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# PROPORTIONALITY PRINCIPLE FOR CUSPED MANIFOLDS

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Abstract. We give a short proof of the proportionality principle for cusped hyperbolic manifolds.

## 1. Introduction

Let $M$ be a hyperbolic manifold of finite volume $\mathrm{Vol}(M)$. The Gromov-Thurston proportionality principle is the following theorem, where $\|M, \partial M\|$ denotes the simplicial volume and $V_n$ is the volume of a regular ideal simplex in hyperbolic n-space.

**Theorem 1** ([6], Theorem 6.5.4.)**.** *Let $M$ be a compact, orientable $n$-manifold such that $\mathrm{int}(M) = M - \partial M$ admits a hyperbolic metric of finite volume $\mathrm{Vol}(M)$. Then*

$$\|M, \partial M\| = \frac{1}{V_n} \mathrm{Vol}(M) \ .$$

The proportionality principle for closed manifolds was proved by Gromov ([3, Section 2.2]), for a more detailed elaboration of the proof see [1], Theorem C.4.2. (An alternative proof using measure homology was proposed by Thurston and completed in [4].)

For cusped manifolds (i.e., non-compact manifolds of finite volume), the proportionality principle was stated in [6], Theorem 6.5.4. The proof in [6] is very brief and uses another definition of the simplicial volume. Thus it remained an open question whether Gromov's proof has a direct generalization to cusped manifolds. Such a generalization is actually possible, all necessary arguments are contained in the recent paper [2].

The aim of our paper is to give yet another, somewhat shorter, proof of the proportionality principle for cusped manifolds, using bounded cohomology and measure chains to give a more direct generalization of Gromov's argument.

We give a short outline of the argument. Consider the map which pinches all boundary tori to points, and let $(M', \partial M')$ be the quotient, with $\partial M'$ a finite number of points. It follows from Gromov's theory of multicomplexes that this

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map induces an isometry of Gromov norms. There is an obvious isometry between $H_*(M', \partial M')$ and the (absolute) homology theory constructed from ideal simplices with all ideal vertices in the cusps of $M$. The main point in the proof of the wanted inequality is then to check that to the latter homology theory one can apply Gromov's smearing construction. A little care is needed because one can not apply the smearing construction to ideal simplices: one is not allowed to have ideal simplices with vertices not in cusps. In section 2.3, which is the main part of this paper, this technical point is surmounted.

## 2. Proof

### 2.1. **Crushing.**

**Definition 1.** For a manifold $M$ with boundary $\partial M$ we denote by $M^{\mathrm{crush}}$ resp. $\partial M^{\mathrm{crush}}$ the quotient $M/\sim$ of $M$ resp. $\partial M/\sim$ of $\partial M$ under the following equivalence relation: $x \sim y$ if and only if $x$ and $y$ belong to the same connected component of $\partial M$ (or if $x = y \notin \partial M$). We denote by $\mathrm{crush}\colon (M, \partial M) \to \left(M^{\mathrm{crush}}, \partial M^{\mathrm{crush}}\right)$ the canonical projection.

**Lemma 1.** *Let $M$ be an $n$-dimensional compact, orientable, connected manifold with boundary $\partial M$. If each connected component of $\partial M$ has amenable fundamental group, then $(\mathrm{crush})_n : H_n(M, \partial M) \to H_n\left(M^{\mathrm{crush}}, \partial M^{\mathrm{crush}}\right)$ is an isometry of Gromov norms.*

**Proof.** We have $H_n(M, \partial M) \simeq H_n\left(M^{\mathrm{crush}}, \partial M^{\mathrm{crush}}\right) \simeq \mathbb{R}$. This implies, by the argument in [3], section 1.1, that for proving equality of Gromov-norms it suffices to show that $(\mathrm{crush})^*$ induces an isometric isomorphism in bounded cohomology.

Because each connected component of $\partial M$ has amenable fundamental group, the Relative Mapping Theorem ([3], section 4.1) states that $(\mathrm{crush})^*$ is indeed an isometric isomorphism in bounded cohomology. □

For a topological space $X$, the measure chain complex $\mathcal{C}_*(X)$ is the vector space of signed measures on $\mathrm{map}(\Delta^*, X)$ with compact determination set and finite variation. The total variation gives a norm on $\mathcal{C}_*(X)$ which descends to a seminorm on the homology of this complex (for the usual boundary operator), the so-called measure homology $\mathcal{H}_*(X)$. (For a detailed account on measure homology see [4], section 2). For a pair of spaces $(X, Y)$, $\mathcal{C}_*(X, Y)$ is the vector space of signed measures on $\mathrm{map}(\Delta^*, X) / \mathrm{map}(\Delta^*, Y)$ with compact determination set and finite variation.

