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# ON THE NON-INVARIANCE OF SPAN AND IMMERSION CO-DIMENSION FOR MANIFOLDS

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Abstract. In this note we give examples in every dimension $m \geq 9$ of piecewise linearly homeomorphic, closed, connected, smooth $m$-manifolds which admit two smoothness structures with differing spans, stable spans, and immersion co-dimensions. In dimension 15 the examples include the total spaces of certain 7-sphere bundles over $S^8$. The construction of such manifolds is based on the topological variance of the second Pontrjagin class: a fact which goes back to Milnor and which was used by Roitberg to give examples of span variation in dimensions $m \geq 18$.

We also show that span does not vary for piecewise linearly homeomorphic smooth manifolds in dimensions less than or equal to 8, or under connected sum with a smooth homotopy sphere in any dimension. Finally, we use results of Morita to show that in all dimensions $m \geq 19$ there are topological manifolds admitting two piecewise linear structures having different $PL$-spans.

## 1. Introduction

We shall use the notation $M$ for a closed, connected, topological manifold, $M_A, M_B, \ldots$ for $M$ together with a given piecewise linear (henceforth $PL$) structure, and $M_\alpha, M_\beta, \ldots$ for $M$ together with a given smoothness structure. Recall that for a smooth $m$-dimensional manifold $M_\alpha$, two basic and classical geometric invariants are its span and its immersion co-dimension. The span is the maximal number $r$ such that $M_\alpha$ admits $r$ pointwise linearly independent vector fields, while the immersion co-dimension is the least $k$ such that $M_\alpha$ immerses in $\mathbb{R}^{m+k}$. Clearly $0 \leq r \leq m$, and from the Whitney Immersion Theorem (together with the fact that a closed $m$-manifold cannot immerse in dimension $m$), one has $1 \leq k \leq m-1$. A fundamental question is whether these two invariants can differ for distinct smooth structures, $M_\alpha$ and $M_\beta$, on the same $PL$-manifold $M_A$. An affirmative answer was first given by Roitberg [22] in 1969, in all dimensions $m \geq 18$. In this paper we use smoothing theory to settle this question in all dimensions: we give an affirmative answer for dimensions $m \geq 9$ and show that span and immersion co-dimension are $PL$ invariants in dimensions less than or equal to 8.

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Let us first fix some definitions and notation. For a vector bundle $\xi$ over a space $X$, we define

$$\operatorname{span}(\xi) := \max\{r : \xi \approx r\varepsilon \oplus \eta\}$$

where $\approx$ denotes isomorphism of vector bundles, $r\varepsilon$ denotes the trivial bundle of rank $r$ and $\eta$ is some other vector bundle over $X$. This is the same as the maximal number of pointwise linearly independent sections of $\xi$, and if $\xi$ is of rank $m$, then clearly $0 \leq \operatorname{span}(\xi) \leq m$. We also write $m - \operatorname{span}(\xi) = \operatorname{gd}(\xi)$, the geometric dimension of $\xi$, and this clearly equals $\operatorname{rank}(\eta)$. Replacing isomorphism $\approx$ by stable isomorphism $\sim$ in the above definitions gives the corresponding notions of stable span and stable geometric dimension, written respectively $\operatorname{span}^0$, $\operatorname{gd}^0$. Writing $\xi^0$ for the stable vector bundle represented by $\xi$ we also define $\operatorname{span}(\xi^0) := \operatorname{span}^0(\xi)$ and similarly for geometric dimension. Evidently

$$0 \leq \operatorname{span}(\xi) \leq \operatorname{span}^0(\xi) = \operatorname{span}(\xi^0) \leq m\,, \quad m \geq \operatorname{gd}(\xi) \geq \operatorname{gd}^0(\xi) = \operatorname{gd}(\xi^0) \geq 0\,.$$

We remark that in the literature "geometric dimension" is often used to denote what we are calling "stable geometric dimension". Let $M_\alpha$ be a smooth $m$-dimensional manifold with underlying topological manifold $M$. With the above definitions, the span (resp. stable span) of $M_\alpha$ is simply the span (resp. stable span) of its tangent bundle $\tau_\alpha = \tau(M_\alpha)$, i.e.

$$\operatorname{span}(M_\alpha) := \operatorname{span}(\tau_\alpha)\,, \quad \operatorname{span}^0(M_\alpha) := \operatorname{span}^0(\tau_\alpha)\,.$$

The manifold $M$ is also a CW-complex of dimension $m = \operatorname{rank}(\tau)$, it is then useful to note that by standard stability properties of vector bundles (cf. [8, Ch. 9]), $\operatorname{span}^0(M_\alpha) = \max\{r : \tau_\alpha \oplus \varepsilon \approx (r+1)\varepsilon \oplus \eta\}$. The notation $M^{(k)}$ will be used, as usual, to denote the $k$-skeleton of $M$.

Turning to the normal bundle $\nu^0_\alpha = \nu^0(M_\alpha)$ (which is a stable bundle), the Hirsch immersion theorem states that the immersion co-dimension $k$ of $M_\alpha$ is given by the formula $k = \max\{1, \operatorname{gd}(\nu^0_\alpha)\}$. The stable isomorphism $\tau^0_\alpha \oplus \nu^0_\alpha \sim 0$ suggests a possible relation between the stable span and the immersion co-dimension. For interesting inequalities relating these with the Lyusternik-Schnirel'man category of $M$ we refer the reader to Korbaš and Szűcs, [12].

Now let $M_A$ be the $PL$-manifold underlying $M_\alpha$ and let $\mathcal{C}(M_A)$ denote the finite set of concordance classes of smooth structures on $M_A$ (see Section 2). We define the *smooth span variation* of $M_A$ to be to be the maximal difference of spans over all the smooth structures on $M_A$ and similarly define the *smooth stable span variation of* $M_A$:

$$\operatorname{ssv}(\mathrm{M_A}) := \max\{\operatorname{span}(M_\alpha) \mid [M_\alpha] \in \mathcal{C}(M_A)\} - \min\left\{\operatorname{span}(M_\alpha) \mid [M_\alpha] \in \mathcal{C}(M_A))\right\},$$

$$\operatorname{ss^0v}(M_A) := \max\{\operatorname{span}^0(M_\alpha) \mid [M_\alpha] \in \mathcal{C}(M_A)\} - \min\{\operatorname{span}^0(M_\alpha) \mid [M_\alpha] \in \mathcal{C}(M_A)\}\,.$$

