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CALCULATIONS IN NEW SEQUENCE SPACES

Bruno de Malafosse

Abstract. In this paper we define new sequence spaces using the concepts of strong summability and boundedness of index $p > 0$ of r-th order difference sequences. We establish sufficient conditions for these spaces to reduce to certain spaces of null and bounded sequences.

1. Introduction and preliminary results.

This paper is organized as follows. First we recall some well known results on matrix transformations. In Section 2 we deal with the identity $E(\Delta) = E$ where E is either of the sets s_α°, $s_\alpha^{(c)}$, or s_α. In Section 3 we recall some results on the sets

$$[A_1(\lambda), A_2(\mu)] = \{X \in s \ : A_1(\lambda)(|A_2(\mu) X|) \in s_\alpha\}$$

where $A_1(\lambda)$ and $A_2(\mu)$ are of the form $C(\xi)$, or $C^+(\xi)$, or $\Delta(\xi)$, or $\Delta^+(\xi)$ and we give sufficient conditions to get $[A_1(\lambda), A_2(\mu)]$ in the form s_γ. The main results are stated in Theorem 9, Theorem 12 and Theorem 13 of Section 4. Among other things we give sufficient conditions to reduce the sets

$$\left[A_1^r(\lambda), A_2^{(p)}(\mu)\right] = \{X \in s : \ A_1^r(\lambda)(|A_2(\mu) X|^p) \in s_\alpha\},$$

for $A_1 = C(\lambda)$, or $C^+(\lambda)$, or $\Delta(\lambda)$, or $\Delta^+(\lambda)$, and $A_2 = \Delta(\mu)$, or $\Delta^+(\mu)$, or $C(\mu)$, or $C^+(\mu)$ to spaces of the form s_ξ.

Now give definitions and notations used in the following. For a given infinite matrix $A = (a_{nm})_{n,m\geq 1}$ the operators A_n are defined for any integer $n \geq 1$, by

$$A_n(X) = \sum_{m=1}^{\infty} a_{nm} x_m \tag{1}$$

where $X = (x_n)_{n\geq 1}$, the series intervening on the right hand being convergent. So we are led to the study of the infinite linear system

$$A_n(X) = b_n \quad n = 1, 2, \dots \tag{2}$$

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where $B = (b_n)_{n\geq 1}$ is a *one-column matrix and* X *the unknown*, see [2-4]. Equation (2) can be written in the form $AX = B$, where $AX = (A_n(X))_{n\geq 1}$. In this paper we shall also consider A as an operator from a sequence space into another sequence space.

A Banach space E of complex sequences with the norm $\| \|_E$ is a BK space if each projection $P_n X = x_n$ for all $X \in E$ is continuous. A *BK space* E is said *to have AK*, (see [12-14]), if for every $B = (b_n)_{n\geq 1} \in E$, then $B = \sum_{m=1}^{\infty} b_m e_m$, (with $e_n = (0, \dots, 1, \dots)$, 1 being in the $n-$th position), i.e.

$$\Big\| \sum_{m=N+1}^{\infty} b_m e_m \Big\|_E \to 0 \quad (n \to \infty) \; .$$

We will write s for the set of all complex sequences, l_∞, c, c_0 for the sets of bounded, convergent and null sequences, respectively. We will denote by cs and l_1 the sets of convergent and absolutely convergent series respectively.

In all that follows we shall use the set

$$U^+ = \left\{ (u_n)_{n\geq 1} \in s \; : u_n > 0 \text{ for all } n \right\}.$$

From Wilansky's notations [14], we define for any sequence $\alpha = (\alpha_n)_{n\geq 1} \in U^+$ and for any set of sequences E, the set

$$\left(\frac{1}{\alpha}\right)^{-1} * E = \left\{ (x_n)_{n\geq 1} \in s \; : \; \left(\frac{x_n}{\alpha_n}\right)_n \in E \right\}.$$

We use the notation

$$\left(\frac{1}{\alpha}\right)^{-1} * E = \begin{cases} \overset{\circ}{s}_\alpha & \text{if} \quad E = c_0 \, , \\ s_\alpha^{(c)} & \text{if} \quad E = c \, , \\ s_\alpha & \text{if} \quad E = l_\infty \, . \end{cases}$$

We have for instance

$$\left(\frac{1}{\alpha}\right)^{-1} * c_0 = \overset{\circ}{s}_\alpha = \left\{ (x_n)_{n\geq 1} \in s \; : \; x_n = o(\alpha_n) \;\; (n \to \infty) \right\}. \tag{3}$$

Each of the spaces $(1/\alpha)^{-1} * E$, where $E \in \{c_0, c, l_\infty\}$, is a *BK space normed by*

$$\|X\|_{s_\alpha} = \sup_{n\geq 1} \left(\frac{|x_n|}{\alpha_n} \right), \tag{4}$$

and $\overset{\circ}{s}_\alpha$ *has* AK, see [9].

Now let $\alpha = (\alpha_n)_{n\geq 1}$, $\beta = (\beta_n)_{n\geq 1} \in U^+$. We denote by $S_{\alpha,\beta}$ the set of infinite matrices $A = (a_{nm})_{n,m\geq 1}$ such that $\sup_{n\geq 1} \left(\sum_{m=1}^{\infty} |a_{nm}| \, \alpha_m / \beta_n \right) < \infty$, see [9]. The set $S_{\alpha,\beta}$ is a *Banach space with the norm*

$$\|A\|_{S_{\alpha,\beta}} = \sup_{n\geq 1} \left(\sum_{m=1}^{\infty} |a_{nm}| \, \frac{\alpha_m}{\beta_n} \right).$$

Let E and F be any subsets of s. When A maps E into F we shall write $A \in (E, F)$, see [2]. So for every $X \in E$, $AX \in F$, ($AX \in F$ will mean that for each $n \geq 1$ the series defined by $y_n = \sum_{m=1}^{\infty} a_{nm} x_m$ is convergent and $(y_n)_{n\geq 1} \in F$). It was shown in [9] that $A \in (s_\alpha, s_\beta)$ if and only if $A \in S_{\alpha,\beta}$. So we can write that $(s_\alpha, s_\beta) = S_{\alpha,\beta}$.

