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APPROXIMATION OF SOLUTIONS OF THE FORCED DUFFING EQUATION WITH NONLOCAL DISCONTINUOUS TYPE INTEGRAL BOUNDARY CONDITIONS

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Abstract. A generalized quasilinearization technique is applied to obtain a sequence of approximate solutions converging monotonically and quadratically to the unique solution of the forced Duffing equation with nonlocal discontinuous type integral boundary conditions.

1. Introduction

Integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics, see for example [16, 17, 24]. In fact, boundary value problems involving integral boundary conditions have received considerable attention, see for instance, [3, 10], [12]–[15], [18, 19, 26] and the references therein. In a recent reference [2], Ahmad, et. al. discussed the existence and uniqueness of the solutions of a boundary value problem with discontinuous type integral boundary conditions.

The monotone iterative technique coupled with the method of upper and lower solutions [5, 8, 20, 23, 25] manifests itself as an effective and flexible mechanism that offers theoretical as well as constructive existence results in a closed set, generated by the lower and upper solutions. In general, the convergence of the sequence of approximate solutions given by the monotone iterative technique is at most linear [11, 21]. To obtain a sequence of approximate solutions converging quadratically, we use the method of quasilinearization (QSL) [9]. This method has been developed for a variety of problems [1, 4, 6, 7, 22]. In view of its diverse applications, this approach is quite an elegant and easier for application algorithms. To the best of our knowledge, the method of quasilinearization has not been developed for Duffing equation with nonlocal discontinuous type integral boundary conditions.

In this paper, we apply a quasilinearization technique to obtain the analytic approximation of the solution of the forced Duffing equation with nonlocal discontinuous type integral boundary conditions. In fact, we obtain a sequence of approximate solutions converging monotonically and quadratically to the unique solution of the problem at hand. The concept of nonlocal discontinuous integral

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boundary conditions corresponds to a situation when some forcing term is present at an arbitrary intermediate point of the boundary segment and thereby generates a discontinuity in the integral boundary conditions.

2. Preliminaries

We consider the following boundary value problem

$$
\begin{aligned}
&u''(t) + \sigma u'(t) + f(t,u) = 0\,, \quad t \in [0,1],\ \sigma \in \mathbb{R}\backslash\{0\}\,,\\
&u(0) - \mu_1 u'(0) = g_1(u(\gamma)) + \int_0^{\gamma-} q_1\big(u(s)\big)\,ds + \int_{\gamma+}^1 q_1\big(u(s)\big)\,ds\,,\\
&u(1) + \mu_2 u'(1) = g_2\big(u(\gamma)\big) + \int_0^{\gamma-} q_2\big(u(s)\big)\,ds + \int_{\gamma+}^1 q_2\big(u(s)\big)\,ds\,,\\
&\qquad 0 < \gamma < 1\,,
\end{aligned}
\tag{2.1}
$$

where $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$, $g_i\colon \mathbb{R}\to\mathbb{R}$ $(i=1,2)$ are continuous functions, q_i are continuous functions on $(0,\gamma)$ and $(\gamma,1)$ and μ_i are nonnegative constants.
The quasilinearization technique is applied to obtain a sequence of approximate solutions converging monotonically and quadratically to the unique solution of the problem (2.1).

Definition 2.1. A function $\alpha\in C^2[0,1]$ is a lower solution of (2.1) if

$$
\begin{aligned}
&\alpha''(t) + \sigma\alpha'(t) + f\big(t,\alpha(t)\big) \ge 0\,, \quad t\in[0,1]\,,\\
&\alpha(0) - \mu_1\alpha'(0) \le g_1\big(\alpha(\gamma)\big) + \int_0^{\gamma-} q_1\big(\alpha(s)\big)\,ds + \int_{\gamma+}^1 q_1\big(\alpha(s)\big)\,ds\,,\\
&\alpha(1) + \mu_2\alpha'(1) \le g_2\big(\alpha(\gamma)\big) + \int_0^{\gamma-} q_2\big(\alpha(s)\big)\,ds + \int_{\gamma+}^1 q_2\big(\alpha(s)\big)\,ds\,.
\end{aligned}
$$