Evidently $\operatorname{ssv}(M_A)$ and $\operatorname{ss^0v}(\mathrm{M_A})$ are invariants of the $PL$-homeomorphism type of $M_A$. We also note that both span variations can be defined to give topological

invariants of $M$ by replacing $\mathcal{C}(M_A)$ with $\mathcal{C}(M)$, the finite set of concordance classes of smooth structures on $M$: we write $\mathrm{ssv}(M)$ and $\mathrm{ss}^0\mathrm{v}(M)$. Of course $\mathrm{ssv}(M) \geq \mathrm{ssv}(M_A)$ and $\mathrm{ss}^0\mathrm{v}(M) \geq \mathrm{ss}^0\mathrm{v}(M_A)$. As an example, if $M$ is a manifold with non-zero Euler characteristic (whence $\dim(M)$ is necessarily even), then the tangent bundle of every smooth structure on $M$ admits no nowhere zero sections so $\mathrm{ssv}(M) = \mathrm{ssv}(M_A) = 0$. If also the Euler characteristic of $M$ is odd then by [13, Theorem 2.2] we even have that $\mathrm{ss}^0\mathrm{v}(M) = \mathrm{ss}^0\mathrm{v}(M) = 0$.

We mention one of the reasons why span variation is surprising: by definition the span of a smooth manifold $M_\alpha$ depends upon its tangent bundle $\tau_\alpha$ and a result of Atiyah [1] says that the stable spherical fibration associated to the tangent bundle of a smooth manifold is in fact a homotopy invariant. This was later strengthened by Dupont [6], and by Benlian-Wagoner [2], so that the word "stable" may be omitted. Thus the examples of Theorem 1.1 below and of Roitberg entail span variation amongst vector bundles in the kernel of the $J$-homomorphism.

We now state our main theorems for span, where we use $\sharp$ to denote the connected sum of locally oriented, smooth manifolds and $S^m_0$ to denote the standard smooth $m$-sphere. Analogous results hold for immersion co-dimension.

**Theorem 1.1.** *In every dimension $m \geq 9$ there are $PL$-manifolds $M_A$ for which $\mathrm{ssv}(M_A) \geq 4$ and $\mathrm{ss}^0\mathrm{v}(M_A) \geq 4$.*

**Theorem 1.2.**

(a) *Let $M$ be a topological manifold with $\dim(M) \leq 8$ which admits a $PL$-structure $M_A$. Then $\mathrm{ssv}(M_A) = \mathrm{ss}^0\mathrm{v}(M_A) = 0$. If also $H^3(M; \mathbb{Z}/2) = 0$ then $\mathrm{ssv}(M) = \mathrm{ss}^0\mathrm{v}(M) = 0$.*

(b) *For every oriented homotopy sphere $S^m_\sigma$, and every locally oriented smooth manifold $M_\alpha$, $\mathrm{span}(M_\alpha) = \mathrm{span}(M_\alpha \# S^m_\sigma)$. In particular for every homotopy sphere $\mathrm{span}(S^m_\sigma) = \mathrm{span}(S^m_0)$.*

**Remark 1.3.** All of the manifolds we find for Theorem 1.1 admit a smooth structure $M_\alpha$ which is parallelisable and another smooth structure $M_\beta$ with non-vanishing second Pontrjagin class, $p_2(M_\beta) \neq 0$. This explains the 4, since $p_2(\xi) = 0$ for any vector bundle with stable geometric dimension less than 4. It was also stated in [19] that the second Pontrjagin class is not a topological invariant for closed manifolds, and a recent proof appears in [15].

One can also define the span and stable span of $CAT$-manifolds for $CAT = PL$ or $Top$ as well as for smooth manifolds where $CAT = O$ (see [25] for the topological case and also [21]). Let $CAT(k)$ be the group of $CAT$-isomorphisms of $\mathbb{R}^k$ fixing zero. An $m$-dimensional $CAT$ manifold $M_{\mathcal{A}}$ has a $CAT$-tangent bundle $\tau(M_{\mathcal{A}})$ and a stable $CAT$-bundle $\tau^0(M_{\mathcal{A}})$. The span of $M_{\mathcal{A}}$ equals $j$ if the principal $CAT(m)$-bundle associated to $\tau(M_{\mathcal{A}})$ has a $CAT(m-j)$ reduction but no $CAT(m-j-1)$-reduction. The stable span of $M_{\mathcal{A}}$ is $j$ if the same is true of the principal $CAT$-bundle associated to $\tau^0(M_{\mathcal{A}})$. Analogously to the case of smooth span variations, we obtain the $PL$-span variations of a topological manifold $M$ by setting $\mathcal{C}_{PL}(M)$ to be the finite set of concordance classes of $PL$-structures on $M$

and defining

$$\mathrm{plsv}(M) := \max\{\mathrm{span}(M_C) \mid [M_C] \in \mathcal{C}_{PL}(M)\} - \min\{\mathrm{span}(M_C) \mid [M_C] \in \mathcal{C}_{PL}(M)\},$$

$$\mathrm{pls}^0\mathrm{v}(M) := \max\{\mathrm{span}^0(M_C) \mid [M_C] \in \mathcal{C}_{PL}(M)\} - \min\{\mathrm{span}^0(M_C) \mid [M_C] \in \mathcal{C}_{PL}(M)\}.$$

In [18] Morita discovered topological manifolds $M$ in each dimension $m \geq 22$ which admit $PL$ structures $M_A$ and $M_B$ which cannot both be smoothed. It is a relatively simple matter to combine Morita's resuls with a theorem of Wall [26] to prove

**Theorem 1.4.** *In all dimensions $m \geq 19$ there are topological manifolds $M$ such that $\mathrm{plsv}(M) > 0$ and $\mathrm{pls}^0\mathrm{v}(M) > 0$.*

The remainder of the paper is organised as follows. In Section 2 we review the smoothing theory we need and prove Theorem 1.2. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.4. We now conclude the introduction with a list of open problems concerning span variation.

**Problem 1.5** (Problems about span variation and span)**.** Let $M$ be a closed topological manifold. We state these problems for $\mathrm{ssv}(M)$ and $\mathrm{plsv}(M)$ for brevity but the analogous problems are open and interesting for $\mathrm{ss}^0\mathrm{v}(M)$ and $\mathrm{pls}^0\mathrm{v}(M)$, as well as for immersion co-dimension.