When $s_\alpha = s_\beta$ we obtain the *Banach algebra with identity* $S_{\alpha,\beta} = S_\alpha$, (see [2-9]) normed by $\|A\|_{S_\alpha} = \|A\|_{S_{\alpha,\alpha}}$.

We also have $A \in (s_\alpha, s_\alpha)$ if and only if $A \in S_\alpha$. If $\|I - A\|_{S_\alpha} < 1$, we shall say that $A \in \Gamma_\alpha$. Since S_α is a *Banach algebra with identity*, it can easily be seen that the condition $A \in \Gamma_\alpha$ implies that A *is bijective from* s_α *into itself and* $A^{-1} \in (s_\alpha, s_\alpha)$.

If $\alpha = (r^n)_{n\geq 1}$, we will write Γ_r, S_r, s_r, $\overset{\circ}{s}_r$ and $s_r^{(c)}$ instead of Γ_α, S_α, s_α, $\overset{\circ}{s}_\alpha$ and $s_\alpha^{(c)}$ respectively (see [2–9]). When $r = 1$, we obtain $s_1 = l_\infty$, $\overset{\circ}{s}_1 = c_0$ and $s_1^{(c)} = c$, and putting $e = (1, 1, \dots)$ we have $S_1 = S_e$. It is well known, see [1] that

$$(s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1 \,.$$

For any subset E of s, we put

$$A(E) = \{Y \in s \,:\, Y = AX \text{ for some } X \in E\} \,. \tag{5}$$

If F is a subset of s, then

$$F(A) = F_A = \{X \in s \,:\, Y = AX \in F\} \,. \tag{6}$$

denotes the *matrix domain* of A in X.

2. Properties of some sets of sequences.

In this section we recall [5] some properties of the sets $s_\alpha(A)$ for $A = \Delta$, or Δ^+, or Σ, or Σ^+, and we give characterizations of the sets $w_\alpha^p(\lambda)$, $w_\alpha^{+p}(\lambda)$, $\overset{\circ}{w}{}_\alpha^p(\lambda)$ and $\overset{\circ}{w}{}_\alpha^{+p}(\lambda)$.

Let U be the set of all sequences $(u_n)_{n\geq 1} \in s$ with $u_n \neq 0$ for all n. We define $C(\lambda) = (c_{nm})_{n,m\geq 1}$ for $\lambda = (\lambda_n)_{n\geq 1} \in U$, by $c_{nm} = 1/\lambda_n$ for $m \leq n$ and $c_{nm} = 0$ for $m > n$. We put $C^+(\lambda) = C(\lambda)^t$. It can be shown that the matrix $\Delta(\lambda) = (c'_{nm})_{n,m\geq 1}$ with

$$c'_{nm} = \begin{cases} \lambda_n & \text{if} \quad m = n \,, \\ -\lambda_{n-1} & \text{if} \quad m = n-1 \quad \text{and} \quad n \geq 2 \,, \\ 0 & \text{otherwise,} \end{cases}$$

is the inverse of $C(\lambda)$, see [9]. Similarly we put $\Delta^+(\lambda) = \Delta(\lambda)^t$. If $\lambda = e$ we get the well known *operator of first-difference represented by* $\Delta(e) = \Delta$ and it is usually written $\Sigma = C(e)$. Note that $\Delta = \Sigma^{-1}$ and Σ belong to any given space S_R with $R > 1$. Writing $D_\lambda = (\lambda_n \delta_{nm})_{n,m\geq 1}$, (where $\delta_{nm} = 0$ for $n \neq m$ and $\delta_{nn} = 1$ otherwise), we have $\Delta^+(\lambda) = D_\lambda \Delta^+$. So for any given $\alpha \in U^+$, we see that if $(\alpha_{n-1}/\alpha_n)\,|\lambda_n/\lambda_{n-1}| = O(1)$ $(n \to \infty)$, then $\Delta^+(\lambda) \in \left(s_{\left(\frac{\alpha}{|\lambda|}\right)}, s_\alpha\right)$. Since per $\Delta^+(\lambda) \neq 0$, we are led to define the set

$$s_\alpha^*\left(\Delta^+(\lambda)\right) = s_\alpha\left(\Delta^+(\lambda)\right) \cap s_{\left(\frac{\alpha}{|\lambda|}\right)} = \left\{X = (x_n)_{n\geq 1} \in s_{\left(\frac{\alpha}{|\lambda|}\right)} \;:\; \Delta^+(\lambda)\, X \in s_\alpha\right\}.$$

It can easily be seen that

$$s^*_{\left(\frac{\alpha}{|\lambda|}\right)}\left(\Delta^+(e)\right) = s^*_{\left(\frac{\alpha}{|\lambda|}\right)}\left(\Delta^+\right) = s^*_{\alpha}\left(\Delta^+(\lambda)\right). \tag{7}$$

