ARCHIVUM MATHEMATICUM (BRNO)
Tomus 44 (2008), 185–189

# LEFT APP-PROPERTY OF FORMAL POWER SERIES RINGS

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Abstract. A ring $R$ is called a left APP-ring if the left annihilator $l_R(Ra)$ is right $s$-unital as an ideal of $R$ for any element $a \in R$. We consider left APP-property of the skew formal power series ring $R[[x;\alpha]]$ where $\alpha$ is a ring automorphism of $R$. It is shown that if $R$ is a ring satisfying descending chain condition on right annihilators then $R[[x;\alpha]]$ is left APP if and only if for any sequence $(b_0, b_1, \dots)$ of elements of $R$ the ideal $l_R\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R\alpha^k(b_j)\right)$ is right $s$-unital. As an application we give a sufficient condition under which the ring $R[[x]]$ over a left APP-ring $R$ is left APP.

Throughout this paper, $R$ denotes a ring with unity. Recall that $R$ is *left principally quasi-Baer* if the left annihilator of every principal left ideal of $R$ is generated by an idempotent. Similarly, right principally quasi-Baer rings can be defined. A ring is called *principally quasi-Baer* if it is both right and left principally quasi-Baer. Observe that biregular rings and quasi-Baer rings (i.e. the rings over which the left annihilator of every left ideal of $R$ is generated by an idempotent of $R$) are principally quasi-Baer. For more details and examples of left principally quasi-Baer rings, see [3], [1], [2], [4], and [7]. A ring $R$ is called a *right* (resp. *left*) *PP-ring* if the right (resp. left) annihilator of every element of $R$ is generated by an idempotent. $R$ is called a *PP-ring* if it is both right and left PP. As a generalization of left principally quasi-Baer rings and right PP-rings, the concept of left APP-rings was introduced in [9]. A ring $R$ is called a *left APP-ring* if the left annihilator $l_R(Ra)$ is right $s$-unital as an ideal of $R$ for any element $a \in R$. For more details and examples of left APP-rings, see [9] and [6].

There are a lot of results concerning left principal quasi-Baerness and right PP-property of polynomial extensions of a ring. It was proved in ([2], Theorem 2.1) that a ring $R$ is left principally quasi-Baer if and only if $R[x]$ is left principally quasi-Baer. If all right semicentral idempotents of $R$ are central, then it was shown in [7] that the ring $R[[x]]$ is left principally quasi-Baer if and only if $R$ is left principally quasi-Baer and every countable family of idempotents in $R$ has a generalized join in $I(R)$, the set of all idempotents of $R$. It was shown in [5] that $R$ is a reduced PP-ring if and only if $R[[x]]$ is a reduced PP-ring. In [8] the PP-property of the rings of generalized power series over a ring $R$ has been

2000 *Mathematics Subject Classification*: primary 16W60; secondary 16P60.

*Key words and phrases*: left APP-ring, skew power series ring, left principally quasi-Baer ring.

Supported by National Natural Science Foundation of China, TRAPOYT and the Cultivation Fund of the Key Scientific and Technical Innovation Project, Ministry of Education of China.

Received September 25, 2007. Editor J. Trlifaj.

considered. For left APP-rings, It was noted in [9] that there exists a commutative von Neumann regular ring $R$ (hence left APP), but the ring $R[[x]]$ is not APP. It was also shown in [9] that if $R$ is a left APP-ring satisfying descending chain condition on left and right annihilators then $R[[x]]$ is left APP. In this note we consider left APP-property of skew formal power series rings. We will show that if $R$ is a ring satisfying descending chain condition on right annihilators then $R[[x;\alpha]]$ is left APP if and only if for any sequence $(b_0, b_1, \dots)$ of elements of $R$ the ideal $l_R\big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R\alpha^k(b_j)\big)$ is right $s$-unital. As an application we give a sufficient condition under which the ring $R[[x]]$ over a left APP-ring $R$ is left APP.