(1) Relate $\mathrm{ssv}(M)$ to other topological invariants of $M$.
(2) For a dimension $m$, determine the largest $\mathrm{ssv}(M)$ for an $m$-dimensional manifold.
(3) If possible, find families of manifolds $M_i$ such that $\lim_{i\to\infty}\mathrm{ssv}(M_i) = \infty$.
(4) Find a manifold $M$ where the spherical fibration associated to $\tau(M)$ is non-trivial and $\mathrm{ssv}(M) > 0$.
(5) Determine the dimensions $m$ for which $\mathrm{plsv}(M^m) = 0$ is always zero. This relates to the next problem.
(6) Determine whether the assumption that $H^3(M; \mathbb{Z}/2) = 0$ can be removed from the second part of Theorem 1.2 (a).
(7) Compute $\mathrm{ssv}(M)$ for well known manifolds. In particular, for the total spaces of 7-bundles over $S^8$. This relates to the next problem.
(8) Determine the span of stably parallelisable topological 15-manifolds. (Bredon and Kosinski calculated the span of stably parallelisable smooth manifolds in [3]. In [25] Varadarajan extended their result to stably parallelisable topological manifolds except in dimension 15.)

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## 2. A rapid review of smoothing theory

Recall the notation established in the introduction: $M_\alpha$ is a closed, connected smooth manifold with underlying $PL$-manifold $M_A$ and underlying topological manifold $M$. In this section we review the implications of Cairns-Hirsch smoothing theory for the question of whether the smooth span of $M_\alpha$ depends upon the choice of smooth structure $\alpha$. We use [16] as our reference for smoothing theory and for further details relating to this brief review.

A concordance between smooth structures $M_\alpha$ and $M_\beta$ is a smooth structure on $M_A \times [0,1]$, compatibile with the $PL$ structure of $M_A \times [0,1]$, which restricts to $M_\alpha$ on $M_A \times \{0\}$ and to $M_\beta$ on $M_A \times \{1\}$. The set of concordance classes of smooth structures on $M_A$ is denoted by $\mathcal{C}(M_A)$, and $[M_\alpha] \in \mathcal{C}(M_A)$ will denote the equivalence class of $M_\alpha$, i.e. the set of all $M_\beta$ refining $M_A$ that are concordant to $M_\alpha$. We are interested in the difference a choice of smooth structure can make to the smooth tangent bundle considered as an abstract vector bundle up to isomorphism. Notice that if $M_\alpha$ and $M_\beta$ are concordant, then their tangent bundles are stably equivalent. The following lemma implies that this remains true unstably.

**Lemma 2.1.** *Let $M_\alpha$ and $M_\beta$ be smooth structures on the topological manifold $M$. Then $\tau(M_\alpha) \sim \tau(M_\beta)$ if and only if $\tau(M_\alpha) \approx \tau(M_\beta)$.*

**Proof.** One implication is trivial, so let $\tau(M_\alpha)$ and $\tau(M_\beta)$ be classified by $f_\alpha \colon M \to BO(m)$ and $f_\beta \colon M \to BO(m)$, and suppose these bundles are stably equivalent. Then they agree over $M^{(m-1)}$. Now let $O_{\alpha,\beta} \in H^m(M;K)$ be the obstruction to a homotopy $f_\alpha \simeq f_\beta$, where $K = \mathrm{Ker}\big(\pi_{m-1}(O(m)) \to \pi_{m-1}(O)\big) \cong 0, \mathbb{Z}/2, \mathbb{Z}$, corresponding to $m \in \{1,3,7\}$, or $m$ odd and $m \notin \{1,3,7\}$, or $m$ even, respectively. We now show this obstruction vanishes in turn for the cases: $m$ is odd, $m$ is even with $M$ orientable, and $m$ is even with $M$ non-orientable.

If $m = 2r+1$ is odd, it follows from [9] that there are either one or two isomorphism classes of rank $m$ vector bundles over $M$, stably equivalent to $\tau(M_\alpha)$, this number being called the James-Thomas number. If the James-Thomas number is one then automatically $\tau(M_\alpha) \approx \tau(M_\beta)$. On the other hand, if this number is two, then the two isomorphism classes are distinguished by the Browder-Dupont invariant $b_B$, cf. [24]. But according to [24], $b_B\big(\tau(M_\alpha)\big)$ and $b_B\big(\tau(M_\beta)\big)$ must both equal the mod-2 Kervaire semi-characteristic $\chi_2(M) := \Sigma_{i=0}^r \mathrm{rank}\big(H^i(M;\mathbb{Z}/2)\big) \pmod 2$, so $O_{\alpha,\beta} = 0$.

If $m$ is even and $M$ is orientable then $O_{\alpha,\beta}$ lies in $H^m(M;\mathbb{Z})$, where the coefficients are untwisted. In this case $O_{\alpha,\beta}$ measures the difference in the Euler classes of the bundles $\tau(M_\alpha)$ and $\tau(M_\beta)$, but these are both determined by the Euler characteristic of $M$ and hence the same. Thus $O_{\alpha,\beta}$ vanishes.

If $m$ is even and non-orientable let $\omega \colon \pi_1(M) \twoheadrightarrow \mathbb{Z}/2 = \{1,-1\}$ be the first Stiefel-Whitney class. In this case $O_{\alpha,\beta} \in H^m(M;\widetilde{\mathbb{Z}})$ where the coefficients are twisted and $\widetilde{\mathbb{Z}}$ denotes the $\mathbb{Z}[\pi_1(M)]$-module with $g \in \pi_1(M)$ acting via multiplication by $\omega(g)$. By twisted Poincaré duality (see, for example, [5, §5]), $H^m(M;\widetilde{\mathbb{Z}}) \cong H_0(M;\mathbb{Z}) \cong \mathbb{Z}$. Now let $p \colon \widetilde{M} \twoheadrightarrow M$ denote the orientation double cover of $M$ and $\widetilde{M_\alpha}$, $\widetilde{M_\beta}$ the corresponding smooth structures on $\widetilde{M}$ induced via $p$. Of course

the classifying map for $\tau(\widetilde{M}_{\widetilde{\alpha}})$ is $f_\alpha \circ p$ and similarly for the classifying map of $\tau(\widetilde{M}_{\widetilde{\beta}})$. We write $O_{\widetilde{\alpha},\widetilde{\beta}}$ for the obstruction to a homotopy of the classifying map for $\tau(\widetilde{M}_{\widetilde{\alpha}})$ to that of $\tau(\widetilde{M}_{\widetilde{\beta}})$, which is zero by the oriented case. The covering map $p$ induces $p^* : H^m(M;\widetilde{\mathbb{Z}}) \to H^m(\widetilde{M};\mathbb{Z})$ where the latter coefficients are untwisted and we have that $p^*(O_{\alpha,\beta}) = O_{\widetilde{\alpha},\widetilde{\beta}}$. Since $p^*$ is induced by a double covering it is isomorphic to $\times 2\colon \mathbb{Z} \to \mathbb{Z}$ and we conclude that $O_{\alpha,\beta} = 0$. □

Let us now define the following sets of isomorphism classes of vector bundles and stable vector bundles:

$$Tv(M_A) := \big\{[\tau(M_\alpha)] \mid [M_\alpha] \in \mathcal{C}(M_A)\big\}$$

and

$$T^0v(M_A) := \big\{[\tau^0(M_\alpha)] \mid [M_\alpha] \in \mathcal{C}(M_A)\big\}.$$

Observe that Lemma 2.1 shows that there is a bijection $T^0v(M_A) \equiv Tv(M_A)$. We first show that $Tv(M_A)$ is a singleton in dimensions $m \leq 4$.