**Lemma 2** ([4])**.** *If $N$ is a compact manifold with $\pi_1$-injective boundary $\partial N$, then the inclusion $i_*\colon C_*(N, \partial N) \to \mathcal{C}_*(N, \partial N)$ induces an isometric isomorphism in homology.*

**Proof.** This is [4], Theorem 1.1, in the case $\partial N = \emptyset$. We explain how the proof can be adapted to the relative case.
Let $p\colon \widetilde{N} \to N$ be the universal covering. The main step in [4] is the construction of a Borel map $s_k\colon \mathrm{map}\left(\Delta^k, N\right) \to \mathrm{map}\left(\Delta^k, \widetilde{N}\right)$ with $p_k s_k = \mathrm{id}$. The construction of $s_k$ consists in the construction of a countable covering $\mathrm{map}\left(\Delta^k, \widetilde{N}\right) = \cup_{j=1}^{\infty} V_j$

such that $p\mid_{V_j}$ is a homeomorphism, and to succesively define $s_k = p^{-1}$ on $p(V_n) - \cup_{j=1}^{n-1} p(V_j)$.

In the proof of [4, Theorem 4.1], $s_k$ is used to define a map $v_k\colon \operatorname{map}\left(\widetilde{N}^{k+1}, \mathbb{R}\right)^{\pi_1 N} \to \operatorname{map}\left(\operatorname{map}(\Delta^k, N), \mathbb{R}\right)$ by $v_k(f) := f\big(s_k(\sigma)(e_0), \dots, s_k(\sigma)(e_k)\big)$.

Let $W_j = V_j \cap \operatorname{map}\left(\Delta^k, p^{-1}(\partial N)\right)$. Since $p\mid_{V_j}$ is a homeomorphism, also its restriction $p\mid_{W_j}\colon W_j \to p(W_j) \subset \operatorname{map}(\Delta^k, \partial N)$ is a homeomorphism. Therefore $s_k$ maps $\operatorname{map}(\Delta^k, \partial N)$ to $\operatorname{map}\left(\Delta^k, p^{-1}(\partial N)\right)$.

This means that $v_k$ maps $\operatorname{map}\left(p^{-1}(\partial N)^{k+1}, \mathbb{R}\right)$ to $\operatorname{map}\left(\operatorname{map}(\Delta^k, \partial N), \mathbb{R}\right)$. Hence we get a relative cochain map of norm $\leq 1$. Since its composition with $i^*$ is an isometric isomorphism (which follows from independence of bounded cohomology from the choice of strong relatively injective resolutions), $i^*$ must actually be an isometry. By duality, the claim follows. □

2.2. **Ideal simplices.**

**Definition 2.** We denote $C_*^{\mathrm{str,ideal}}(\mathbb{H}^n)$ the chain complex of straight, possibly ideal, simplices in hyperbolic space $\mathbb{H}^n$. For a hyperbolic manifold $N = \Gamma\backslash\mathbb{H}^n$ we denote

$$C_*^{\mathrm{str,ideal}}(N) = \Gamma\backslash C_*^{\mathrm{str,ideal}}(\mathbb{H}^n)\,.$$

Assume that $M$ is an orientable manifold with boundary such that its interior $N = M - \partial M$ admits a complete hyperbolic metric, i.e. $N = \Gamma\backslash\mathbb{H}^n$ for some discrete, torsionfree subgroup $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$. Assume that this hyperbolic metric has finite volume. It is then well-known that points of $\partial M^{\mathrm{crush}}$ (the so-called cusps) correspond to $\Gamma$-orbits of points of the ideal boundary $\partial_\infty\mathbb{H}^n$ which are fixed points for some parabolic element of $\Gamma$. Therefore each n+1-tuple of vertices in $M^{\mathrm{crush}}$ corresponds to a $\Gamma$-orbit of straight simplices with vertices either in $\mathbb{H}^n$ or in parabolic fixed points of $\Gamma$. In particular, we get a straight ideal simplex $\mathrm{str}(\sigma)$ with the same vertices and homotopic rel. vertices to a given simplex $\sigma$ in $M^{\mathrm{crush}}$.