For a nonempty subset $Y$ of $R$, $l_R(Y)$ and $r_R(Y)$ denote the left and right annihilator of $Y$ in $R$, respectively.

An ideal $I$ of $R$ is said to be *right $s$-unital* if, for each $a \in I$ there exists an element $x \in I$ such that $ax = a$. It follows from ([11, Theorem 1]) that $I$ is right $s$-unital if and only if for any finitely many elements $a_1, a_2, \dots, a_n \in I$ there exists an element $x \in I$ such that $a_i = a_i x$, $i = 1, 2, \dots, n$. A submodule $N$ of a left $R$-module $M$ is called a *pure submodule* if $L \otimes_R N \to L \otimes_R M$ is a monomorphism for every right $R$-module $L$. By ([10], Proposition 11.3.13), an ideal $I$ is right $s$-unital if and only if $R/I$ is flat as a left $R$-module if and only if $I$ is pure as a left ideal of $R$.

**Lemma 1.** *Let $R[[x;\alpha]]$ be a left APP-ring and $b_0, b_1, \dots$ in $R$. If $a_0, a_1, \dots, a_n \in R$ are such that for any $r \in R$ and any $s = 0, 1, \dots,$*

$$
\begin{aligned}
&a_0 r\alpha^s(b_0) = 0\\
&a_0 r\alpha^s(b_1) + a_1\alpha(r)\alpha^{1+s}(b_0) = 0\\
&\quad\vdots\\
&a_0 r\alpha^s(b_{n-1}) + a_1\alpha(r)\alpha^{1+s}(b_{n-2}) + \cdots + a_{n-1}\alpha^{n-1}(r)\alpha^{n-1+s}(b_0) = 0\\
&a_0 r\alpha^s(b_n) + a_1\alpha(r)\alpha^{1+s}(b_{n-1}) + \cdots + a_n\alpha^{n}(r)\alpha^{n+s}(b_0) = 0\,,
\end{aligned}
$$

*then for any $s$,*

$$a_0 R\alpha^s(b_j) = 0\,, \quad j = 0, 1, \dots n\,.$$

**Proof.** We prove this result by induction on $n$.

Suppose that $n = 1$. For any $\phi(x) = c_0 + c_1 x + c_2 x^2 + \cdots \in R[[x;\alpha]]$, $a_0\phi(x)b_0 = a_0c_0b_0 + a_0c_1\alpha(b_0)x + a_0c_2\alpha^2(b_0)x^2 + \cdots = 0$ since $a_0R\alpha^s(b_0) = 0$ for any $s$. Thus $a_0R[[x;\alpha]]b_0 = 0$. Since $R[[x;\alpha]]$ is a left APP-ring, there exists $h(x) = h_0 + h_1x + h_2x^2 + \cdots \in l_{R[[x;\alpha]]}(R[[x;\alpha]]b_0)$ such that $a_0 = a_0h(x)$. Clearly $a_0 = a_0h_0$ and for any $r \in R$ and any $s$, $h(x)(rx^s)b_0 = 0$. Thus $h_0r\alpha^s(b_0) = 0$ for any $s$. Take $r = h_0r'$ in $a_0r\alpha^s(b_1) + a_1\alpha(r)\alpha^{1+s}(b_0) = 0$. Then $a_0r'\alpha^s(b_1) = a_0h_0r'\alpha^s(b_1) = a_0h_0r'\alpha^s(b_1) + a_1\alpha\big(h_0r'\alpha^s(b_0)\big) = a_0h_0r'\alpha^s(b_1) + a_1\alpha(h_0r')\alpha^{1+s}(b_0) = 0$. Thus $a_0R\alpha^s(b_1) = 0$.

Now suppose that $n \geq 2$. From the first $n$ equations and the induction hypothesis, it follows that $a_0R\alpha^s(b_j) = 0$, $j = 0, 1, \dots n-1$. Thus for any $r \in R$ and any $s$, $a_0(rx^s)(b_0 + b_1x + \cdots + b_{n-1}x^{n-1}) = a_0r\alpha^s(b_0)x^s + a_0r\alpha^s(b_1)x^{s+1} + \cdots + a_0r\alpha^s(b_{n-1})x^{s+n-1} = 0$. Hence $a_0R[[x;\alpha]](b_0 + b_1x + \cdots + b_{n-1}x^{n-1}) = 0$.