2.1. **Properties of the sequence $C(\alpha)\alpha$.** We shall use the following sets

$$\widehat{C_1} = \left\{\alpha \in U^+ : \quad \frac{1}{\alpha_n}\Big(\sum_{k=1}^{n}\alpha_k\Big) = O(1) \ \ (n\to\infty)\right\},$$

$$\widehat{C} = \left\{\alpha \in U^+ : \quad \frac{1}{\alpha_n}\Big(\sum_{k=1}^{n}\alpha_k\Big) \in c\right\},$$

$$\widehat{C_1^+} = \left\{\alpha \in U^+\cap cs \ : \quad \frac{1}{\alpha_n}\Big(\sum_{k=n}^{\infty}\alpha_k\Big) = O(1) \ \ (n\to\infty)\right\},$$

$$\Gamma = \left\{\alpha \in U^+ : \quad \overline{\lim_{n\to\infty}}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1\right\}$$

and

$$\Gamma^+ = \left\{\alpha \in U^+ : \quad \overline{\lim_{n\to\infty}}\left(\frac{\alpha_{n+1}}{\alpha_n}\right) < 1\right\}.$$

Note that $\alpha\in\Gamma^+$ if and only if $1/\alpha\in\Gamma$. We shall see in Proposition 1 that if $\alpha\in\widehat{C_1}$, then α_n tends to infinity. On the other hand we see that $\Delta\in\Gamma_\alpha$ implies $\alpha\in\Gamma$, and $\alpha\in\Gamma$ if and only if there is an integer $q\geq 1$ such that

$$\gamma_q(\alpha) = \sup_{n\geq q+1}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1.$$

We obtain the following results in which we put $[C(\alpha)\alpha]_n = \Big(\sum\limits_{k=1}^{n}\alpha_k\Big)/\alpha_n$.

Proposition 1 ([7])**.** *Let $\alpha\in U^+$. Then*

i) $\alpha_{n-1}/\alpha_n \to 0$ *if and only if* $[C(\alpha)\alpha]_n \to 1$.

ii) a) $\alpha\in\widehat{C}$ *implies* $(\alpha_{n-1}/\alpha_n)_{n\geq 1}\in c$,

b) $[C(\alpha)\alpha]_n \to l$ *implies* $\alpha_{n-1}/\alpha_n \to 1-1/l$.

iii) *If $\alpha\in\widehat{C_1}$ there are $K>0$ and $\gamma>1$ such that*

$$\alpha_n \geq K\gamma^n \quad \textit{for all} \quad n.$$

iv) *The condition $\alpha\in\Gamma$ implies $\alpha\in\widehat{C_1}$ and there exists a real $b>0$ such that*

$$[C(\alpha)\alpha]_n \leq \frac{1}{1-\chi} + b\chi^n \quad \textit{for} \quad n\geq q+1 \quad \textit{and} \quad \chi = \gamma_q(\alpha)\in\,]0,1[\,.$$

v) *The condition $\alpha\in\Gamma^+$ implies $\alpha\in\widehat{C_1^+}$.*

Put now

$$\widehat{\Gamma} = \left\{\alpha\in U^+ : \lim_{n\to\infty}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1\right\}.$$

We then have the next result, see [10].

Proposition 2. $\widehat{C} = \widehat{\Gamma}\subset\Gamma\subset\widehat{C_1}$.

Proof. The inclusion $\widehat{C} \subset \widehat{\Gamma}$ comes from Proposition 1 ii) b); and the inclusion $\widehat{\Gamma} \subset \widehat{C}$ was shown in [10]. The inclusion $\widehat{\Gamma} \subset \Gamma$ is obvious and $\Gamma \subset \widehat{C_1}$ comes from Proposition 1 iv). $\square$

Remark 1. Note that as a direct consequence of Proposition 1, we have $\widehat{C_1} \cap \widehat{C_1^+} = \Gamma \cap \Gamma^+ = \phi$.

Remark 2. The condition $\alpha \in \widehat{C_1}$ does not imply $\alpha \in \Gamma$, see [7].

2.2. **Some new properties of the operators Δ and Δ^+.** We can assert the following result, in which we put $\alpha^+ = (\alpha_{n+1})_{n \geq 1}$ and $\overset{\circ}{s}{}_{\alpha}^{*}(\Delta^+) = \overset{\circ}{s}_{\alpha}(\Delta^+) \cap \overset{\circ}{s}_{\alpha}$; note that from (7), we have

$$s_{\alpha}^{*}\left(\Delta^{+}(e)\right) = s_{\alpha}^{*}\left(\Delta^{+}\right) = s_{\alpha}\left(\Delta^{+}\right) \cap s_{\alpha}\,.$$

Theorem 3 ([8])**.** *Let $\alpha \in U^+$. Then*

i) a) $s_{\alpha}(\Delta) = s_{\alpha}$ *if and only if* $\alpha \in \widehat{C_1}$;
b) $\overset{\circ}{s}_{\alpha}(\Delta) = \overset{\circ}{s}_{\alpha}$ *if and only if* $\alpha \in \widehat{C_1}$;
c) $s_{\alpha}^{(c)}(\Delta) = s_{\alpha}^{(c)}$ *if and only if* $\alpha \in \widehat{\Gamma}$.

ii) a) $\alpha \in \widehat{C_1}$ *if and only if* $s_{\alpha^+}(\Delta^+) = s_{\alpha}$ *and* Δ^+ *is surjective from* s_{α} *into* s_{α^+};
b) $\alpha \in \widehat{C_1^+}$ *if and only if* $s_{\alpha}^{*}(\Delta^+) = s_{\alpha}$ *and* Δ^+ *is bijective from* s_{α} *into* s_{α}.
c) $\alpha \in \widehat{C_1^+}$ *implies* $\overset{\circ}{s}{}_{\alpha}^{*}(\Delta^+) = \overset{\circ}{s}_{\alpha}$ *and* Δ^+ *is bijective from* $\overset{\circ}{s}_{\alpha}$ *into* $\overset{\circ}{s}_{\alpha}$.

iii) $\alpha \in \widehat{C_1^+}$ *if and only if* $s_{\alpha}(\Sigma^+) = s_{\alpha}$ *and* $s_{\alpha}(\Sigma^+) = s_{\alpha}$ *implies* $\overset{\circ}{s}_{\alpha}(\Sigma^+) = \overset{\circ}{s}_{\alpha}$.