**Lemma 2.2.** *Let $h\colon M_\alpha \to N_\beta$ be a homotopy equivalence between smooth $m$-manifolds with $m \leq 4$. Then $h$ preserves the tangent bundles; i.e. $h^*\big(\tau(N_\beta)\big) \approx \tau(M_\alpha)$.*

**Proof.** By Lemma 2.1 it is enough to show that $h^*\big(\tau^0(N_\beta)\big) \sim \tau^0(M_\alpha)$. Let $f_\alpha\colon M \to BO$ and $g_\beta\colon N \to BO$ classify the stable tangent bundles of $M_\alpha$ and $N_\beta$, let $p\colon BO \to BG$ be the canonical fibration, and let $i\colon G/O \to BO$ be the inclusion of a fibre. By [1], $h$ preserves the stable spherical fibrations underlying $\tau^0(M_\alpha)$ and $\tau^0(N_\beta)$ and so $p \circ f_\alpha$ is homotopic to $p \circ g_\beta \circ h$. As $p$ is an isomorphism on $\pi_1$ and $\pi_2$ and as $\pi_3(BO) = 0$, $f_\alpha$ and $g_\beta \circ h$ agree on $M^{(3)}$. Hence the lemma holds in dimensions $m \leq 3$.

Now assume that $\dim(M) = 4$. There is a cohomology class $O_{\alpha,\beta} \in H^4\big(M;\pi_4(BO)\big)$ which is the obstruction to a homotopy from $f_\alpha$ to $g_\beta \circ h$. The coefficients are untwisted since $\pi_1(BO)$ acts trivially on $\pi_4(BO)$. Moreover we see that $O_{\alpha,\beta}$ lies in the image of the map from $H^4\big(M;\pi_4(G/O)\big)$. If $M$ is not orientable then $H^4\big(M;\pi_4(G/O)\big)$ and $H^4\big(M;\pi_4(BO)\big)$ are both isomorphic to $\mathbb{Z}/2$ but the map $\pi_4(G/O) \to \pi_4(BO)$ is multiplication by 24, and since $O_{\alpha,\beta}$ lifts to $H^4\big(M;\pi_4(G/O)\big)$ it must vanish. If $M$ and $N$ are orientable then orient them so that $h$ is orientation preserving and repeat the above argument replacing $BO$ and $BG$ respectively by $BSO$ and $BSG$, and using the classifying maps of the oriented tangent bundles. The class $O_{\alpha,\beta}$ is now detected by the difference of the Pontrjagin classes $p_1\big(\tau^0(M_\alpha)\big) - h^*\big(p_1(\tau^0(N_\beta))\big)$ but by the signature theorem these classes agree since $h$ is an orientation preserving homotopy equivalence from $M$ to $N$. Hence $\tau^0(M_\alpha)$ and $h^*\big(\tau^0(M_\beta)\big)$ may be oriented so that they become isomorphic oriented stable vector bundles and so, in particular, they are isomorphic. □

We now recall how smoothing theory calculates $T^0v(M_A)$ and hence $Tv(M_A)$ in dimensions $m \geq 5$. Fixing a smooth structure, $M_\alpha$, makes $\mathcal{C}(M_A)$ into a pointed set denoted $\mathcal{C}(M_\alpha)$. A fundamental result of smoothing theory is the following

**Theorem 2.3** (Cairns-Hirsch, see [16, Theorem 7.2])**.** *Let $M_\alpha$ be a smooth manifold of dimension at least* 5*, then there is a bijection*

$$\Psi_\alpha \colon \mathcal{C}(M_A) \equiv [M, PL/O]$$

*which takes the base point $[M_\alpha]$ to the homotopy class of the constant map.*

Recall that $PL/O$ has a commutative $H$-space structure which makes the fibration $PL/O \to BO \to BPL$ into a sequence of $H$-space maps where $BO$ and $BPL$ have compatible commutative $H$-space structures coming from the Whitney sum of bundles [16][p 92]. Associated to this fibration we have the long exact Puppe sequence of abelian groups, for any space $X$,

$$\ldots \longrightarrow [X, PL] \longrightarrow [X, PL/O] \xrightarrow{\partial_X} [X, BO] \longrightarrow [X, BPL].$$

When $X = M$ is homeomorphic to a smooth manifold $M_\alpha$, $\partial_M$ computes the difference a smooth structure makes to the isomorphism class of the stable tangent bundle. That is, for the appropriate choice of $\Psi_\alpha$,

$$\partial_M\big(\Psi_\alpha(M_\beta)\big) = [\tau^0(M_\alpha)] - [\tau^0(M_\beta)] \in \widetilde{KO}(M) = [M, BO]\,.$$

Combining Lemma 2.2, the fact that $PL/O$ is 6-connected and the above identity we deduce

**Lemma 2.4.** *The group* $\mathrm{Im}(\partial_M)$ *acts freely and transitively on* $T^0v(M_A)$*.*

Applying Lemma 2.1 we immediately obtain

**Corollary 2.5.** If $\partial_M = 0$ then $Tv(M_A)$ and $T^0v(M_A)$ are singletons and so $\mathrm{ssv}(M_A) = \mathrm{ss}^0\mathrm{v}(M_A) = 0$.