**Lemma 3.**

$$\mathrm{str}\colon C_*\left(M^{\mathrm{crush}}, \partial M^{\mathrm{crush}}\right) \to C_*^{\mathrm{str,ideal}}(N)$$

*is a chain map of norm 1.*

**Proof.** Obvious.

**Corollary 1.** $\|M, \partial M\| \geq \frac{1}{V_n}\operatorname{Vol}(M)$.

**Proof.** Let $\sum a_i\sigma_i$ be a representative of $\left[M^{\mathrm{crush}}, \partial M^{\mathrm{crush}}\right]$. We consider the volume form of the hyperbolic metric on $N = M - \partial M = M^{\mathrm{crush}} - \partial M^{\mathrm{crush}}$. It defines a measure $\nu$ on $N$ which extends to a measure on $M^{\mathrm{crush}}$ by putting $\nu\left(\partial M^{\mathrm{crush}}\right) = 0$. With respect to this measure we have $\operatorname{Vol}(N) = \sum a_i \operatorname{Vol}(\mathrm{str}(\sigma_i))$ and $|\operatorname{Vol}(\mathrm{str}(\sigma_i))| \leq V_n$, from which we get

$$\frac{1}{V_n}\operatorname{Vol}(M) \leq \|\operatorname{str}\left[M^{\mathrm{crush}}, \partial M^{\mathrm{crush}}\right]\| \leq \|M^{\mathrm{crush}}, \partial M^{\mathrm{crush}}\| = \|M, \partial M\|\,.$$

(The first inequality is true because $\sum a_i \operatorname{str}(\sigma_i)$ represents $\operatorname{str}_* \left[M^{\operatorname{crush}}, \partial M^{\operatorname{crush}}\right]$, the second by Lemma 3, the last equality is Lemma 1.) □

2.3. **Smearing construction.** The purpose of this section is to construct representatives of $[M, \partial M] \in \mathcal{H}_n(M, \partial M)$ of norm close to $\frac{1}{V_n}\operatorname{Vol}(M)$. We assume that $N = M - \partial M$ is a complete hyperbolic manifold of finite volume. For $\epsilon > 0$ let $N_{[\epsilon,\infty]} = \{x \in N : \operatorname{inj}(x) \geq \epsilon\}$ be the $\epsilon$-thick part and $N_{[0,\epsilon]} = \{x \in N : \operatorname{inj}(x) \leq \epsilon\}$ the $\epsilon$-thin part.

For some small $\delta > 0$, let $\sigma\colon \Delta^n \to N$ be some fixed straight regular simplex with $\operatorname{Vol}(\sigma) = V_n - \delta$. Abbreviate $G = \operatorname{Isom}^+(\mathbb{H}^n)$. We define $\sigma^*\colon \Gamma\backslash G \to \operatorname{map}(\Delta^n, N)$ by $\sigma^*[g] = g\sigma$. We denote by Haar the Haar measure on $\Gamma\backslash G$, normalised such that $\operatorname{Haar}(\Gamma\backslash G) = \operatorname{Vol}(N)$.

As in [5], we define $\operatorname{smr}(\sigma) \in \mathcal{C}_*(N)$ to be the measure chain given by $\operatorname{smr}(\sigma)(B) := \operatorname{Haar}\left((\sigma^*)^{-1}B\right)$ for $B \subset \operatorname{map}(\Delta^n, N)$, and $\operatorname{Avg}(\sigma) := \frac{1}{2}\operatorname{smr}(\sigma) - \frac{1}{2}\operatorname{smr}(r\sigma)$ for some fixed orientation-reversing isometry $r \in \operatorname{Isom}(\mathbb{H}^n)$. By the same proof as in [5], we have $\partial \operatorname{Avg}(\sigma) = 0$.

Let $\epsilon_0$ be smaller than the Margulis constant for $G$. It follows from the Margulis lemma (see [1, D.3.3]) that, for each $\epsilon_1 \leq \epsilon_0$, $N_{[0,\epsilon_1]}$ is a product and one thus has a homotopy-equivalence $h_{\epsilon_1} : \left(N, N_{[0,\epsilon_1]}\right) \to \left(N_{[\epsilon_1,\infty]}, \partial N_{[\epsilon_1,\infty]}\right)$.