Similarly, $\beta\in C^2[0,1]$ is an upper solution of (2.1) if the inequalities in the definition of lower solution are reversed.
Since the associated homogeneous problem of (2.1) has only the trivial solution, therefore, by Green's function method, the solution $u(t)$ of (2.1) can be written as

$$
\begin{aligned}
u(t) = {}& \frac{1}{(1+\sigma\mu_1)-(1-\sigma\mu_2)e^{-\sigma}}\Big[\big((-1+\sigma\mu_2)e^{-\sigma}+e^{-\sigma t}\big)\\
&\times\Big\{g_1\big(u(\gamma)\big) + \int_0^{\gamma-} q_1\big(u(s)\big)\,ds + \int_{\gamma+}^1 q_1\big(u(s)\big)\,ds\Big\}\\
&+\big((1+\sigma\mu_1)-e^{-\sigma t}\big)\Big\{g_2\big(u(\gamma)\big) + \int_0^{\gamma-} q_2\big(u(s)\big)\,ds + \int_{\gamma+}^1 q_2\big(u(s)\big)\,ds\Big\}\Big]\\
&+\int_0^1 G(t,s)f\big(s,u(s)\big)\,ds\,,
\end{aligned}
$$

where

$$G(t,s)=\Lambda\begin{cases}\left[(1-\sigma\mu_2)-e^{\sigma(1-s)}\right]\left[(1+\sigma\mu_1)-e^{-\sigma t}\right], & 0\le t\le s,\\ \left[(1-\sigma\mu_2)-e^{\sigma(1-t)}\right]\left[(1+\sigma\mu_1)-e^{-\sigma s}\right], & s\le t\le 1,\end{cases}$$

$$\Lambda=\frac{e^{\sigma s}}{\sigma[(1-\sigma\mu_2)-(1+\sigma\mu_1)e^{\sigma}]}\,.$$

Observe that $G(t,s)>0$ on $(0,1)\times(0,1)$. We state the following results which lay a foundation to establish the main result. We omit the proof as the method of proof is similar to the one employed in [2].

Theorem 2.1. *Let α and β be lower and upper solutions of the boundary value problem* (2.1) *respectively. Let $f\colon[0,1]\times\mathbb{R}\to\mathbb{R}$ be such that $f_u(t,u)<0$ and q_i are continuous functions on $(0,\gamma)$ and $(\gamma,1)$ satisfying one sided Lipschitz condition: $q_i(u)-q_i(v)\le L_i(u-v)$, $0\le L_i<1$, $i=1,2$, and $g_i\colon\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying one sided Lipschitz condition: $g_i(u)-g_i(v)\le L_i^*(u-v)$, $0\le L_i^*<1$, $i=1,2$. Then $\alpha(t)\le\beta(t)$.*

Theorem 2.2. *Assume that α and β are lower and upper solutions of the boundary value problem* (2.1) *respectively such that $\alpha(t)\le\beta(t)$. If $f\colon[0,1]\times\mathbb{R}\to\mathbb{R}$, $g_i\colon\mathbb{R}\to\mathbb{R}$ are continuous functions and q_i are continuous functions on $(0,\gamma)$ and $(\gamma,1)$ with g_i and q_i satisfying one sided Lipschitz condition, then there exists a solution $u(t)$ of* (2.1) *such that $\alpha(t)\le u(t)\le\beta(t)$, $t\in[0,1]$.*

3. Main result

Theorem 3.1. *Assume that*

- $(\mathbf{A_1})$ *α and $\beta\in C^2[0,1]$ are respectively lower and upper solutions of* (2.1) *such that $\alpha(t)\le\beta(t)$;*
- $(\mathbf{A_2})$ *$f(t,u)\in C^2([0,1]\times\mathbb{R})$ be such that $f_u(t,u)<0$ and $\big(f_{uu}(t,u)+\phi_{uu}(t,u)\big)\ge 0$, where $\phi_{uu}(t,u)\ge 0$ for some continuous function $\phi(t,u)$ on $[0,1]\times\mathbb{R}$;*
- $(\mathbf{A_3})$ *q_i are continuous functions on $(0,\gamma)$ and $(\gamma,1)$ satisfying $0\le q_i'(u)<1$, and $\big(q_i''(u)+\chi_i''(u)\big)\ge 0$ with $\chi_i''\ge 0$, $i=1,2$;*
- $(\mathbf{A_4})$ *$g_i\in C^2(\mathbb{R})$ be such that $0\le g_i'(u)<1$ and $\big(g_i''(u)+\psi_i''(u)\big)\le 0$ with $\psi_i''\le 0$, $i=1,2$.*

Then, there exists a sequence $\{\alpha_n\}$ of approximate solutions converging monotonically and quadratically to the unique solution of the problem (2.1).