**Proof of Theorem 1.2.** Lemma 2.2 implies both parts in dimensions $m \leq 4$. So we now assume that $m \geq 5$ and start with part (b). If $M = S^m$, then it is known [**?**] that $\pi_m(PL) \to \pi_m(PL/O)$ is surjective and so $\partial_{S^m} = 0$. It follows that every exotic sphere gives rise to the same tangent bundle as the usual one (a fact already observed in [20]). Now for any smooth locally oriented manifold $M_\alpha$ and any homotopy $m$-sphere $S^m_\sigma$ we have $M_{\alpha+\sigma} := M_\alpha \sharp S^m_\sigma$. Using smoothing theory we identify the smooth structure $\alpha + \sigma$ as follows. Identify $\mathcal{C}(S^m) = \pi_m(PL/O)$ using the standard smooth structure $S^m_0$ on the sphere so that $\sigma \in \pi_m(PL/O)$ corresponds to the exotic sphere $S^m_\sigma$ under the bijection $\Psi_0$, and let $c : M \to S^m$ be the collapse map taking an open $m$-disc in $M$ homeomorphically onto $S^m \setminus \{\mathrm{pt}\}$ and all points outside the open $m$-disc to pt. By definition we have that $\Psi_\alpha^{-1}(c^*\sigma) = M_{\alpha+\sigma}$. Now the induced maps $c^* : \pi_m(PL/O) \to [M, PL/O]$ and $c^* : \pi_m(BO) \to [M, BO]$ give rise to the following commutative diagram:

$$\begin{array}{ccc} \pi_m(PL/O) & \xrightarrow{\partial_{S^m}} & \pi_m(BO) \\ \downarrow{\scriptstyle c^*} & & \downarrow{\scriptstyle c^*} \\ [M, PL/O] & \xrightarrow{\partial_M} & [M, BO]\,. \end{array}$$

It follows that

$$\partial_M\big(\Psi_\alpha(M_{\alpha+\sigma})\big) = \partial_M\big(c^*(\sigma)\big) = c^*\big(\partial_{S^m}(\sigma)\big) = c^*(0) = 0\,.$$

Thus $\tau^0(M_\alpha) \sim \tau^0(M_{\alpha+\sigma})$. By Lemma 2.1 we have that $\tau(M_\alpha) \approx \tau(M_{\alpha+\sigma})$ and so $\mathrm{span}(M_\alpha) = \mathrm{span}(M_{\alpha+\sigma})$. This concludes the proof of part $(b)$.

We now prove part $(a)$. For the $PL$-statement, since $m \geq 5$ we apply Theorem 2.3. As $PL/O$ is 6-connected, if $M_A$ is 5 or 6 dimensional then $M_A$ admits a unique smooth structure. If $M_A$ is of dimension 7 then Theorem 2.3 implies that all smooth structures are obtained from a fixed one by connected sum with a homotopy 7-sphere and so by part (b) don't alter the span. If $M$ is 8-dimensional it suffices, by Corollary 2.5, to show that $\partial_M = 0$. As usual, let $M$ be the topological manifold underlying $M_A$ and let $M^{(6)}$ be the 6-skeleton of a CW-decomposition for $M$ containing just one 8-cell. Such a decomposition exists by [27]. As $PL/O$ is 6-connected, $[M/M^{(6)}, PL/O] \twoheadrightarrow [M, BO]$ is surjective and thus the image of $\partial_M$ lies in $\mathrm{Im}([M/M^{(6)}, BO] \to [M, BO])$. If $M$ is orientable then $M/M^{(6)} \simeq (\vee S^7)\vee S^8$ is homotopy equivalent to a wedge of 7-spheres and an 8-sphere, then $\partial_M$ splits as the sum of $\partial_{S^7}$'s and $\partial_{S^8}$ but these are zero. If $M$ is not orientable then $M/M^{(6)} \simeq M(\mathbb{Z}/2,7)\vee(\vee S^7)$ is homotopy equivalent to a degree 7 Moore space wedged with a wedge of 7-spheres. Since the short exact sequence $\pi_7(O) \to \pi_7(PL) \to \pi_7(PL/O)$ (see Section 2) splits at the prime 2 it again follows that $\partial_M = 0$.

It remains to prove that $\mathrm{ssv}(M) = 0$ if $H^3(M;\mathbb{Z}/2) = 0$, in dimensions $5 \leq m \leq 8$. In dimensions $m \geq 5$ there is a smoothing theory for $PL$-structures on topological manifolds which is analogous to the smoothing theory for smooth structures on $PL$-manifolds we sketched above. In particular the set of concordance classes of $PL$-structures on $M$, $\mathcal{C}_{PL}(M)$, corresponds bijectively with $[M, TOP/PL]$. Moreover, the fundamental work of [11] shows that $TOP/PL$ is homotopy equivalent to the Eilenberg-MacLane space $K(\mathbb{Z}/2,3)$. Hence the assumption that $H^3(M;\mathbb{Z}/2) = 0$ ensures that there is a unique concordance class $[M_A]$ of $PL$ structures on $M$. Thus the span variations for $M$ and the span variations for $M_A$ are zero by the $PL$ case. □

We remark that our proof in fact shows

**Corollary 2.6.** Let $M_A$ be a $PL$-manifold of dimension $m \leq 8$. Then $|Tv(M_A)| = 1$.

Turning our attention now to higher dimensions, if there is a $PL$-manifold $M_A$ with $\partial_M \neq 0$ and which admits a parallelisable smooth structure $M_\alpha$, i.e. $\tau(M_\alpha) \approx m\varepsilon$, then there will be a smooth structure $M_\beta$ such that $\tau^0(M_\beta)$ is non-trivial and so $\mathrm{span}(M_\beta) \leq \mathrm{span}^0(M_\beta) < m$. However, $\mathrm{span}(M_\alpha) = \mathrm{span}^0(M_\alpha) = m$, so in such a case both $\mathrm{ssv}(M_A) > 0$ and $\mathrm{ss}^0\mathrm{v}(M_A) > 0$. In the next section we produce examples of this sort.

## 3. $PL$-Manifolds with varying smooth spans

In this section we give examples of $PL$-manifolds $M_A$ in dimensions 9 and higher with $\mathrm{ssv}(M_A) \geq 4$ and $\mathrm{ss}^0\mathrm{v}(M_A) \geq 4$. Let $M(C_k,1) = S^1 \cup_k e^2$ be the degree 1 Moore space with first homology group cyclic of order $k$. As $M(C_k,1)$ is a 2-dimensional complex it can be embedded into $\mathbb{R}^5$; we take an embedding

into $\mathbb{R}^{10}$ and then take a regular neighbourhood of $M(C_k, 1)$, $T^{10}_\alpha(k)$, which is a compact, smooth, parallelisable 10-manifold with boundary. Here $\alpha$ is the induced smoothness structure coming from the standard one on $\mathbb{R}^{10}$. Let $N^9_\alpha(k)$ be the boundary of $T^{10}_\alpha(k)$. We see that $N^9_\alpha(k)$ is a closed, connected, smooth stably parallelisable 9-manifold and we write $N^9_A(k)$ for the underlying $PL$-manifold.