As a direct consequence of the preceding result we get

Corollary 4. *Let $R > 0$ be any real. The following assertions are equivalent*
(i) $R > 1$,
(ii) $s_R(\Delta) = s_R$,
(iii) $\overset{\circ}{s}_R(\Delta) = \overset{\circ}{s}_R$,
(iv) $s_R(\Delta^+) = s_R$.

2.3. **The spaces $w_{\alpha}^{p}(\lambda)$ and $w_{\alpha}^{+p}(\lambda)$ for given real $p > 0$.** Here we shall define sets generalizing the well known sets

$$w_{\infty}^{p}(\lambda) = \{X \in s \ : C(\lambda)(|X|^p) \in l_{\infty}\}\ ,$$
$$w_{0}^{p}(\lambda) = \{X \in s \ : C(\lambda)(|X|^p) \in c_0\}\ ,$$

see [13, 14]. It was shown each of the sets $w_0^p = w_0^p((n)_n)$, and $w_{\infty}^p = w_{\infty}^p((n)_n)$ is a $p-$normed FK space for $0 < p < 1$, (that is a complete linear metric space for which each projection P_n is continuous). The set w_0^p *has the property* AK and every sequence $X = (x_n)_{n \geq 1} \in w^p$ has a unique representation $X = le^t + \sum_{n=1}^{\infty}(x_n - l)\, e_n^t$, where $l \in \mathbb{C}$ is such that $X - le^t \in w_0^p$, (see [13]). Now, let α, $\lambda \in U^+$. We put

$$\begin{aligned} w_\alpha^p(\lambda) &= \{X \in s : C(\lambda)(|X|^p) \in s_\alpha\}, \\ w_\alpha^{+p}(\lambda) &= \left\{X \in s : C^+(\lambda)(|X|^p) \in s_\alpha\right\}, \\ w_\alpha^{\circ p}(\lambda) &= \left\{X \in s : C(\lambda)(|X|^p) \in s_\alpha^\circ\right\}, \\ w_\alpha^{\circ +p}(\lambda) &= \left\{X \in s : C^+(\lambda)(|X|^p) \in s_\alpha^\circ\right\}. \end{aligned}$$

We deduce from the previous section the following.

Theorem 5 ([5]). *Let α, $\lambda \in U^+$ and $p > 0$ real. Then*

i) a) *The condition $\alpha \in \widehat{C_1^+}$ is equivalent to $w_\alpha^{+p}(\lambda) = s_{(\alpha\lambda)^{\frac{1}{p}}}$;*

b) *if $\alpha \in \widehat{C_1^+}$, then $w_\alpha^{\circ +p}(\lambda) = s^\circ_{(\alpha\lambda)^{\frac{1}{p}}}$.*

ii) a) *The condition $\alpha\lambda \in \widehat{C_1}$ is equivalent to $w_\alpha^p(\lambda) = s_{(\alpha\lambda)^{\frac{1}{p}}}$;*

b) *if $\alpha\lambda \in \widehat{C_1}$, then $w_\alpha^{\circ p}(\lambda) = s^\circ_{(\alpha\lambda)^{\frac{1}{p}}}$.*

3. New sets of sequences of the form $[A_1, A_2]$.

In this section we recall some results given in [8]. We are led to use the sets

$$[A_1(\lambda), A_2(\mu)] = \{X \in s : A_1(\lambda)(|A_2(\mu)X|) \in s_\alpha\}$$

where $A_1(\lambda)$ and $A_2(\mu)$ are of the form $C(\xi)$, or $C^+(\xi)$, or $\Delta(\xi)$, or $\Delta^+(\xi)$ and we give sufficient conditions to get $[A_1(\lambda), A_2(\mu)]$ in the form s_γ.

Let λ, $\mu \in U^+$. For simplification, we shall write throughout this section

$$[A_1, A_2] = [A_1(\lambda), A_2(\mu)] = \{X \in s : A_1(\lambda)(|A_2(\mu)X|) \in s_\alpha\}$$

for any matrices

$$A_1(\lambda) \in \left\{\Delta(\lambda), \Delta^+(\lambda), C(\lambda), C^+(\lambda)\right\}$$

and

$$A_2(\mu) \in \left\{\Delta(\mu), \Delta^+(\mu), C(\mu), C^+(\mu)\right\}.$$

So we have for instance

$$[C, \Delta] = \{X \in s : C(\lambda)(|\Delta(\mu)X|) \in s_\alpha\} = (w_\alpha(\lambda))_{\Delta(\mu)}, \quad \text{etc} \ldots$$

In all that follows, the conditions $\xi \in \Gamma$, or $1/\eta \in \Gamma$ for any given sequences ξ and η, can be replaced by the conditions $\xi \in \widehat{C_1}$ and $\eta \in \widehat{C_1^+}$.

3.1. **The sets** $[C,\Delta]$**,** $[C,\Delta^+]$**,** $[C,C]$**,** $[C^+,\Delta]$**,** $[C^+,\Delta^+]$**,** $[C^+,C]$ **and** $[C^+,C^+]$**.** For the convenience of a reader we shall write the following identities.