Proof. Let $F\colon[0,1]\times\mathbb{R}\to\mathbb{R}$ and $G_i,K_i\colon\mathbb{R}\to\mathbb{R}$ be defined by $F(t,u)=f(t,u)+\phi(t,u)$, $G_i(u)=g_i(u)+\psi_i(u)$, $K_i(u)=q_i(u)+\chi_i(u)$ so that $F_{uu}(t,u)\ge 0$, $G_i''(u)\le 0$, $K_i''(u)\ge 0$. Using the generalized mean value theorem together with (A_2), (A_3) and (A_4), we obtain

$$f(t,u)\ge f(t,v)+F_u(t,v)(u-v)+\phi(t,v)-\phi(t,u)\,, \tag{3.1}$$

$$g_i(u)\le g_i(v)+G_i'(v)(u-v)+\psi_i(v)-\psi_i(u)\,,\quad u,v\in\mathbb{R}\,, \tag{3.2}$$

$$q_i(u)\ge q_i(v)+K_i'(v)(u-v)+\chi_i(v)-\chi_i(u)\,,\quad u,v\in\mathbb{R}\,. \tag{3.3}$$

We set

$$\bar{F}(t,u,\alpha) = f(t,\alpha) + F_u(t\alpha)(u-\alpha) + \phi(t,\alpha) - \phi(t,u)\,,$$
$$Q_i(u,\alpha) = q_i(\alpha) + K_i'(\alpha)(u-\alpha) + \chi_i(\alpha) - \chi_i(u)\,,$$
$$\bar{g}_i(u(\gamma),\alpha,\beta) = g_i(\alpha(\gamma)) + G_i'\big(\beta(\gamma)\big)\big(u(\gamma)-\alpha(\gamma)\big) + \psi_i(\alpha) - \psi_i(u)\,,$$

and note that

$$\bar{F}_u(t,u,\alpha) < 0\,, \quad 0 \le (\partial/\partial u) Q_i(u,\alpha) < 1\,, \quad 0 \le (\partial/\partial u)\bar{g}_i\big(u(\gamma),\alpha,\beta\big) < 1\,.$$

Now, we fix $\alpha = \alpha_0$ and consider the problem

$$\begin{aligned} & u''(t) + \sigma u'(t) + \bar{F}(t,u,\alpha_0) = 0\,, \qquad t \in [0,1]\,, \\ & u(0) - \mu_1 u'(0) = \bar{g}_1\big(u(\gamma),\alpha_0,\beta\big) + \int_0^{\gamma-} Q_1\big(u(s),\alpha_0(s)\big)\,ds \\ & \qquad + \int_{\gamma+}^1 Q_1\big(u(s),\alpha_0(s)\big)\,ds\,, \\ & u(1) + \mu_2 u'(1) = \bar{g}_2\big(u(\gamma),\alpha_0,\beta\big) + \int_0^{\gamma-} Q_2\big(u(s),\alpha_0(s)\big)\,ds \\ & \qquad + \int_{\gamma+}^1 Q_2\big(u(s),\alpha_0(s)\big)\,ds\,. \end{aligned} \tag{3.4}$$

As a first step, it will be shown that α_0, β are respectively lower and upper solutions of (3.4). Using (A_1) together with the fact that $\bar{F}(t,\alpha_0,\alpha_0) = f(t,\alpha_0)$, $\bar{g}_1\big(\alpha(\gamma),\alpha_0,\beta\big) = g_i\big(\alpha_0(\gamma)\big)$ and $Q_i(\alpha_0,\alpha_0) = q_i(\alpha_0)$, we have

$$\alpha_0''(t) + \sigma\alpha_0'(t) + \bar{F}(t,\alpha_0,\alpha_0) = \alpha_0''(t) + \sigma\alpha_0'(t) + f(t,\alpha_0) \ge 0\,, \qquad t \in [0,1]\,,$$

$$\begin{aligned} \alpha_0(0) - \mu_1\alpha_0'(0) &\le g_1\big(\alpha_0(\gamma)\big) + \int_0^{\gamma-} q_1\big(\alpha_0(s)\big)\,ds + \int_{\gamma+}^1 q_1\big(\alpha_0(s)\big)\,ds \\ &= \bar{g}_1(\alpha_0(\gamma),\alpha_0,\beta) + \int_0^{\gamma-} Q_1\big(\alpha_0(s),\alpha_0(s)\big)\,ds \\ &\quad + \int_{\gamma+}^1 Q_1\big(\alpha_0(s),\alpha_0(s)\big)\,ds\,, \\ \alpha_0(1) + \mu_2\alpha_0'(1) &\le g_2\big(\alpha_0(\gamma)\big) + \int_0^{\gamma-} q_2\big(\alpha_0(s)\big)\,ds + \int_{\gamma+}^1 q_2\big(\alpha_0(s)\big)\,ds \\ &= \bar{g}_2\big(\alpha_0(\gamma),\alpha_0,\beta\big) + \int_0^{\gamma-} Q_2\big(\alpha_0(s),\alpha_0(s)\big)\,ds \\ &\quad + \int_{\gamma+}^1 Q_2\big(\alpha_0(s),\alpha_0(s)\big)\,ds \end{aligned}$$