Before starting the next theorem, we recall (following [3]) the definitions of the semi-characteristic $\chi^*(M)$ and the reduced semi-characteristic $\widehat{\chi}(M)$ of a manifold $M$. If $\dim(M)$ is even then $\chi^*(M)$ is the half-integer $\chi(M)/2$ where $\chi(M)$ is as usual the Euler characteristic of $M$. If $\dim(M)$ is odd then $\chi^*(M) \in \mathbb{Z}/2$ is equal to $\chi_2(M)$, the mod-2 Kervaire semi-characteristic (defined in the proof of Lemma 2.1). The reduced semi-characteristic is defined to be $\widehat{\chi}(M) = 1 - \chi^*(M)$ and satisfies $\widehat{\chi}(M_0 \sharp M_1) = \widehat{\chi}(M_0) + \widehat{\chi}(M_1)$. For example: $\widehat{\chi}(S^1 \times S^m) = 1$ if $m \geq 1$ and $\widehat{\chi}\big(N^9_\alpha(k)\big) = 0$. We also orient the manifolds $N^9_A(k)$ and use the notation $M\#_j T = M\#T\# \underset{j}{\cdot\,\cdot\,\cdot} \#T$ for the connected sum of $M$ with $j$ copies of an oriented manifold $T$, for any choice of $CAT = O, PL, Top$.

**Theorem 3.1.**

(1) *Let $n \geq 0$ and $W^n_B$ be any closed, oriented $PL$-$n$-manifold admitting a stably parallelisable smooth structure. Assume that 7 divides $k$ and set $l = \chi^*\big(N^9_A(k) \times W^n_B\big)$. Then for all $j \geq 0$*

$$\mathrm{ss}^0\mathrm{v}\big((N^9_A(k)\times W^n_B)\sharp_j(S^1\times S^{n+8})\big) \geq 4 \quad \textit{and} \quad \mathrm{ssv}\big((N^9_A(k)\times W^n_B)\sharp_l(S^1\times S^{n+8})\big) \geq 4\,,$$

*where we regard $S^1 \times S^{n+8}$ as a $PL$ manifold.*

(2) *Let $\xi$ be a linear 7-sphere bundle over $S^8$ and let $P^{15}_A$ be the $PL$-manifold underlying the total space of $\xi$. If the total space of $\xi$ is stably parallelisable and 14 divides the Euler class of $\xi$, $e(\xi) \in H^8(S^8; \mathbb{Z}) \cong \mathbb{Z}$, then $\mathrm{ssv}(P^{15}_A) \geq 4$ and $\mathrm{ss}^0\mathrm{v}(P^{15}_A) \geq 4$.*

**Remark 3.2.** Of course in part (1) above one may take $W^0_B$ to be a point, and $W^n_B = S^n$, $n > 0$. Furthermore, $l \in \mathbb{Z}$ because $\mathrm{span}^0(N^9_A(k) \times W^n_B) = 9 + n > 0$ implies $\chi(N^9_A(k) \times W^n_B)$ is even. The idea of taking neighbourhoods of appropriate Moore spaces to find examples of homeomorphic smooth manifolds with differing tangent bundles goes back to Milnor [17]. Roitberg [22] doubled compact neighbourhoods of Moore spaces of degree at least 7 to exhibit smooth span variation for closed manifolds in dimensions 18 and higher. We are able to get examples down to dimension 9 by using a degree 1 Moore space so that a "dual" Moore space appears in dimension 7. In (2), note that $E(\xi)$ has a standard smoothness structure because it is a linear 7-sphere bundle.

**Remark 3.3.** Total spaces as in Theorem 3.1 (2) exist: in the notation of [23, §2] take any 7-sphere bundle $\xi_{h,j} \in \pi_7\big(SO(8)\big) \cong \mathbb{Z} \oplus \mathbb{Z}$ with $(h, j) = (7k, 7k)$ and $k \neq 0$. By [23] the corresponding total spaces are almost parallelisable and hence stably parallelisable since $\pi_{14}(O) = 0$ (or cf. [14, Ch. 9 (8.5)]). We do not resolve whether the non-stably parallelisable smooth structures in this case are also realised as the total spaces of 7-sphere bundles over $S^8$.

**Proof of Theorem 3.1.** Let $M^m_A$ be any manifold satisfying the hypotheses of the theorem. By assumption $M_A$ admits a stably parallelisable smooth structure $M_\alpha$, so $\mathrm{span}^0(M_\alpha) = m$. If, in addition, the semi-characteristic $\chi^*(M)$ vanishes then [3] asserts that $\mathrm{span}(M_\alpha) = m$ and it is a simple matter (using the addition formula for the reduced semicharacteristic $\widehat{\chi}$ under connected sums, as well as $\widehat{\chi}(S^1 \times S^{n+8}) = 1$) to check that the additional hypotheses in the theorem ensure that the semi-characteristic vanishes. We will show that each $M_A$ admits a smooth structure $M_\beta$ with non-zero second Pontrjagin class, $p_2(M_\beta) \neq 0$. The theorem then follows since any smooth $m$-manifold with stable span greater than $m-4$ has vanishing second Pontrjagin class, which shows

$$\mathrm{span}(M_\beta) \leq \mathrm{span}^0(M_\beta) \leq m - 4\,.$$

It remains to show the existence of a smooth structure $\beta$ with $p_2(M_\beta) \neq 0$. We may therefore specialize to the case where $M^m_A$ is one of $N^9_A(k)$ or $P^{15}_A$ using the product formula for the Pontrjagin classes of the manifolds in Theorem 3.1 (1). First recall [4, 7] that the homotopy exact sequence

$$0 \to \pi_7(O) \longrightarrow \pi_7(PL) \longrightarrow \pi_7(PL/O) \to 0$$

is isomorphic to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(7,1)} \mathbb{Z} \oplus \mathbb{Z}/4 \xrightarrow{\binom{-1}{7}} \mathbb{Z}/28 \longrightarrow 0\,.$$

We denote the Bockstein homomorphism associated to the first short exact sequence by Bk. We shall relate Bk to $\partial_M : [M, PL/O] \to [M, BO]$.