Further we fix some positive constants $\epsilon < \epsilon_1 < \epsilon_0$ such that

$$d\left(\partial N_{[\epsilon_0,\infty]}, \partial N_{[\epsilon_1,\infty]}\right) > l(\delta),$$

$$d\left(\partial N_{[\epsilon_1,\infty]}, \partial N_{[\epsilon,\infty]}\right) > l(\delta),$$

where $l(\delta)$ denotes the edge-length of the regular simplex of volume $V_n - \delta$.

**Definition 3.** For a hyperbolic manifold $N$ and $\sigma, \epsilon$ as above we define

$$C_\epsilon(N) = \left\{\tau \in \operatorname{map}(\Delta^n, N) : \tau(e_0), \ldots, \tau(e_n) \in N_{[\epsilon,\infty]}\right\}$$

and a signed measure $\operatorname{Avg}_\epsilon(\sigma) \in \mathcal{C}_*(N)$ by

$$\operatorname{Avg}_\epsilon(\sigma)(B) = \operatorname{Avg}(\sigma)(B \cap C_\epsilon(N)).$$

**Lemma 4.** *$\partial \operatorname{Avg}_\epsilon(\sigma)$ is determined on* $\operatorname{map}\left(\Delta^n, N_{[0,\epsilon_1]}\right)$.

**Proof.** We know that $\partial \operatorname{Avg}(\sigma) = 0$. Therefore $\partial \operatorname{Avg}_\epsilon(\sigma)$ is determined on simplices which are, at the same time, face of some n-simplex in $C_\epsilon(N)$ and some n-simplex not in $C_\epsilon(N)$, both simplices being regular simplices with volume $V_n - \delta$ and edgelength $l(\delta)$. Since $d\left(\partial N_{[\epsilon_1,\infty]}, \partial N_{[\epsilon,\infty]}\right) > l(\delta)$, this implies that $\partial \operatorname{Avg}_\epsilon(\sigma)$ is determined on simplices with vertices in $N_{[0,\epsilon_1]}$. It is well-known that $N_{[0,\epsilon_1]}$ is convex, therefore $\partial \operatorname{Avg}_\epsilon(\sigma)$ is determined on simplices in $N_{[0,\epsilon_1]}$. □

Therefore $\operatorname{Avg}_\epsilon(\sigma) \in \mathcal{C}_n\left(N, N_{[0,\epsilon_1]}\right)$ is a relative cycle. Applying $h_{\epsilon_1}$, we get a relative cycle

$$(h_{\epsilon_1})_* \operatorname{Avg}_\epsilon(\sigma) \in \mathcal{C}_n\left(N_{[\epsilon_1,\infty]}, \partial N_{[\epsilon_1,\infty]}\right) \simeq \mathcal{C}_n(M, \partial M).$$

**Lemma 5.** *Let $M$ be a compact, orientable $n$-manifold such that $N = \operatorname{int}(M)$ admits a hyperbolic metric of finite volume. Then*

(i) $\| (h_{\epsilon_1})_* \operatorname{Avg}_\epsilon(\sigma) \| \leq \operatorname{Vol}\left(N_{[\epsilon,\infty]}\right)$,

(ii) $(h_{\epsilon_1})_* \operatorname{Avg}_\epsilon(\sigma)$ *represents* $\frac{K}{\operatorname{Vol}(N)}\left[N_{[\epsilon_1,\infty]}, \partial N_{[\epsilon_1,\infty]}\right]$ *with* $K \geq (V_n - \delta)\left(\operatorname{Vol}(N) - (n+1)\operatorname{Vol}(N_{[0,\epsilon_0]})\right)$.