and

$$\beta''(t) + \sigma\beta'(t) + \bar{F}(t,\beta,\alpha_0) \le \beta''(t) + \sigma\beta'(t) + f(t,\beta) \le 0\,, \qquad t \in [0,1]\,.$$

Moreover, there exists c_0, $c_1 \in \big(\alpha_0(\gamma),\beta(\gamma)\big)$ and c_2, $c_3 \in (\alpha_0,\beta)$ so that

$$\begin{aligned}
&g_1\big(\beta(\gamma)\big) - \bar{g}_1\big(\beta(\gamma),\alpha_0,\beta\big)\\
&\quad = g_1\big(\beta(\gamma)\big) - g_1\big(\alpha_0(\gamma)\big) - G_1'\big(\beta(\gamma)\big)\big(\beta(\gamma)-\alpha_0(\gamma)\big) - \psi_1\big(\alpha_0(\gamma)\big) + \psi_1\big(\beta(\gamma)\big)\\
&\quad = \big[g_1'(c_0) - g_1'\big(\beta(\gamma)\big)\big]\big(\beta(\gamma)-\alpha_0(\gamma)\big) + \big[\psi_1'(c_1)-\psi_1'\big(\beta(\gamma)\big)\big]\big(\beta(\gamma)-\alpha_0(\gamma)\big) \ge 0\,,
\end{aligned}$$

$$\begin{aligned}
&q_1\big(\beta(s)\big) - Q_1\big(\beta(s),\alpha_0(s)\big)\\
&\quad = q_1\big(\beta(s)\big) - q_1\big(\alpha_0(s)\big) - K_1'\big(\alpha_0(s)\big)\big(\beta(s)-\alpha_0(s)\big) - \chi_1\big(\alpha_0(s)\big) + \chi_1\big(\beta(s)\big)\\
&\quad = \big[q_1'(c_2) - q_1'\big(\alpha_0(s)\big)\big]\big(\beta(s)-\alpha_0(s)\big) + \big[\chi_1'(c_3)-\chi_1'\big(\alpha_0(s)\big)\big]\big(\beta(s)-\alpha_0(s)\big) \ge 0
\end{aligned}$$

and consequently, we obtain

$$\begin{aligned}
\beta(0) - \mu_1\beta'(0) &\ge g_1\big(\beta(\gamma)\big) + \int_0^{\gamma-} q_1\big(\beta(s)\big)\,ds + \int_{\gamma+}^1 q_1\big(\beta(s)\big)\,ds\\
&\ge \bar{g}_1\big(\beta(\gamma),\alpha_0,\beta\big) + \int_0^{\gamma-} Q_1\big(\beta(s),\alpha_0(s)\big)\,ds + \int_{\gamma+}^1 Q_1\big(\beta(s),\alpha_0(s)\big)\,ds.
\end{aligned}$$

Similarly, it can be shown that

$$\beta(1)+\mu_2\beta'(1) \ge \bar{g}_2\big(\beta(\gamma),\alpha_0,\beta\big) + \int_0^{\gamma-} Q_2\big(\beta(s),\alpha_0(s)\big)\,ds + \int_{\gamma+}^1 Q_2\big(\beta(s),\alpha_0(s)\big)\,ds\,.$$

Thus we conclude that α_0 and β are respectively lower and upper solutions of (3.4). Hence, by Theorems 2.1 and 2.2, there exists the unique solution α_1 of (3.4) such that

$$\alpha_0(t) \le \alpha_1(t) \le \beta(t)\,, \qquad t\in[0,1]\,.$$

Next, we consider

$$\tag{3.5}\begin{aligned}
&u''(t) + \sigma u'(t) + \bar{F}(t,u,\alpha_1) = 0\,, \qquad t\in[0,1]\,,\\
&u(0) - \mu_1 u'(0) = \bar{g}_1\big(u(\gamma),\alpha_1,\beta\big) + \int_0^{\gamma-} Q_1\big(u(s),\alpha_1(s)\big)\,ds\\
&\qquad\qquad + \int_{\gamma+}^1 Q_1\big(u(s),\alpha_1(s)\big)\,ds\,,\\
&u(1) + \mu_2 u'(1) = \bar{g}_2\big(u(\gamma),\alpha_1,\beta\big) + \int_0^{\gamma-} Q_2\big(u(s),\alpha_1(s)\big)\,ds\\
&\qquad\qquad + \int_{\gamma+}^1 Q_2\big(u(s),\alpha_1(s)\big)\,ds\,.
\end{aligned}$$