Since $M_\alpha$ is stably parallelisable and $PL/O$ is 6-connected it follows for any smooth structure, $M_\gamma$, that $\tau^0(M_\gamma)$ is trivial when restricted to $M^{(6)}$. Further, since $\pi_7(BO) = 0$, we can extend this statement to $M^{(7)}$. Thus the primary obstruction to the triviality of $\tau^0(M_\gamma)$, $\mathrm{Ob}_O\big(\tau^0(M_\gamma)\big)$, lies in $H^8(M; \pi_7(O))$ and there is a commutative diagram

$$\begin{array}{ccc} [M, PL/O] & \xrightarrow{\quad\partial_M\quad} & \mathrm{Im}(\partial_M) \\ \big\downarrow{\scriptstyle \mathrm{Ob}_{PL/O}} & & \big\downarrow{\scriptstyle \mathrm{Ob}_O} \\ H^7(M; \pi_7(PL/O)) & \xrightarrow{\mathrm{Bk}} & H^8(M; \pi_7(O)) \end{array}$$

where we have used $\Psi_\alpha$ to identify $\mathcal{C}(M_A) \equiv [M, PL/O]$ and $\mathrm{Ob}_{PL/O} \colon [M, PL/O] \to H^7\big(M; \pi_7(PL/O)\big)$ as the primary obstruction to a null-homotopy. Now for all the $M$ to which we have specialized, $H^8\big(M; \pi_7(O)\big) \cong H^8(M; \mathbb{Z})$ contains a cyclic summand of order $7^a$ with $a \geq 1$. Let $y$ be a generator for this summand. We claim that there is an element $x \in [M, PL/O]$ such that $\mathrm{Bk} \circ \mathrm{Ob}_{PL/O}(x) = 7^{a-1}y$. Firstly we observe that $\mathrm{Ob}_{PL/O}$ is onto the 7-torsion in $H^7\big(M; \pi_7(PL/O)\big)$ since the Atiyah-Hirzeburch spectral sequence to compute $[M, PL/O]$ gives an exact sequence

$$\cdots \longrightarrow [M, PL/O] \xrightarrow{\mathrm{Ob}_{PL/O}} H^7\big(M; \pi_7(PL/O)\big) \longrightarrow H^m(M; \pi_{m-1}PL/O) \longrightarrow \ldots$$

and $H^m(M;\pi_{m-1}PL/O) \cong \pi_{m-1}(PL/O)$ is prime to 7 ($m = 9$ or 15, and $\pi_8(PL/O) \cong \pi_{14}(PL/O) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$). Secondly, from the coefficient sequence above, we see that when restricted to the summand generated by $y$, the map $H^8\big(M;\pi_7(O)\big) \to H^8\big(M;\pi_7(PL/O)\big)$ is isomorphic to multiplication by 7. It follows that $7^{a-1}y \neq 0$ lies in the image of Bk and since it is 7-torsion it also lies in the image of $\mathrm{Bk} \circ \mathrm{Ob}_{PL/O}$.

From the claim and the commutativity of the above diagram we have an $x \in [M, PL/O]$ such that $\mathrm{Ob}_O \circ \partial_M(x) = 7^{a-1}y$. Setting $\beta = \Psi_\alpha^{-1}(x)$ we obtain a smooth structure $\beta$ on $M_A$ with $\mathrm{Ob}_O\big(\tau^0(M_\beta)\big) = 7^{a-1}y$. Finally, Kervaire [10] has shown that $p_2 = 6 \cdot \mathrm{Ob}_O$ for vector bundles which are trivial over $M^{(7)}$ and hence □

$$p_2(M_\beta) = 6 \cdot \mathrm{Ob}_O\big(\tau^0(M_\beta)\big) = 6 \cdot 7^{a-1}y \neq 0\,.$$

## 4. Topological manifolds with varying $PL$ spans

In this section we prove Theorem 1.4. We assume that the reader is familiar with the simply connected surgery exact sequences for smooth and $PL$-manifolds.

In every dimension $m \geq 22$, Morita [18, Theorem 6.1] defines a simply connected topological manifold $M = M^m(K)$ by embedding a 10-skeleton $K$ of $PL/O \simeq K(\mathbb{Z}/2, 3)$ in $\mathbb{R}^m$, $m \geq 22$, taking a regular neighbourhood $T = T^m(K)$ of $K$ and letting $M$ be the trivial double of $T$: $M = T \cup_{\mathrm{Id}} T$. The manifold $M$ admits two $PL$ structures, $M_A$ and $M_B$, such that $M_A$ admits a stably parallelisable smooth structure and $M_B$ is not smoothable (we explain this below). We first explain how to find examples of this type in dimensions 19 and higher. We observe that $M^m(K)$ is the boundary $T^m(K) \times [0,1]$ and hence is a closed, stably parallelisable, topological manifold which contains $K$ as a retract. We observe also that these properties along with $K \to M$ being an 8-equivalence are all that is required in Morita's arguments to show that $PL$-structures $A$ and $B$ exist as above. Now by [26] $K$ embedds into $\mathbb{R}^{19}$. Let $T^{19}(K)$ be a regular neighbourhood of such an embedding and let $M^{19}(K)$ be the boundary of $T^{19}(K) \times [0,1]$. Then $M^{19}(K)$ is a closed, stably parallelisable, topological manifold containing $K$ as an 8-connected retract and hence admits $PL$ structures $A$ and $B$ as above. We first prove the following

**Lemma 4.1.** *For all the manifolds $M = M^m(K)$, $m \geq 19$, $M_A$ is stably parallelisable and $M_B$ is not smoothable. Hence* $\mathrm{pls}^0\mathrm{v}(M) > 0$.

**Proof.** Morita's arugments show the following. Consider the $PL$-structure, in the sense of surgery theory, $f : M_B \to M$, $f$ the identity map. This gives an element $[f]$ in the $PL$-structure set of $M$. As $M$ is simply connected, the $PL$-structure set injects into the normal invariant set and so we obtain an element $[f] \in [M, G/PL]$ (where we use $\mathrm{Id_M} : M_A \to M$ as the base point to identify the normal invariants of $M$ with $[M, G/PL]$ ). Morita showed that $[f]$ does not belong to the image of the canonical map $q : [M, G/O] \to [M, G/PL]$.

