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MULTIPLE POSITIVE SOLUTIONS FOR NONLINEAR BOUNDARY VALUE PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper we investigate the existence of multiple positive solutions for nonlinear boundary value problems with integral boundary conditions. We shall rely on the Leggett-Williams fixed point theorem.

1. Introduction

This paper is concerned with the existence of three nonnegative solutions for nonlinear boundary value problems with integral boundary conditions. More precisely, in Section 3, we consider the following nonlinear boundary value problem with integral boundary conditions:

$$-x''(t) = f\big(x(t)\big)\,, \quad \text{for each} \quad t \in [0,1]\,, \tag{1}$$

$$x(0) - k_1 x'(0) = \int_0^1 h_1\big(x(s)\big)\, ds\,, \tag{2}$$

$$x(1) + k_2 x'(1) = \int_0^1 h_2\big(x(s)\big)\, ds \tag{3}$$

where $f, h_1, h_2 : [0,\infty) \to [0,\infty)$ are continuous and nondecreasing functions, and k_1, k_2 are nonnegative constants.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint and nonlocal boundary value problems as special cases. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers [7, 9, 13] and the references therein. Moreover, boundary value problems with integral boundary conditions have been studied by a number of authors, for example [3, 5, 6, 11, 10].

Recently the existence of multiple solutions for differential, difference and integral equations has been investigated by several authors (see for instance [1, 2, 4] and the references cited therein). The main theorem of this note extends the particular

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problem (1)–(3) with $h_1 = h_2 \equiv 0$ considered in [2] and the references therein. Our approach here is based on the Leggett-Williams fixed point theorem in cones [12].

2. Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper.
$C([0,1],\mathbb{R})$ is the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|x\|_\infty = \sup\{|x(t)| : 0 \le t \le 1\}\,.$$

$C^2((0,1),\mathbb{R})$ is the space of functions $x : (0,1) \to \mathbb{R}$, whose first and second derivatives are continuous.

Let $(E, \|\cdot\|)$ be a Banach space and $C \subset E$ be a cone in E. By a concave nonnegative continuous functional ψ on C we mean a continuous mapping $\psi\colon C \to [0,\infty)$ with

$$\psi(\lambda x + (1-\lambda)y) \ge \lambda\psi(x) + (1-\lambda)\psi(y) \quad \text{for all} \quad x, y \in C \quad \text{and} \quad \lambda \in [0,1]\,.$$

For K, L, $r > 0$ be constants with C and ψ as above, let

$$C_K = \{y \in C : \|y\| < K\}$$

and

$$C(\psi, L, K) = \{y \in C : \psi(y) \ge L \quad \text{and} \quad \|y\| \le K\}\,.$$

Our consideration is based on the following fixed point theorem given by Leggett and Williams in 1979 [12] (see also Guo and Lakshmikantham [8]).

Theorem 2.1. *Let E be a Banach space, $C \subset E$ a cone in E and $R > 0$ a constant. Suppose there exists a concave nonnegative continuous functional ψ on C with $\psi(y) \le \|y\|$ for all $y \in \overline{C}_R$ and let $N\colon \overline{C}_R \to \overline{C}_R$ be a continuous compact map. Assume that there are numbers r, L and K with $0 < r < L < K \le R$ such that*

- (A1) *$\{y \in C(\psi, L, K) : \psi(y) > L\} \neq \emptyset$ and $\psi\big(N(y)\big) > L$ for all $y \in C(\psi, L, K)$;*
- (A2) *$\|N(y)\| < r$ for all $y \in \overline{C}_r$;*
- (A3) *$\psi\big(N(y)\big) > L$ for all $y \in C(\psi, L, R)$ with $\|N(y)\| > K$.*

Then N has at least three fixed points y_1, y_2, y_3 in $\overline{C}_R$. Furthermore, we have

$$y_1 \in C_r\,, \quad y_2 \in \big\{y \in C(\psi, L, R) : \psi(y) > L\big\}$$

and

$$y_3 \in \overline{C}_R - \big\{C(\psi, L, R) \cup \overline{C}_r\big\}\,.$$

3. Main result

We start by defining what we mean by a solution of problem (1)–(3).