Using the earlier arguments, it can be shown that α_1 and β are lower and upper solutions of (3.5) respectively and by Theorems 2.1 and 2.2, there exists the unique solution α_2 of (3.5) such that

$$\alpha_1(t) \le \alpha_2(t) \le \beta(t)\,, \qquad t\in[0,1]\,.$$

Continuing this process successively yields a sequence $\{\alpha_n\}$ of solutions satisfying

$$\alpha_0(t) \le \alpha_1(t) \le \alpha_2(t) \le \cdots \le \alpha_n(t) \le \beta(t)\,, \quad t\in[0,1]\,,$$

where the element α_n of the sequence $\{\alpha_n\}$ is a solution of the problem

$$\begin{aligned} &u''(t) + \sigma u'(t) + \bar{F}(t, u, \alpha_{n-1}) = 0\,, \quad t \in [0,1]\,, \\ &\qquad u(0) - \mu_1 u'(0) = \bar{g}_1\big(u(\gamma), \alpha_{n-1}, \beta\big) + \int_0^{\gamma-} Q_1\big(u(s), \alpha_{n-1}(s)\big)\, ds \\ &\qquad\qquad + \int_{\gamma+}^1 Q_1\big(u(s), \alpha_{n-1}(s)\big)\, ds\,, \\ &\qquad u(1) + \mu_2 u'(1) = \bar{g}_2\big(u(\gamma), \alpha_{n-1}, \beta\big) + \int_0^{\gamma-} Q_2\big(u(s), \alpha_{n-1}(s)\big)\, ds \\ &\qquad\qquad + \int_{\gamma+}^1 Q_2\big(u(s), \alpha_{n-1}(s)\big)\, ds \end{aligned}$$

and is given by

$$\begin{aligned} \alpha_n(t) = {} & \frac{-(1-\sigma\mu_2)e^{-\sigma} + e^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)e^{-\sigma}} \Big[\bar{g}_1\big(\alpha_n(\gamma), \alpha_{n-1}, \beta\big) \\ & + \int_0^{\gamma-} Q_1\big(\alpha_n(s), \alpha_{n-1}(s)\big)\, ds + \int_{\gamma+}^1 Q_1\big(\alpha_n(s), \alpha_{n-1}(s)\big)\, ds\Big] \\ & + \frac{(1+\sigma\mu_1) - e^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)e^{-\sigma}} \Big[\bar{g}_2\big(\alpha_n(\gamma), \alpha_{n-1}, \beta\big) \\ & + \int_0^{\gamma-} Q_2\big(\alpha_n(s), \alpha_{n-1}(s)\big)\, ds + \int_{\gamma+}^1 Q_2\big(\alpha_n(s), \alpha_{n-1}(s)\big)\, ds\Big] \\ & + \int_0^1 G(t,s)\bar{F}\big(s, \alpha_n(s), \alpha_{n-1}(s)\big)\, ds\,. \end{aligned} \tag{3.6}$$

Using the fact that $[0,1]$ is compact and the monotone convergence of the sequence $\{\alpha_n\}$ is pointwise, it follows that the convergence of the sequence is uniform. If $u(t)$ is the limit point of the sequence, taking the limit $n \to \infty$ in (3.6), we obtain

$$\begin{aligned} u(t) = {} & \frac{-(1-\sigma\mu_2)e^{-\sigma} + e^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)e^{-\sigma}} \Big[g_1\big(u(\gamma)\big) + \int_0^{\gamma-} q_1\big(u(s)\big)\, ds + \int_{\gamma+}^1 q_1\big(u(s)\big)\, ds\Big] \\ & + \frac{(1+\sigma\mu_1) - e^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)e^{-\sigma}} \Big[g_2\big(u(\gamma)\big) + \int_0^{\gamma-} q_2\big(u(s)\big)\, ds + \int_{\gamma+}^1 q_2\big(u(s)\big)\, ds\Big] \\ & + \int_0^1 G(t,s) f\big(s, u(s)\big)\, ds\,. \end{aligned}$$