$$\left[C,\Delta^+\right]=\left\{X:\frac{1}{\lambda_n}\Big(\sum_{k=1}^{n}|\mu_k x_k-\mu_k x_{k+1}|\Big)=\alpha_n O(1)\ \ (n\to\infty)\right\},$$

$$[C,C]=\left\{X\ :\ \frac{1}{\lambda_n}\Big(\sum_{m=1}^{n}\Big|\frac{1}{\mu_m}\Big(\sum_{k=1}^{m}x_k\Big)\Big|\Big)=\alpha_n O(1)\ \ (n\to\infty)\right\},$$

$$\left[C^+,\Delta\right]=\left\{X\ :\ \sum_{k=n}^{\infty}\Big(\frac{1}{\lambda_k}|\mu_k x_k-\mu_{k-1}x_{k-1}|\Big)=\alpha_n O(1)\ \ (n\to\infty)\right\},$$

$$\left[C^+,\Delta^+\right]=\left\{X\ :\ \sum_{k=n}^{\infty}\Big(\frac{1}{\lambda_k}|\mu_k x_k-\mu_k x_{k+1}|\Big)=\alpha_n O(1)\ \ (n\to\infty)\right\},$$

$$\left[C^+,C\right]=\left\{X\ :\ \sum_{k=n}^{\infty}\Big(\frac{1}{\lambda_k}\Big|\frac{1}{\mu_k}\sum_{i=1}^{k}x_i\Big|\Big)=\alpha_n O(1)\ \ (n\to\infty)\right\},$$

$$\left[C^+,C^+\right]=\left\{X\ :\ \sum_{k=n}^{\infty}\Big(\frac{1}{\lambda_k}\Big|\sum_{i=k}^{\infty}\frac{x_i}{\mu_i}\Big|\Big)=\alpha_n O(1)\ \ (n\to\infty)\right\}.$$

Note that for $\alpha=e$ and $\lambda=\mu$, $[C,\Delta]$ is the well known set of *sequences that are strongly bounded,* denoted $c_\infty(\lambda)$, see [12, 14]. We get the following result where we put $\alpha^-=(\alpha_{n-1})_n$ with the convention $\alpha_n=1$ for $n\leq 1$.

Theorem 6. *Let* $\alpha,\ \lambda\in U^+$. *Then*
i) *if* $\alpha\lambda\in\Gamma$, *then* $[C,\Delta]=s_{\left(\alpha\frac{\lambda}{\mu}\right)}$.
ii) *The conditions* $\alpha\lambda,\ \alpha\lambda/\mu\in\Gamma$ *imply together* $[C,\Delta^+]=s_{\left(\alpha\frac{\lambda}{\mu}\right)^-}$.
iii) *If* $\alpha\lambda,\ \alpha\lambda\mu\in\Gamma$, *then* $[C,C]=s_{(\alpha\lambda\mu)}$.
iv) *If* $1/\alpha,\ \alpha\lambda\in\Gamma$, *then* $[C^+,\Delta]=s_{\left(\alpha\frac{\lambda}{\mu}\right)}$.
v) *If* $1/\alpha,\ \alpha\lambda/\mu\in\Gamma$, *then* $[C^+,\Delta^+]=s_{\left(\alpha\frac{\lambda}{\mu}\right)^-}$.
vi) *If* $1/\alpha,\ \alpha\lambda\mu\in\Gamma$, *then* $[C^+,C]=s_{(\alpha\lambda\mu)}$.
vii) *If* $1/\alpha,\ 1/\alpha\lambda\in\Gamma$, *then* $[C^+,C^+]=s_{(\alpha\lambda\mu)}$.

Remark 3. If we define

$$[A_1,A_2]_0=\left\{X\in s\ :\ A_1(\lambda)\left(|A_2(\mu)X|\right)\in s_\alpha^\circ\right\},$$

we get the same results as in Theorem 6, replacing s_ξ by s_ξ° in each case i), ii), iii) and iv).

3.2. **The sets $[\Delta,\Delta]$, $[\Delta,\Delta^+]$, $[\Delta,C]$, $[\Delta,C^+]$, $[\Delta^+,\Delta]$, $[\Delta^+,C]$, and $[\Delta^+,\Delta^+]$.** From the definitions of the operators $\Delta(\xi)$, $\Delta^+(\xi)$, $C(\xi)$ and $C^+(\xi)$, we immediatly get the following

$$\begin{aligned}
[\Delta,\Delta] &= \big\{X\ :\ -\lambda_{n-1}\left|\mu_{n-1}x_{n-1}-\mu_{n-2}x_{n-2}\right| + \lambda_n\left|\mu_n x_n-\mu_{n-1}x_{n-1}\right| \\
&= \alpha_n O(1)\,(n\to\infty)\big\}\,, \\
\left[\Delta,\Delta^+\right] &= \big\{X\ :\ \lambda_n\left|\mu_n\left(x_n-x_{n+1}\right)\right| - \lambda_{n-1}\left|\mu_{n-1}\left(x_{n-1}-x_n\right)\right| \\
&= \alpha_n O(1)\,(n\to\infty)\big\} \\
[\Delta,C] &= \Big\{X\ :\ -\lambda_{n-1}\Big|\frac{1}{\mu_{n-1}}\Big(\sum_{k=1}^{n-1}x_k\Big)\Big| + \lambda_n\Big|\frac{1}{\mu_n}\Big(\sum_{k=1}^{n}x_k\Big)\Big| \\
&= \alpha_n O(1)\,(n\to\infty)\Big\}\,, \\
\left[\Delta,C^+\right] &= \Big\{X\ :\lambda_n\Big|\sum_{i=n}^{\infty}\frac{x_i}{\mu_i}\Big| - \lambda_{n-1}\Big|\sum_{i=n-1}^{\infty}\frac{x_i}{\mu_i}\Big| = \alpha_n O(1)\,(n\to\infty)\Big\}\,, \\
\left[\Delta^+,\Delta\right] &= \big\{X\ :\ \lambda_n\left(\left|\mu_n x_n-\mu_{n-1}x_{n-1}\right| - \left|\mu_{n+1}x_{n+1}-\mu_n x_n\right|\right) \\
&= \alpha_n O(1)\,(n\to\infty)\big\}\,, \\
\left[\Delta^+,\Delta^+\right] &= \big\{X\ :\ \lambda_n\left|\mu_n\left(x_n-x_{n+1}\right)\right| - \lambda_n\left|\mu_{n+1}\left(x_{n+1}-x_{n+2}\right)\right| \\
&= \alpha_n O(1)\,(n\to\infty)\big\}\,, \\
\left[\Delta^+,C\right] &= \Big\{X\ :\ \frac{\lambda_n}{\mu_n}\Big|\sum_{i=1}^{n}x_i\Big| - \frac{\lambda_n}{\mu_{n+1}}\Big|\sum_{i=1}^{n+1}x_i\Big| = \alpha_n O\left(1\right)\ (n\to\infty)\Big\}\,.
\end{aligned}$$

