frac{a}{q} + \frac{A}{Q^2} \Big] \Big| \\ &\le \sum_{q < Q/A} \frac{\varphi(q)(2Q - 2Aq)}{qQ^2} \le \frac{1}{Q^2} \int_0^{Q/A} \big(2Q - 2At\big) \, dt = \frac{1}{A} \,. \end{aligned}$$

We can now estimate the right hand side of (4). The first summand is $\frac{Q^2}{A} \sum_{n \le N} |a_n|^2$, while the second is by definition at most $\frac{Q^2}{A} M(1/A)$. For the third we apply the Cauchy-Schwarz-inequality to obtain

$$\begin{aligned} \Big(\int_0^1 |S(u) S'(u)| \, du \Big)^2 &\le \Big(\int_0^1 |S(u)|^2 \, du \Big) \Big(\int_0^1 |S'(u)|^2 \, du \Big) \\ &= \Big(\sum_{n \le N} |a_n^2| \Big) \Big(\sum_{n \le N} (2\pi n)^2 |a_n^2| \Big) \\ &\le (2\pi N)^2 \Big(\sum_{n \le N} |a_n^2| \Big)^2 . \end{aligned}$$

Hence, the last term in (4) is bounded above by $3A(2\pi N)\sum_{n\leq N}|a_n|^2$, and inserting our bounds into (4) yields the claim of our lemma. □

Now we deduce Theorem 1. Let $S(\alpha)$ be a random sum in the sense that the coefficients $a_n \in \{1,-1\}$ are chosen at random. We compute the expectation of the fourth moment of $S(\alpha)$.

$$\begin{aligned}\mathrm{E}\int_0^1 |S(u)|^4\,du &= \mathrm{E}\sum_{\substack{\mu_1+\mu_2=\nu_1+\nu_2\\ \mu_1,\mu_2,\nu_1,\nu_2\leq N}} a_{\nu_1}a_{\nu_2}a_{\mu_1}a_{\mu_2}\\ &= \#\big\{\mu_1,\mu_2,\nu_1,\nu_2\leq N : \{\mu_1,\mu_2\}=\{\nu_1,\nu_2\}\big\}\\ &= 2N^2-N.\end{aligned}$$

If $m\subseteq[0,1]$ is of measure x, then $\int\limits_m |S(u)|^2\,du \leq \sqrt{x}\big(\int\limits_m |S(u)|^4\,du\big)^{1/2}$, thus $\mathrm{E}M(x)\leq\sqrt{2x}$. In particular, we have $M(x)\leq 1/2$ with probability $\geq 1-\sqrt{8x}$. Let $\delta>0$ be given, and set $A=8\delta^{-2}$. Then with probability $\geq 1-\delta$ we have $M(1/A)\leq 1/2$, and (3) becomes

$$\begin{aligned}\sum_{q\leq Q}\sum_{(a,q)=1}\left|S\left(\frac{a}{q}\right)\right|^2 &\geq \left(\frac{Q^2\delta^2}{16}-48\delta^{-2}\pi N\right)\sum_{n\leq N}|a_n|^2\\ &\geq \frac{Q^2\delta^2}{32}\sum_{n\leq N}|a_n|^2\,,\end{aligned}$$

provided that $Q^2>1536\delta^4 N$. Hence, for fixed ϵ, the relation (2) becomes true with probability $1-\sqrt{1024\epsilon}$, provided that Q^2/N is sufficiently large. Hence, our claim follows.

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