Thus, $u(t)$ is a solution of (2.1). Now, we show that the convergence of the sequence is quadratic. For that we set $\omega_n(t) = \big(u(t) - \alpha_n(t)\big) \geq 0$, $t \in [0,1]$. In view of (A_2), it follows by Taylor's theorem that

$$
\begin{aligned}
\omega_n''(t)+\sigma\omega_n'(t) &= u''+\sigma u'-(\alpha_n''+\sigma\alpha_n') = -f(t,u)+\bar{F}(t,\alpha_n,\alpha_{n-1})\\
&= -f(t,u)+f(t,\alpha_{n-1})+F_u(t,\alpha_{n-1})(\alpha_n-\alpha_{n-1})\\
&\quad +\phi(t,\alpha_{n-1})-\phi(t,\alpha_n)\\
&= -f_u(t,c_4)(u-\alpha_{n-1})-F_u(t,\alpha_{n-1})(u-\alpha_n)\\
&\quad +F_u(t,\alpha_{n-1})(u-\alpha_{n-1})-\phi_u(t,c_5)(\alpha_n-\alpha_{n-1})\\
&= \big[-f_u(t,c_4)+F_u(t,\alpha_{n-1})-\phi_u(t,c_5)\big]\omega_{n-1}\\
&\quad +\big[-F_u(t,\alpha_{n-1})+\phi_u(t,c_5)\big]\omega_n\\
&= \big[-F_u(t,c_4)+F_u(t,\alpha_{n-1})+\phi_u(t,c_4)-\phi_u(t,c_5)\big]\omega_{n-1}\\
&\quad +\big[-F_u(t,\alpha_{n-1})+\phi_u(t,c_5)\big]\omega_n\\
&\geq \big[-F_u(t,u)+F_u(t,\alpha_{n-1})+\phi_u(t,\alpha_{n-1})-\phi_u(t,\alpha_n)\big]\omega_{n-1}\\
&\quad +\big[-F_u(t,\alpha_{n-1})+\phi_u(t,\alpha_{n-1})\big]\omega_n\\
&\geq \big[-F_{uu}(t,c_6)-\phi_{uu}(t,c_7)\big]\omega_{n-1}^2-f_u(t,\alpha_{n-1})\omega_n \geq -A_1\|\omega_{n-1}\|^2\,, \qquad (3.7)
\end{aligned}
$$

where $\alpha_{n-1}\leq c_4,\ c_6\leq u,\ \alpha_{n-1}\leq c_5,\ c_7\leq \alpha_n$, A is a bound on $\|F_{uu}\|$, B is a bound on $\|\phi_{uu}\|$ and $A_1=A+B$. Further, we have

$$
\begin{aligned}
\omega_n(0)-\mu_1\omega_n'(0) &= g_1\big(u(\gamma)\big)-\bar{g}_1\big(\alpha_n(\gamma),\alpha_{n-1},\beta\big)\\
&\quad +\int_0^{\gamma-}\big[q_1\big(u(s)\big)-Q_1\big(\alpha_n(s),\alpha_{n-1}(s)\big)\big]\,ds\\
&\quad +\int_{\gamma+}^1\big[q_1\big(u(s)\big)-Q_1\big(\alpha_n(s),\alpha_{n-1}(s)\big)\big]\,ds\\
&= g_1\big(u(\gamma)\big)-g_1\big(\alpha_{n-1}(\gamma)\big)-G_1'\big(\beta(\gamma)\big)(\alpha_n-\alpha_{n-1})\\
&\quad +\int_0^{\gamma-}\big[q_1\big(u(s)\big)-q_1=\big(\alpha_{n-1}(s)\big)-K_1'\big(\alpha_{n-1}(s)\big)(\alpha_n-\alpha_{n-1})-\chi_1(\alpha_{n-1})\\
&\quad +\chi_1(\alpha_n)\big]\,ds+\int_{\gamma+}^1\big[q_1\big(u(s)\big)-q_1\big(\alpha_{n-1}(s)\big)-K_1'\big(\alpha_{n-1}(s)\big)(\alpha_n-\alpha_{n-1})\\
&\quad -\chi_1(\alpha_{n-1})+\chi_1(\alpha_n)\big]\,ds\\
&\leq \left[\frac{1}{2}g_1''(\xi_1)+\psi_1''(\eta_2)\right]\omega_{n-1}^2(\gamma)+\big[G_1'\big(\beta(\gamma)\big)-\psi_1'(\eta_1)\big]\omega_n(\gamma)\\
&\quad +\int_0^{\gamma-}\left[\big(K_1'\big(\alpha_{n-1}(s)\big)-\chi_1'(\eta_3)\big)\,\omega_n(s)+\left(\frac{1}{2}q_1''(\xi_2)+\chi_1''(\eta_4)\right)\omega_{n-1}^2(s)\right]ds\\
&\quad +\int_{\gamma+}^1\left[\big(K_1'\big(\alpha_{n-1}(s)\big)-\chi_1'(\eta_3)\big)\,\omega_n(s)+\left(\frac{1}{2}q_1''(\xi_2)+\chi_1''(\eta_4)\right)\omega_{n-1}^2(s)\right]ds
\end{aligned}
$$