Definition 3.1. A function $x \in C^2((0,1),\mathbb{R})$ is said to be a solution of (1)–(3) if x satisfies $-x''(t) = f\big(x(t)\big)$ for each $t \in J$ and the conditions (2) and (3).

We need the following auxiliary result. Its proof uses standard argument.

Lemma 3.1. *x is a solution of (1)–(3) if and only if*

$$x(t) = P(t) - \int_0^1 G(t,s) f\big(x(s)\big)\, ds\,,$$

where

$$P(t) = \frac{1}{1+k_1+k_2}\Big\{(1-t+k_2)\int_0^1 h_1\big(x(s)\big)\,ds + (k_1+t)\int_0^1 h_2\big(x(s)\big)\,ds\Big\}$$

is the unique solution of the problem

$$\begin{aligned} -x''(t) &= 0\,, \quad \text{for each} \quad t \in [0,1]\,,\\ x(0) - k_1 x'(0) &= \int_0^1 h_1\big(x(s)\big)\,ds\,,\\ x(1) + k_2 x'(1) &= \int_0^1 h_2\big(x(s)\big)\,ds\,, \end{aligned}$$

and

$$G(t,s) = \frac{-1}{k_1+k_2+1}\begin{cases} (k_1+t)(1-s+k_2)\,, & 0 \le t \le s \le 1\,,\\ (k_1+s)(1-t+k_2)\,, & 0 \le s \le t \le 1 \end{cases}$$

is the Green function of the corresponding homogeneous problem.

We note that $-G(t,s) < -G(s,s)$ on $[0,1]\times[0,1]$. For $\rho > 0$ let

$$P_\rho = \frac{1}{1+k_1+k_2}\Big\{(1+k_2)\int_0^1 h_1(\rho)\,ds + (1+k_1)\int_0^1 h_2(\rho)\,ds\Big\}\,.$$

Theorem 3.1. *Assume*

(H1) *There exists a constant $r > 0$ such that*

$$P_r + f(r)\sup_{t\in[0,1]}\Big(\int_0^1 G(t,s)ds\Big) < r\,;$$

(H2) *there exist $L > r$ and an interval $[a,b] \subset (0,1)$ such that*

$$f(L)\min_{t\in[a,b]}\Big(-\int_0^1 G(t,s)\,ds\Big) > L\,;$$

(H3) *there exist $0 < r < L < K < R$, $M^{-1}L \le R$ such that*

$$P_R - f(R)\sup_{t\in[0,1]}\Big(\int_0^1 G(t,s)\,ds\Big) \le R\,,$$

where

$$P(t) \geq c_1 P_R\,, \quad -G(t,s) \geq -c_2 G(s,s) \quad \textit{for all} \quad t,s \in [0,1],\ c_1, c_2 \in (0,1)\,,$$

and

$$M = \min\{c_1, c_2\}\,,$$

are satisfied. Then the problem (1)–(3) *has at least three positive solutions.*

Proof. Transform the problem into a fixed point problem. Consider the operator, $N : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ defined by

$$N\big(x(t)\big) = P(t) - \int_0^1 G(t,s) f\big(x(s)\big)\, ds\,.$$

We shall show that N satisfies the assumptions of the Leggett-Williams fixed point theorem (see [12]). The proof will be given in several steps.
Step 1. N is continuous.
Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C([0,1],\mathbb{R})$. Then for each $t \in [0,1]$

$$\begin{aligned}\big|N\big(x_n(t)\big) - N\big(x(t)\big)\big| \leq & \int_0^1 \big|G(t,s)\big|\, \big|f\big(x_n(s)\big) - f\big(x(s)\big)\big|\, ds \\ & + \frac{1+k_2}{1+k_1+k_2} \int_0^1 \big|h_1\big(x_n(s)\big) - h_1\big(x(s)\big)\big|\, ds \\ & + \frac{1+k_1}{1+k_1+k_2} \int_0^1 \big|h_2\big(x_n(s)\big) - h_2\big(x(s)\big)\big|\, ds\,.\end{aligned}$$