Similarly to Section 2, the map $\delta_M^{PL} : [M, G/PL] \to [M, BPL]$ maps $[f]$ to the difference of the stable $PL$-tangent bundles $\tau^0(M_A) - \tau^0(M_B) \in \widetilde{KPL}(M) = [M, BPL]$ and a similar statment holds for $\delta_M^O : [M, G/O] \to [M, BO]$ and the

smooth normal invariant set. There is a commuting diagram of long exact sequences

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & [M,G] & \longrightarrow & [M,G/O] & \xrightarrow{\delta_M^O} & [M,BO] & \xrightarrow{BJ} & [M,BG] & \ldots \\ & & \downarrow= & & \downarrow q & & \downarrow & & \downarrow= & \\ \ldots & \longrightarrow & [M,G] & \longrightarrow & [M,G/PL] & \xrightarrow{\delta_M^{PL}} & [M,BPL] & \longrightarrow & [M,BG] & \ldots \end{array}$$

where $BJ$ denotes the map induced on classifying spaces by the $J$-homomorphism $J : O \to G$. Suppose that $\tau^0(M_B)$ has a smooth reduction. Since $\tau^0(M_A)$ is trivial this means that $\delta_M([f])$ lifts to $x \in [M, BO]$. As $BJ(x)$ is defined by the stable spherical fibration of $M$ and this is trivial we conclude that $x \in \mathrm{Im}(\delta_M^O)$. Now a simple diagram chase ensures that $y \in [M, G/O]$ can be chosen such that $q(y) = [f]$, contradicting Morita's results. Hence $\tau^0(M_B)$ cannot be smoothed, so it must be non-trivial and $\mathrm{span}^0(M_B) < m$. But $\mathrm{span}^0(M_A) = m$, so $\mathrm{pls}^0\mathrm{v}(M) > 0$. □

**Proof of Theorem 1.4.** Let $M = M^{19}(K)$ and let $M_\alpha$ be a stably parallelisable smooth structure refining $M_A$. By the Bredon-Kosinski theorem we know that $\tau(M_\alpha)$ is trivial if and only if $\chi_2(M) = 0$. However, we do not know $\chi_2(M)$ so similarly to Theorem 3.1 we let $N_\alpha = M_\alpha \sharp_l (S^1 \times S^{18})$ where $l = \chi_2(M)$ is 1 or 0. It follows that $N_\alpha$ is stably parallelisable and that $\chi_2(N) = 0$. Thus $N_\alpha$ is parallelisable and so $N_A = M_A \sharp_l (S^1 \times S^{18})$ is too. The manifold $N$ also admits the $PL$-structure $N_B = M_B \sharp_l (S^1 \times S^{18})$ which is not smoothable. Hence $\mathrm{plsv}(N) > 0$ and $\mathrm{pls}^0\mathrm{v}(N) > 0$. In dimensions $m > 19$ we take $Q = N \times S^n$ for $n > 0$, for then $Q$ admits a $PL$-structure $Q_A = N_A \times S^n$ which is parallelisable and another $PL$-structure $Q_B = N_B \times S^n$ which is not smoothable. Hence $\mathrm{plsv}(Q) > 0$ and $\mathrm{pls}^0\mathrm{v}(Q) > 0$. □

## References


[1] Atiyah, M., *Thom complexes*, Proc. London Math. Soc. **11** (3) (1961), 291–310.

[2] Benlian, R., Wagoner, J., *Type d'homotopie et réduction structurale des fibrés vectoriels*, C. R. Acad. Sci. Paris Sér. A-B 207-209. **265** (1967), 207–209.

[3] Bredon, G. E., Kosinski, A., *Vector fields on π-manifolds*, Ann. of Math. (2) **84** (1966), 85–90.

[4] Brumfiel, G., *On the homotopy groups of* B$PL$ *and* P$L/O$, Ann. of Math. (2) **88** (1968), 291–311.

[5] Davis, J. F., Kirk, P., *Lecture notes in algebraic topology*, Grad. Stud. Math. **35** (2001).

[6] Dupont, J., *On the homotopy invariance of the tangent bundle II*, Math. Scand. **26** (1970), 200–220.

[7] Frank, D., *The signature defect and the homotopy of* B$PL$ *and* P$L/O$, Comment. Math. Helv. **48** (1973), 525–530.

[8] Husemoller, D., *Fibre Bundles*, Grad. Texts in Math. **20** (1993), (3rd edition).

[9] James, I. M., Thomas, E., *An approach to the enumeration problem for non-stable vector bundles*, J. Math. Mech. **14** (1965), 485–506.

[10] Kervaire, M. A., *A note on obstructions and characteristic classes*, Amer. J. Math. **81** (1959), 773–784.

[11] Kirby, R. C., Siebenmann, L. C., *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*, Ann. of Math. Stud. **88** (1977).

[12] Korbaš, J., Szücs, A., *The Lyusternik-Schnirel'man category, vector bundles, and immersions of manifolds*, Manuscripta Math. **95** (1998), 289–294.

[13] Korbaš, J., Zvengrowski, P., *The vector field problem: a survey with emphasis on specific manifolds*, Exposition. Math. **12** (1) (1994), 3–20.

[14] Kosinski, A. A., *Differential Manifolds*, pure and applied mathematics ed., Academic Press, San Diego, 1993.

[15] Kreck, M., Lück, W., *The Novikov Conjecture, Geometry and Algebra*, Oberwolfach Seminars **33**, Birkhäuser Verlag, Basel, 2005.

[16] Lance, T., *Differentiable Structures on Manifolds, in Surveys on Surgery Theory*, Ann. of Math. Stud. **145** (2000), 73–104.

[17] Milnor, J., *Microbundles I*, Topology **3** Suppl. 1 (1964), 53–80.

[18] Morita, S., *Smoothability of* P*L manifolds is not topologically invariant*, Manifolds—Tokyo 1973, 1975, pp. 51–56.

[19] Novikov, S. P., *Topology in the 20th century: a view from the inside*, Uspekhi Mat. Nauk (translation in Russian Math. Surveys **59** (5) (2004), 803-829 **59** (5) (2004), 3–28.

[20] Pedersen, E. K., Ray, N., *A fibration for* Diff $\Sigma^n$, Topology Symposium, Siegen 1979, Lecture Notes in Math. **788**, 1980, pp. 165–171.

[21] Randall, D., *CAT* 2*-fields on nonorientable CAT manifolds*, Quart. J. Math. Oxford Ser. (2) **38** (151) (1987), 355–366.

[22] Roitberg, J., *On the* PL *noninvariance of the span of a smooth manifold*, Proc. Amer. Math. Soc. **20** (1969), 575–579.

[23] Shimada, N., *Differentiable structures on the* 15*-sphere and Pontrjagin classes of certain manifolds*, Nagoya Math. J. **12** (1957), 59–69.

[24] Sutherland, W. A., *The Browder-Dupont invariant*, Proc. Lond. Math. Soc. (3) **33** (1976), 94–112.

[25] Varadarajan, K., *On topological span*, Comment. Math. Helv. **47** (1972), 249–253.

[26] Wall, C. T. C., *Classification problems in differential topology - VI*, Topology **6** (1967), 273–296.

[27] Wall, C. T. C., *Poincaré complexes I*, Ann. of Math. (2) **86** (1967), 213–245.

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