and

$$\begin{aligned}\omega_n(1)+\mu_2\omega_n'(1)&=g_2(u(\gamma))-\bar{g}_2(\alpha_n(\gamma),\alpha_{n-1},\beta)\\&\quad+\int_0^{\gamma-}\big[q_2\big(u(s)\big)-Q_2\big(\alpha_n(s),\alpha_{n-1}(s)\big)\big]\,ds\\&\quad+\int_{\gamma+}^1\big[q_2\big(u(s)\big)-Q_2\big(\alpha_n(s),\alpha_{n-1}(s)\big)\big]\,ds\\&\le\Big[\frac{1}{2}g_2''(\xi_3)+\psi_2''(\eta_5)\Big]\,\omega_{n-1}^2(\gamma)+\big[G_2'\big(\beta(\gamma)\big)-\psi_2'(\eta_6)\big]\,\omega_n(\gamma)\\&\quad+\int_0^{\gamma-}\Big[\big(K_2'\big(\alpha_{n-1}(s)\big)-\chi_2'(\eta_7)\big)\,\omega_n(s)+\Big(\frac{1}{2}q_2''(\xi_4)+\chi_2''(\eta_8)\Big)\,\omega_{n-1}^2(s)\Big]\,ds\\&\quad+\int_{\gamma+}^1\Big[(K_2'(\alpha_{n-1}(s))-\chi_2'(\eta_7))\omega_n(s)+\Big(\frac{1}{2}q_2''(\xi_4)+\chi_2''(\eta_8)\Big)\,\omega_{n-1}^2(s)\Big]\,ds\,,\end{aligned}$$

where $\alpha_{n-1}\le\xi_j\le u$, $j=1,\dots,4$, $\alpha_{n-1}\le\eta_\nu\le\alpha_n\le u$, $\nu=1,\dots,8$. In view of (A_3) and (A_4), there exists $\lambda_i<1$, $\lambda_i^*<1$, $M_i\ge 0$ and $M_i^*\ge 0$ such that $|G_i'-\psi_i'|\le\lambda_i^*$, $|K_i'-\chi_i'|\le\lambda_i$, $|\frac{1}{2}q_i''+\chi_i''|\le M_i$ and $\big|\frac{1}{2}g_i''+\psi_i''\big|\le M_i^*$. Letting $\lambda=\max\{\lambda_1,\lambda_2\}$, $\lambda^*=\max\{\lambda_1^*,\lambda_2^*\}$, $M^*=\max\{M_1^*,M_2^*\}$, and $M=\max\{M_1,M_2\}$, we get

$$\tag{3.8}\begin{cases}\omega_n(0)-\mu_1\omega_n'(0)\le M^*\omega_{n-1}^2(\gamma)+\lambda^*\omega_n(\gamma)\\\qquad\qquad+\lambda\Big[\displaystyle\int_0^{\gamma-}\omega_n(s)\,ds+\int_{\gamma+}^1\omega_n(s)\,ds\Big]\\\qquad\qquad+M\Big[\displaystyle\int_0^{\gamma-}\omega_{n-1}^2(s)\,ds+\int_{\gamma+}^1\omega_{n-1}^2(s)\,ds\Big]\,,\\\omega_n(1)+\mu_2\omega_n'(1)\le M^*\omega_{n-1}^2(\gamma)+\lambda^*\omega_n(\gamma)\\\qquad\qquad+\lambda\Big[\displaystyle\int_0^{\gamma-}\omega_n(s)\,ds+\int_{\gamma+}^1\omega_n(s)\,ds\Big]\\\qquad\qquad+M\Big[\displaystyle\int_0^{\gamma-}\omega_{n-1}^2(s)\,ds+\int_{\gamma+}^1\omega_{n-1}^2(s)\,ds\Big]\,.\end{cases}$$