**Proof.** (i) We have $\|(h_{\epsilon_1})_* \operatorname{Avg}_\epsilon(\sigma)\| \leq \|\operatorname{Avg}_\epsilon(\sigma)\|$. Let $\operatorname{smr}_\epsilon(\sigma)$ be the positive part of $\operatorname{Avg}_\epsilon(\sigma)$. Then $\|\operatorname{Avg}_\epsilon(\sigma)\| = \frac{1}{2}\|\operatorname{smr}_\epsilon(\sigma)\| + \frac{1}{2}\|\operatorname{smr}_\epsilon(r\sigma)\| = \|\operatorname{smr}_\epsilon(\sigma)\|$, such that it suffices to compute the latter. This is

$$\begin{aligned}\|\operatorname{smr}_\epsilon(\sigma)\| &= \operatorname{smr}_\epsilon(\sigma)\big(\operatorname{map}(\Delta^n, N)\big) = \operatorname{smr}(\sigma)\big(C_\epsilon(N)\big)\\ &= \operatorname{Haar}\left((\sigma^*)^{-1}C_\epsilon(N)\right) \leq \operatorname{Vol}\left(N_{[\epsilon,\infty]}\right).\end{aligned}$$

(ii) To determine the homology class of a relative n-cycle it suffices to evaluate it against the volume form dvol of $N_{[\epsilon_1,\infty]}$. We compute

$$\begin{aligned}\left\langle \operatorname{dvol}, (h_{\epsilon_1})_* \operatorname{smr}_\epsilon(\sigma)\right\rangle &= \left\langle (h_{\epsilon_1})^* \operatorname{dvol}, \operatorname{smr}_\epsilon(\sigma)\right\rangle\\ &= \int_{\operatorname{map}(\Delta^n,N)} \Big(\int_\tau (h_{\epsilon_1})^* \operatorname{dvol}\Big) \operatorname{dsmr}_\epsilon(\sigma)(\tau)\\ &= \int_{C_\epsilon(N)} \Big(\int_\tau (h_{\epsilon_1})^* \operatorname{dvol}\Big) \operatorname{dsmr}(\sigma)(\tau)\\ &= \int_{C_\epsilon(N)} \operatorname{vol}(h_{\epsilon_1}\tau) \operatorname{dsmr}(\sigma)(\tau) = \int_{\Gamma\backslash G} \operatorname{vol}(h_{\epsilon_1} g\sigma) \operatorname{dHaar}(\Gamma g).\end{aligned}$$

Let $v_0 = \sigma(e_0), \ldots, v_n = \sigma(e_n)$ be the vertices of $\sigma$. Let $g \in G$. If all $gv_i \in N_{[0,\epsilon_1]}$, then $g\sigma \subset N_{[0,\epsilon_1]}$ because $d\left(\partial N_{[0,\epsilon]}, \partial N_{[0,\epsilon_1]}\right) > l(\delta)$. Thus $\operatorname{vol}(h_{\epsilon_1} g\sigma) = \operatorname{vol}(g\sigma) = V_n - \delta$ except possibly if $gv_i \notin N_{[0,\epsilon_1]}$ for some $v_i$. But the set

$$\left\{g \in \Gamma\backslash G : gv_i \notin N_{[0,\epsilon_1]} \text{ for some } i\right\}$$

has Haar measure smaller than $(n+1)\operatorname{Vol}\left(N_{[0,\epsilon_0]}\right)$. This implies the claimed inequality

$$\left\langle \operatorname{dvol}, (h_{\epsilon_1})_* \operatorname{smr}_\epsilon(\sigma)\right\rangle \geq (V_n - \delta)\left(\operatorname{Vol}(N) - (n+1)\operatorname{Vol}(N_{[0,\epsilon_0]})\right).$$

□

**Theorem 2.** *Let $M$ be a compact, orientable $n$-manifold such that $\operatorname{int}(M)$ admits a hyperbolic metric of finite volume.*

*Then $\|M\| = \frac{1}{V_n}\operatorname{Vol}(M)$.*

**Proof.** The lower bound follows from Corollary 1. To get the upper bound we observe that, using $(M, \partial M) \simeq \left(N_{[\epsilon_1,\infty]}, \partial N_{[\epsilon_1,\infty]}\right)$, Lemma 5 implies

$$\|M, \partial M\| \leq \frac{\operatorname{Vol}(M)}{V_n - \delta} \frac{\operatorname{Vol}(N_{[\epsilon,\infty]})}{\left(\operatorname{Vol}(N) - (n+1)\operatorname{Vol}(N_{[0,\epsilon_0]})\right)}.$$

This implies the wanted inequality if we let $\delta, \epsilon_0, \epsilon \to 0$. □

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