Since $R[[x;\alpha]]$ is a left APP-ring, there exists $h(x) = h_0 + h_1x + h_2x^2 + \cdots \in l_{R[[x;\alpha]]}\big(R[[x;\alpha]](b_0+b_1x+\cdots+b_{n-1}x^{n-1})\big)$ such that $a_0 = a_0h(x)$. Thus $a_0 = a_0h_0$ and $h(x)(rx^s)(b_0 + b_1x + \cdots + b_{n-1}x^{n-1}) = 0$ for any $r \in R$ and any $s$. Now we have

$$\begin{aligned}
&h_0r\alpha^s(b_0) = 0\\
&h_0r\alpha^s(b_1) + h_1\alpha(r)\alpha^{1+s}(b_0) = 0\\
&\quad\vdots\\
&h_0r\alpha^s(b_{n-1}) + h_1\alpha(r)\alpha^{1+s}(b_{n-2}) + \cdots + h_{n-1}\alpha^{n-1}(r)\alpha^{n-1+s}(b_0) = 0\,.
\end{aligned}$$

By the induction hypothesis, it follows that $h_0R\alpha^s(b_j) = 0$, $j = 0, 1, \dots n-1$. Thus, for any $r' \in R$, taking $r = h_0r'$ in the last equation yields

$$\begin{aligned}
0 &= a_0h_0r'\alpha^s(b_n) + a_1\alpha(h_0r')\alpha^{1+s}(b_{n-1}) + \cdots + a_n\alpha^n(h_0r')\alpha^{n+s}(b_0)\\
&= a_0r'\alpha^s(b_n) + a_1\alpha(h_0r'\alpha^s(b_{n-1})) + \cdots + a_n\alpha^n(h_0r'\alpha^s(b_0))\\
&= a_0r'\alpha^s(b_n)\,.
\end{aligned}$$

Hence $a_0R\alpha^s(b_n) = 0$. Now the result follows. □

**Theorem 2.** *Let $R$ be a ring satisfying descending chain condition on right annihilators and $\alpha$ a ring automorphism of $R$. Then the following conditions are equivalent:*

(1) *$R[[x;\alpha]]$ is a left APP-ring.*

(2) *For any sequence $(b_0, b_1, \dots)$ of elements of $R$, $l_R(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}R\alpha^k(b_j))$ is right $s$-unital.*

**Proof.** (1)⇒(2). Suppose that $(b_0, b_1, \dots)$ is a sequence of elements of $R$. Set $g(x) = b_0 + b_1x + b_2x^2 + \cdots \in R[[x;\alpha]]$. Let $a \in l_R\big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}R\alpha^k(b_j)\big)$. Then $aR[[x;\alpha]]g(x) = 0$. Since $R[[x;\alpha]]$ is a left APP-ring, there exists $h(x) = h_0 + h_1x + h_2x^2 + \cdots \in l_{R[[x;\alpha]]}(R[[x;\alpha]]g(x))$ such that $a = ah(x)$. Thus we have $a = ah_0$ and $h(x)(rx^s)g(x) = 0$. Hence

$$\sum_{i+j=n} h_i\alpha^i(r)\alpha^{i+s}(b_j) = 0\,, \quad \forall\ n\,.$$

By Lemma 1, $h_0R\alpha^s(b_j) = 0$ for any $j$. Thus $h_0 \in l_R\big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}R\alpha^k(b_j)\big)$.