Using the estimates (3.7) and (3.8), we obtain

$$\begin{aligned}\omega_n(t)=\frac{-(1-\sigma\mu_2)e^{-\sigma}+e^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)e^{-\sigma}}&\Big(g_1\big(u(\gamma)\big)-\bar{g}_1\big(\alpha_n(\gamma),\alpha_{n-1},\beta\big)\\&+\int_0^{\gamma-}\big[q_1\big(u(s)\big)-Q_1\big(\alpha_n(s),\alpha_{n-1}(s)\big)\big]\,ds\\&+\int_{\gamma+}^1\big[q_1\big(u(s)\big)-Q_1\big(\alpha_n(s),\alpha_{n-1}(s)\big)\big]\,ds\Big)\\&+\frac{(1+\sigma\mu_1)-e^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)e^{-\sigma}}\Big(g_2\big(u(\gamma)\big)-\bar{g}_2\big(\alpha_n(\gamma),\alpha_{n-1},\beta\big)\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{\gamma-} \big[q_2\big(u(s)\big) - Q_2\big(\alpha_n(s), \alpha_{n-1}(s)\big)\big]\, ds \\
&+ \int_{\gamma+}^1 \big[q_2\big(u(s)\big) - Q_2\big(\alpha_n(s), \alpha_{n-1}(s)\big)\big]\, ds\Big) \\
&+ \int_0^1 G(t,s)\big[f\big(s,u(s)\big) - \bar{F}(t,\alpha_n,\alpha_{n-1})\big]\, ds \\
\le{}& \frac{-(1-\sigma\mu_2)e^{-\sigma} + e^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)e^{-\sigma}} \Big[M^*\omega_{n-1}^2(\gamma) + \lambda^*\omega_n(\gamma) \\
&+ \lambda\Big(\int_0^{\gamma-} \omega_n(s)\, ds + \int_{\gamma+}^1 \omega_n(s)\, ds\Big) + M\Big(\int_0^{\gamma-} \omega_{n-1}^2(s)\, ds + \int_{\gamma+}^1 \omega_{n-1}^2(s)\, ds\Big)\Big] \\
&+ \frac{(1+\sigma\mu_1) - e^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)e^{-\sigma}} \Big[M^*\omega_{n-1}^2(\gamma) + \lambda^*\omega_n(\gamma) \\
&+ \lambda\Big(\int_0^{\gamma-} \omega_n(s)\, ds + \int_{\gamma+}^1 \omega_n(s)\, ds\Big) + M\Big(\int_0^{\gamma-} \omega_{n-1}^2(s)\, ds + \int_{\gamma+}^1 \omega_{n-1}^2(s) ds\Big)\Big] \\
&- \int_0^1 G(t,s)\big[\omega_n''(s) + \sigma\omega_n'(s)\big]\, ds \\
\le{}& M^*\omega_{n-1}^2(\gamma) + \lambda^*\omega_n(\gamma) + \lambda\Big(\int_0^{\gamma-} \omega_n(s)\, ds + \int_{\gamma+}^1 \omega_n(s)\, ds\Big) \\
&+ M\Big(\int_0^{\gamma-} \omega_{n-1}^2(s)\, ds + \int_{\gamma+}^1 \omega_{n-1}^2(s)\, ds\Big) + A_1\|\omega_{n-1}\|^2 \int_0^1 G(t,s)\, ds \\
\le{}& M^*\|\omega_{n-1}\|^2 + \lambda^*\|\omega_n\| + \lambda\|\omega_n\| + M\|\omega_{n-1}\|^2 + A_2\|\omega_{n-1}\|^2 \\
={}& \lambda^{**}\|\omega_n\| + M^{**}\|\omega_{n-1}\|^2
\end{aligned}
$$

where A_2 provides a bound on $A_1 \int_0^1 G(t,s)$. We choose λ^* and λ so that $\lambda^{**} = \lambda^* + \lambda < 1$ and $M^{**} = M^* + M + A_2$. Taking the maximum over $[0,1]$, we get

$$
\|\omega_n\| \le \frac{M^{**}}{1-\lambda^{**}} \|\omega_{n-1}\|^2,
$$

where $\|u\| = \max\big\{|u(t)| \colon t \in [0,1]\big\}$. This establishes the quadratic convergence of the sequence of iterates. □

Remark. The results obtained in [2] appear as a special case of our results if we take $\gamma = 1/2$ in (2.1) and $\psi_i \equiv 0 \equiv \chi_i$, $i = 1,2$ in the assumptions (A_3) and (A_4) of Theorem 3.1.

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