We can state the following result

Theorem 7. *Let α, $\lambda\in U^+$. Then*

i) *if α, $\alpha/\lambda\in\Gamma$, then $[\Delta,\Delta]=s_{\left(\alpha\frac{\mu}{\lambda}\right)}$.*

ii) *Assume $\alpha\in\Gamma$. Then $[\Delta,\Delta^+]=s_{\left(\frac{\alpha}{\lambda\mu}\right)^-}$ if $\alpha/\lambda\mu\in\Gamma$;*

iii) *if α, $\alpha\mu/\lambda\in\Gamma$, then $[\Delta,C]=s_{\left(\alpha\frac{\mu}{\lambda}\right)}$ and $[\Delta,C^+]=s_{\left(\alpha\frac{\mu}{\lambda}\right)}$ if $\lambda/\alpha\in\Gamma$.*

iv) *The condition $\alpha/\lambda\in\Gamma$ implies*

$$\left[\Delta^+,\Delta\right] = s_{\left(\frac{\alpha_{n-1}}{\mu_n\lambda_{n-1}}\right)_n} = s_{\left(\frac{1}{\mu}\left(\frac{\alpha}{\lambda}\right)^-\right)}.$$

v) *If α/λ, $\mu^{-1}\left(\alpha/\lambda\right)^-=\left(\alpha_{n-1}/\left(\mu_n\lambda_{n-1}\right)\right)_n\in\Gamma$, then*

$$\left[\Delta^+,\Delta^+\right] = s_{\left(\frac{\left(\frac{\alpha}{\lambda}\right)^-}{\mu}\right)^-} = s_{\left(\frac{\alpha_{n-2}}{\lambda_{n-2}\mu_{n-1}}\right)_n}.$$

vi) *If α/λ, $\mu\left(\alpha/\lambda\right)^-=\left(\mu_n\alpha_{n-1}/\lambda_{n-1}\right)_n\in\Gamma$, then $[\Delta^+,C]=s_{\mu\left(\frac{\alpha}{\lambda}\right)^-}$.*

Remark 4. Note that in Theorem 7, we have $[A_1,A_2]=s_\alpha\left(A_1A_2\right)=\left(s_\alpha\left(A_1\right)\right)_{A_2}$ for $A_1\in\left\{\Delta\left(\lambda\right),\Delta^+\left(\lambda\right),C\left(\lambda\right),C^+\left(\lambda\right)\right\}$ and $A_2\in\left\{\Delta\left(\mu\right),\Delta^+\left(\mu\right),C\left(\mu\right),C^+\left(\mu\right)\right\}$.

For instance we have

$$[\Delta, C] = \left\{ X \, : \, \left(\frac{\lambda_n}{\mu_n} - \frac{\lambda_{n-1}}{\mu_{n-1}} \right) \sum_{i=1}^{n-1} x_i + \frac{\lambda_n}{\mu_n} x_n = \alpha_n O(1) \;\; (n \to \infty) \right\}$$

for α, $\frac{\alpha\mu}{\lambda} \in \Gamma$.

4. The sets $\left[A_1^r(\lambda), A_2^{(p)}(\mu) \right]$.

In this section we give a generalization of the results obtained in the previous sections. We will write the sets $[A_1(\lambda), A_2(\mu)]$ for $A_1 = \Delta^r(\lambda)$, or $\Delta^{+r}(\lambda)$, or $C^r(\lambda)$ or $C^{+r}(\lambda)$ and $A_2 = \Delta^{(p)}(\mu)$, or $\Delta^{+p}(\mu)$, or $C^p(\mu)$, or $C^{+p}(\mu)$ in the form s_ξ.

First we need to study the following sets.

4.1. The sets $w_\alpha^{r,p}(\lambda)$, $w_\alpha^{+r,p}(\lambda)$, $\overset{\circ}{w}{}_\alpha^{r,p}(\lambda)$ and $\overset{\circ}{w}{}_\alpha^{+r,p}(\lambda)$. In the following we will consider the operator $C^r(\lambda)$ for $r \in \mathbb{N}$, which is defined by $C^r(\lambda)X = C(\lambda)\left(C^{r-1}(\lambda)X\right)$ for all $X \in s$. So we obtain

$$C^r(\lambda)X = \frac{1}{\lambda_{n_r}} \Big[\sum_{n_{r-1}=1}^{n_r} \Big[\frac{1}{\lambda_{n_{r-1}}} \sum_{n_{r-2}=1}^{n_{r-1}} \cdots \frac{1}{\lambda_{n_2}} \sum_{n_1=1}^{n_2} \Big(\frac{1}{\lambda_{n_1}} \sum_{i=1}^{n_1} x_i \Big) \Big] \Big] \quad \text{for all } X \in s\,.$$

We define $C^{+r}(\lambda)X$ in the same way.