Since the functions f, h_1 and h_2 are continuous, we have

$$\begin{aligned}\big\|N(x_n) - N(x)\big\|_\infty \leq & \frac{(1+k_1)(1+k_2)}{1+k_1+k_2} \big\|f(x_n) - f(x)\big\|_\infty \\ & + \frac{1+k_2}{1+k_1+k_2} \big\|h_1(x_n) - h_1(x)\big\|_\infty \\ & + \frac{1+k_1}{1+k_1+k_2} \big\|h_2(x_n) - h_2(x)\big\|_\infty \to 0 \quad \text{as} \quad n \to \infty\,.\end{aligned}$$

Step 2. N maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.
Indeed, it is enough to show that there exists a positive constant ℓ such that for each $x \in B_q = \big\{x \in C([0,1],\mathbb{R}) : \|x\|_\infty \leq q\big\}$, one has $\big\|N(x)\big\|_\infty \leq \ell$.
Let $x \in B_q$. Then for each $t \in [0,1]$, we have

$$N\big(x(t)\big) = P(t) - \int_0^1 G(t,s) f\big(x(s)\big)\, ds\,.$$

Since f, h_1 and h_2 are nondecreasing functions, we have

$$\begin{aligned}\big|N\big(x(t)\big)\big| &\leq \big|P(t)\big| + \int_0^1 \big|G(t,s)\big|\,\big|f\big(x(s)\big)\big|\,ds \\ &\leq \frac{\max(1+k_1,1+k_2)}{1+k_1+k_2}\big\{h_1(q)+h_2(q)\big\} \\ &\quad + \sup_{(t,s)\in[0,1]\times[0,1]} \big|G(t,s)\big| f(q)\,.\end{aligned}$$

Therefore

$$\begin{aligned}\big\|N(x)\big\|_\infty &\leq \frac{\max(1+k_1,1+k_2)}{1+k_1+k_2}\big\{h_1(q)+h_2(q)\big\} \\ &\quad + \sup_{(t,s)\in[0,1]\times[0,1]} \big|G(t,s)\big| f(q) := \ell\,.\end{aligned}$$

Step 3. N maps bounded sets into equicontinuous sets of $C\big([0,1],\mathbb{R}\big)$.

Let $r_1, r_2 \in [0,1]$, $r_1 < r_2$, B_q be a bounded set of $C\big([0,1],\mathbb{R}\big)$ as in Step 2 and $x \in B_q$. Then

$$\begin{aligned}\big|N\big(x(r_2)\big) - N\big(x(r_1)\big)\big| &\leq \big|P(r_2)-P(r_1)\big| + \int_0^1 \big|G(r_2,s)-G(r_1,s)\big|\,\big|f\big(x(s)\big)\big|\,ds \\ &\leq \big|P(r_2)-P(r_1)\big| + \int_0^1 \big|G(r_2,s)-G(r_1,s)\big| f(q)\,ds \\ &\leq \frac{|r_2-r_1|}{1+k_1+k_2}\int_0^1 h_1\big(x(s)\big)\,ds + \frac{|r_2-r_1|}{1+k_1+k_2}\int_0^1 h_2(x(s))ds \\ &\quad + \int_0^1 \big|G(r_2,s)-G(r_1,s)\big| f(q)\,ds \\ &\leq \frac{|r_2-r_1|}{1+k_1+k_2}\big[h_1(q)+h_2(q)\big] \\ &\quad + \int_0^1 \big|G(r_2,s)-G(r_1,s)\big| f(q)\,ds\,.\end{aligned}$$

The right-hand side of the above inequality tends to zero as $r_2 - r_1 \to 0$.

As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $N\colon C\big([0,1],\mathbb{R}\big) \to C\big([0,1],\mathbb{R}\big)$ is continuous and compact.