(2)⇒(1). Suppose that $f(x) = a_0+a_1x+a_2x^2+\dots$, $g(x) = b_0+b_1x+b_2x^2+\cdots \in R[[x;\alpha]]$ are such that $f(x)R[[x;\alpha]]g(x) = 0$. Then for any $r \in R$, $f(x)(rx^s)g(x) = 0$. It follows that

$$\sum_{i+j=k} a_i\alpha^i(r)\alpha^{i+s}(b_j) = 0\,, \qquad k = 0, 1, 2, \dots\,, \tag{1}$$

where $r$ is an arbitrary element of $R$. Thus, since $a_0r\alpha^s(b_0) = 0$ for any $s$, one has $a_0 \in l_R\big(\sum_{s=0}^{\infty}R\alpha^s(b_0)\big)$. By the hypothesis for the sequence $(b_0, 0, 0, \dots)$ of elements of $R$, there exists $p_0 \in l_R\big(\sum_{k=0}^{\infty}R\alpha^k(b_0)\big)$ such that $a_0 = a_0p_0$.

Suppose that $c_0, c_1, \cdots \in R$ are such that $a_i = \alpha^i(c_i)$. Let $r' \in R$ and take $r = p_0r'$ in $a_1\alpha(r)\alpha^{1+s}(b_0)+a_0r\alpha^s(b_1) = 0$. Then $a_1\alpha(p_0r')\alpha^{1+s}(b_0)+a_0p_0r'\alpha^s(b_1) = 0$.

Since $p_0 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0)\big)$, we have $a_1\alpha(p_0r')\alpha^{1+s}(b_0) = a_1\alpha(p_0r'\alpha^s(b_0)) = 0$. Thus $a_0r'\alpha^s(b_1) = a_0p_0r'\alpha^s(b_1) = 0$ for any $s = 0, 1, \dots$, which implies that $a_0 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_1)\big)$. Also $a_1\alpha(r)\alpha^{1+s}(b_0) = 0$ for any $r \in R$. Thus $\alpha(c_1r\alpha^s(b_0)) = 0$. Since $\alpha$ is an automorphism, it follows that $c_1r\alpha^s(b_0) = 0$ for any $s = 0, 1, \dots$. This means that $c_1 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0)\big)$.

Inductively, assume that $q \geq 1$ is such that

$$c_i \in l_R\Big(\sum_{k=0}^{\infty} R\alpha^k(b_j)\Big), \qquad i + j = 0, 1, 2, \dots, q-1\,.$$

Note that $c_0 = a_0$.

Since $c_0, c_1, \dots, c_{q-1} \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0)\big)$ and $l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0)\big)$ is right $s$-unital, there exists $r_0 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0)\big)$ such that $c_i = c_ir_0$, $i = 0, 1, \dots, q-1$. Let $r' \in R$ and take $r = r_0r'$. Then by the equation of (1) for the case when $k = q$, we have

$$a_0r_0r'\alpha^s(b_q) + \cdots + a_{q-1}\alpha^{q-1}(r_0r')\alpha^{q-1+s}(b_1) + a_q\alpha^q(r_0r')\alpha^{q+s}(b_0) = 0\,.$$

For any $i$ with $0 \leq i \leq q-1$, we have $a_i\alpha^i(r_0r')\alpha^{i+s}(b_{q-i}) = \alpha^i\big(c_ir_0r'\alpha^s(b_{q-i})\big) = \alpha^i\big(c_ir'\alpha^s(b_{q-i})\big) = a_i\alpha^i(r')\alpha^{i+s}(b_{q-i})$. Also $a_q\alpha^q(r_0r')\alpha^{q+s}(b_0) = a_q\alpha^q\big(r_0r'\alpha^s(b_0)\big) = 0$ since $r_0 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0)\big)$. Thus

$$a_0r'\alpha^s(b_q) + a_1\alpha(r')\alpha^{1+s}(b_{q-1}) + \cdots + a_{q-1}\alpha^{q-1}(r')\alpha^{q-1+s}(b_1) = 0\,. \tag{2}$$

By (1) it follows that $a_q\alpha^q(r)\alpha^{q+s}(b_0) = 0$. Thus $\alpha^q\big(c_qr\alpha^s(b_0)\big) = 0$, which implies that $c_qr\alpha^s(b_0) = 0$ for any $s$ and any $r \in R$. Hence $c_q \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0)\big)$.