Lemma 8. *Let α, $\lambda \in U^+$ and $r \geq 1$ be an integer.* (i) *If $\alpha\lambda^i \in \widehat{C_1}$ for $i = 1, 2, \dots, r$, then*

$$s_\alpha(C^r(\lambda)) = s_{\alpha\lambda^r} \text{ and } \overset{\circ}{s}_\alpha(C^r(\lambda)) = \overset{\circ}{s}_{\alpha\lambda^r}\,.$$

(ii) *If $\alpha\lambda^i \in \widehat{C_1^+}$ for $i = 0, 1, \dots, r-1$, then*

$$s_\alpha\left(C^{+r}(\lambda)\right) = s_{\alpha\lambda^r} \quad \text{and} \quad \overset{\circ}{s}_\alpha\left(C^{+r}(\lambda)\right) = \overset{\circ}{s}_{\alpha\lambda^r}\,.$$

Proof. (i) First the condition $\alpha\lambda \in \widehat{C_1}$ implies that Δ is bijective from $s_{\alpha\lambda}$ to itself. So

$$s_\alpha(C(\lambda)) = \Delta(\lambda)s_\alpha = \Delta D_\lambda s_\alpha = \Delta s_{\alpha\lambda} = s_{\alpha\lambda}\,.$$

Now let j with $1 \leq j \leq r-1$ and assume $s_\alpha\left(C^j(\lambda)\right) = s_{\alpha\lambda^j}$ for $\alpha\lambda^i \in \widehat{C_1}$ with $i = 1, 2, \dots, j$. Then

$$\begin{aligned} s_\alpha\left(C^{j+1}(\lambda)\right) &= \left\{ X \in s \, : \, C^j(\lambda)(C(\lambda)X) \in s_\alpha \right\} \\ &= \left\{ X \in s \, : \, C(\lambda)X \in s_\alpha\left(C^j(\lambda)\right) \right\}. \end{aligned}$$

Since $s_\alpha\left(C^j(\lambda)\right) = s_{\alpha\lambda^j}$ we then have

$$s_\alpha\left(C^{j+1}(\lambda)\right) = s_\alpha\left(C^j(\lambda)\right)(C(\lambda)) = s_{\alpha\lambda^j}(C(\lambda)) = \Delta(\lambda)s_{\alpha\lambda^j}\,.$$

Now the condition $\alpha\lambda^{j+1} \in \widehat{C_1}$ implies $\Delta(\lambda)s_{\alpha\lambda^j} = \Delta D_\lambda s_{\alpha\lambda^j} = \Delta s_{\alpha\lambda^{j+1}}$ and $s_\alpha\left(C^{j+1}(\lambda)\right) = s_{\alpha\lambda^{j+1}}$ for $\alpha\lambda^i \in \widehat{C_1}$ with $i = 1, 2, \dots, j+1$. This concludes the proof of (i). Similarly we get $\overset{\circ}{s}_\alpha(C^r(\lambda)) = \overset{\circ}{s}_{\alpha\lambda^r}$ if $\alpha\lambda^i \in \widehat{C_1}$ for $i = 1, 2, \dots, r$.

(ii) By Theorem 3 iii), the condition $\alpha \in \widehat{C_1^+}$ implies

$$\begin{aligned} s_\alpha\big(C^+(\lambda)\big) &= \big\{X \in s : \Sigma^+ D_{1/\lambda} X \in s_\alpha\big\} \\ &= \big\{X \in s \,:\, D_{1/\lambda} X \in s_\alpha\left(\Sigma^+\right)\big\} \\ &= \big\{X \in s \,:\, D_{1/\lambda} X \in s_\alpha\big\} = s_{\alpha\lambda} \,. \end{aligned}$$

So $\alpha \in \widehat{C_1^+}$ implies $s_\alpha\left(C^+\left(\lambda\right)\right) = s_{\alpha\lambda}$. Assume now $s_\alpha\left(C^{+j}\left(\lambda\right)\right) = s_{\alpha\lambda^j}$ for $\alpha\lambda^i \in \widehat{C_1^+}$ for $i = 0, 1, \ldots, j-1$ for given $j \geq 1$ integer. Then

$$\begin{aligned} s_\alpha\big(C^{+(j+1)}(\lambda)\big) &= \big\{X \in s \,:\; C^{+j}\left(\lambda\right)\left(C^+\left(\lambda\right)X\right) \in s_\alpha\big\} \\ &= \big\{X \in s \,:\; C^+\left(\lambda\right)X \in s_\alpha\left(C^{+j}\left(\lambda\right)\right)\big\} \,. \end{aligned}$$

Since $s_\alpha\left(C^{+j}\left(\lambda\right)\right) = s_{\alpha\lambda^j}$ we then have $s_\alpha\left(C^{+(j+1)}\left(\lambda\right)\right) = s_{\alpha\lambda^j}\left(C^+\left(\lambda\right)\right)$. Now if $\alpha\lambda^j \in \widehat{C_1^+}$ then

$$s_\alpha\big(C^{+(j+1)}\left(\lambda\right)\big) = s_{(\alpha\lambda^j)\lambda} = s_{\alpha\lambda^{j+1}} \,.$$

Thus if $\alpha\lambda^i \in \widehat{C_1^+}$ for $i = 0, 1, \ldots, j$, then $s_\alpha\left(C^{+(j+1)}\left(\lambda\right)\right) = s_{\alpha\lambda^{j+1}}$. Similarly we can show $\overset{\circ}{s}_\alpha\left(C^{+r}\left(\lambda\right)\right) = \overset{\circ}{s}_{\alpha\lambda^r}$. This concludes the proof. $\square$