Let $C = \big\{x \in C([0,1],\mathbb{R}) : x(t) \geq 0 \text{ for each } t \in [0,1]\big\}$ be a cone in $C\big([0,1],\mathbb{R}\big)$. Since h_i, $i = 1,2$ and f are positive functions, then $N(C) \subset C$ and $N\colon \overline{C}_R \to \overline{C}_R$ is completely continuous. By (H1), (H2) and (H3) we can show that if $x \in \overline{C}_R$ then $N(x) \in \overline{C}_R$.

Let $\psi\colon C \to [0,\infty)$ defined by $\psi(x) = \min\limits_{t\in[a,b]} x(t)$. It is clear that ψ is a nonnegative concave continuous functional and $\psi(x) \leq \|x\|_\infty$ for $x \in \overline{C}_R$. Now it remains to show that the hypotheses of Theorem 2.1 are satisfied. First notice that condition

(A2) of Theorem 2.1 holds since for $x \in \overline{C}_r$ we have

$$N\big(x(t)\big) = P(t) - \int_0^1 G(t,s) f\big(x(s)\big)\, ds\,.$$

Thus

$$\begin{aligned}\big|N\big(x(t)\big)\big| &\le \big|P(t)\big| + \int_0^1 \big|G(t,s)\big|\,\big|f\big(x(s)\big)\big|\, ds\\ &\le P_r + f(r)\sup_{t\in[0,1]}\Big(\int_0^1 G(t,s)\, ds\Big) < r\,.\end{aligned}$$

Let $x(t) = \dfrac{L+K}{2}$ for $t \in [0,1]$. By the definition of $C(\psi, L, K)$, x belongs to $C(\psi, L, K)$. Then $x \in \big\{x \in C(\psi, L, K) : \psi(x) > L\big\}$. Also if $x \in C(\psi, L, K)$, then

$$\psi\big(N(x)\big) = \min_{t\in[a,b]}\Big(P(t) - \int_0^1 G(t,s) f\big(x(s)\big)\, ds\Big)\,.$$

Then from (H2) we have

$$\begin{aligned}\psi(N(x)) =& \min_{t\in[a,b]}\Big(P(t) - \int_0^1 G(t,s) f\big(x(s)\big)\, ds\Big)\\ &\ge \min_{t\in[a,b]}\Big(-\int_0^1 G(t,s) f\big(x(s)\big)\, ds\Big)\\ &\ge f(L)\min_{t\in[a,b]}\Big(-\int_0^1 G(t,s)\, ds\Big) > L\,.\end{aligned}$$

So the condition (A1) of Theorem 2.1 is satisfied.

Finally, we will prove that (A3) of Theorem 2.1 holds. Let $x \in C(\psi, L, R)$ with $\|N(x)\|_\infty > K$. Then

$$N\big(x(t)\big) = P(t) - \int_0^1 G(t,s) f\big(x(s)\big)\, ds \quad \text{for} \quad t \in [0,1]\,.$$

Thus by (H3) we have

$$\begin{aligned}\psi\big(N(x)\big) &= \min_{t\in[a,b]}\Big(P(t) - \int_0^1 G(t,s) f\big(x(s)\big)\, ds\Big)\\ &\ge c_1 P_R - c_2\int_0^1 G(s,s) f\big(x(s)\big)\, ds\\ &\ge M\big\|N(x)\big\|_\infty > MK \ge L\,.\end{aligned}$$

Thus, the condition (A3) of Theorem 2.1 holds. Then, the Leggett-Williams fixed point theorem implies that N has at least three fixed points x_1, x_2 and x_3 which are solutions to the problem (1)–(3). Furthermore, we have

$$x_1 \in C_r\,, \quad x_2 \in \big\{x \in C(\psi, L, R) : \psi(x) > L\big\}$$

and

$$x_3 \in \overline{C}_R - \big\{C(\psi, L, R) \cup \overline{C}_r\big\}\,.$$

□

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