Since $c_0, c_1, \dots, c_{q-2} \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0) + \sum_{k=0}^{\infty} R\alpha^k(b_1)\big)$, there exists $r_1 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0) + \sum_{k=0}^{\infty} R\alpha^k(b_1)\big)$ such that $c_i = c_ir_1$ for any $i$ with $0 \leq i \leq q-2$. Thus $a_i\alpha^i(r_1r'')\alpha^{i+s}(b_{q-i}) = \alpha^i\big(c_ir_1r''\alpha^s(b_{q-i})\big) = \alpha^i\big(c_ir''\alpha^s(b_{q-i})\big) = a_i\alpha^i(r'')\alpha^{i+s}(b_{q-i})$ for any $i$ with $0 \leq i \leq q-2$. Now setting $r' = r_1r''$ in (2) yields

$$a_0r''\alpha^s(b_q) + a_1\alpha(r'')\alpha^{1+s}(b_{q-1}) + \cdots + a_{q-2}\alpha^{q-2}(r'')\alpha^{q-2+s}(b_2) = 0$$

for any $r'' \in R$ since $a_{q-1}\alpha^{q-1}(r_1r'')\alpha^{q-1+s}(b_1) = a_{q-1}\alpha^{q-1}(r_1r''\alpha^s(b_1)) = 0$. Thus, by (2), $a_{q-1}\alpha^{q-1}(r')\alpha^{q-1+s}(b_1) = 0$. This means that $c_{q-1}r'\alpha^s(b_1) = 0$ since $\alpha$ is an automorphism. Hence $c_{q-1} \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_1)\big)$. Continuing this procedure yields $c_{q-2} \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_2)\big) \dots, c_1 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_{q-1})\big), c_0 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_q)\big)$.

Hence we have shown that for any $i$ and $j$, $c_i \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_j)\big)$. Thus $c_i \in l_R\big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R\alpha^k(b_j)\big)$. Consider the descending chain as following:

$$r_R(c_0) \supseteq r_R(c_0, c_1) \supseteq r_R(c_0, c_1, c_2) \supseteq \dots\,,$$

there exists $n$ such that $r_R(c_0, c_1, \dots, c_n) = r_R(c_0, c_1, \dots, c_n, c_{n+1}) = \dots$. By the hypothesis, $l_R\big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R\alpha^k(b_j)\big)$ is right $s$-unital by considering sequence $(b_0, b_1, \dots)$. Thus there exists $e \in l_R\big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R\alpha^k(b_j)\big)$ such that $c_i = c_ie$, $i = 0, 1, \dots, n$. Clearly $1 - e \in r_R(c_0, c_1, \dots, c_n)$. Thus $c_k = c_ke$ for all $k = 0, 1, \dots$. Now $f(x) = a_0 + \alpha(c_1)x + \alpha^2(c_2)x^2 + \cdots = a_0e + \alpha(c_1e)x + \alpha^2(c_2e)x^2 + \cdots =$

$a_0e+a_1\alpha(e)x+a_2\alpha^2(e)x^2+\cdots=f(x)e$ and $e\in l_{R[[x;\alpha]]}\big(R[[x;\alpha]]g(x)\big)$. This means that $R[[x;\alpha]]$ is a left APP-ring. □

It was shown in [9] that if $R$ is a left APP-ring satisfying descending chain condition on left and right annihilators then $R[[x]]$ is left APP. By Theorem 2 we have the following result.

**Corollary 3.** *Let $R$ be a ring satisfying descending chain condition on right annihilators. Then the following conditions are equivalent:*

(1) *$R[[x]]$ is a left APP-ring.*

(2) *For any sequence $(b_0,b_1,\dots)$ of elements of $R$, $l_R(\sum_{j=0}^{\infty}Rb_j)$ is right s-unital.*

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