For $r \in \mathbb{N}$ and $p > 0$ real, put now

$$w_\alpha^{r,p}\left(\lambda\right) = \big\{X \in s \,:\, C^r\left(\lambda\right)\left(|X|^p\right) \in s_\alpha\big\}\,, \tag{8}$$

$$\overset{\circ}{w}_\alpha^{r,p}\left(\lambda\right) = \big\{X \in s \,:\, C^r\left(\lambda\right)\left(|X|^p\right) \in \overset{\circ}{s}_\alpha\big\} \tag{9}$$

$$w_\alpha^{+r,p}\left(\lambda\right) = \big\{X \in s \,:\, C^{+r}\left(\lambda\right)\left(|X|^p\right) \in s_\alpha\big\} \tag{10}$$

$$\overset{\circ}{w}_\alpha^{+r,p}\left(\lambda\right) = \big\{X \in s \,:\, C^{+r}\left(\lambda\right)\left(|X|^p\right) \in \overset{\circ}{s}_\alpha\big\}\,. \tag{11}$$

Theorem 9. *Let α, $\lambda \in U^+$. Then*

(i) *if $\alpha\lambda^i \in \widehat{C_1}$ for $i = 1, 2, \ldots, r$, then*

$$w_\alpha^{r,p}\left(\lambda\right) = s_{(\alpha\lambda^r)^{1/p}} \text{ and } \overset{\circ}{w}_\alpha^{r,p}\left(\lambda\right) = \overset{\circ}{s}_{(\alpha\lambda^r)^{1/p}}\,;$$

(ii) *if $\alpha\lambda^i \in \widehat{C_1^+}$ for $i = 0, 1, \ldots, r-1$, then*

$$w_\alpha^{+r,p}\left(\lambda\right) = s_{(\alpha\lambda^r)^{1/p}} \text{ and } \overset{\circ}{w}_\alpha^{+r,p}\left(\lambda\right) = \overset{\circ}{s}_{(\alpha\lambda^r)^{1/p}}\,.$$

Proof. (i) First

$$w_\alpha^{r,p}\left(\lambda\right) = \left\{X \in s \,:\, |X|^p \in s_\alpha\left(C^r\left(\lambda\right)\right)\right\}\,;$$

and if $\alpha\lambda^i \in \widehat{C_1}$ for $i = 1, 2, \ldots, r$, then $s_\alpha\left(C^r\left(\lambda\right)\right) = s_{\alpha\lambda^r}$. Since $|X|^p \in s_{\alpha\lambda^r}$ if and only if

$$\frac{|x_n|}{\left(\alpha_n\lambda_n^r\right)^{\frac{1}{p}}} = O\left(1\right) \qquad \left(n \to \infty\right)\,,$$

we conclude

$$w_\alpha^{r,p}\left(\lambda\right) = s_{(\alpha\lambda^r)^{1/p}}\,.$$

Similarly we get $\overset{\circ}{w}_{\alpha}^{r,p}(\lambda) = \overset{\circ}{s}_{(\alpha\lambda^r)^{1/p}}$.

(ii) We see that

$$w_{\alpha}^{+r,p}(\lambda) = \left\{X \in s : |X|^p \in s_{\alpha}\left(C^{+r}(\lambda)\right)\right\}.$$

From Lemma 8, if $\alpha\lambda^i \in \widehat{C_1^+}$ for $i = 0, 1, \dots, r-1$, then $s_{\alpha}(C^{+r}(\lambda)) = s_{\alpha\lambda^r}$ and $|X|^p \in s_{\alpha}(C^{+r}(\lambda))$ if and only if $|X| \in s_{(\alpha\lambda^r)^{1/p}}$. Similarly we get $\overset{\circ}{w}_{\alpha}^{+r,p}(\lambda) = \overset{\circ}{s}_{(\alpha\lambda^r)^{1/p}}$. This concludes the proof.

To express the next corollary we require a lemma.

Lemma 10. *Let $q > 0$ be any real and $\alpha \in U^+$ a nondecreasing sequence. Then*
(i) $\alpha \in \widehat{C_1}$ *implies* $\alpha^q \in \widehat{C_1}$ *for* $q \geq 1$,
(ii) $\alpha^q \in \widehat{C_1}$ *implies* $\alpha \in \widehat{C_1}$ *for* $0 < q \leq 1$.

Proof. Let $q \geq 1$. Since α is nondecreasing we immediatly see that for any given integer

$$\sum_{i=1}^{n} \alpha_i\left(\alpha_n^{q-1} - \alpha_i^{q-1}\right) = \sum_{i=1}^{n}\left(\alpha_n^{q-1}\alpha_i - \alpha_i^q\right) \geq 0 \quad \text{for all} \quad n,$$

and

$$\frac{1}{\alpha_n}\left(\sum_{i=1}^{n}\alpha_i\right) \geq \frac{1}{\alpha_n^q}\left(\sum_{i=1}^{n}\alpha_i^q\right). \tag{12}$$

Since $\alpha \in \widehat{C_1}$ implies $\left(\sum_{i=1}^{n}\alpha_i\right)/\alpha_n = O(1)$ $(n \to \infty)$, we obtain (i) using the inequality (12). Now, writing $\nu = \alpha^q \in \widehat{C_1}$ and applying (i), we get $\alpha = \nu^{1/q} \in \widehat{C_1}$ for $0 < q \leq 1$. This permits us to conclude for (ii). □

Corollary 11. *Let $\lambda \in U^+$. Assume $\lambda \in \widehat{C_1}$ and λ_n is a non decreasing sequence. Then $w_e^{r,p}(\lambda) = s_{\lambda^{r/p}}$.*

Proof. Since $\alpha = e$, it follows from Lemma 10 and Theorem 9, that $\lambda \in \widehat{C_1}$ implies $\lambda^i \in \widehat{C_1}$ for all $i \geq 1$ integer and $w_e^{r,p}(\lambda) = s_{\lambda^{r/p}}$. □

4.2. The sets $\chi(\Delta^r(\lambda))$ with $\chi = s_{\alpha}$ or $\overset{\circ}{s}_{\alpha}$ and $s_{\alpha}(\Delta^{+r}(\lambda))$.

Here we have $\Delta^r(\lambda)X = \Delta\left(\Delta^{r-1}(\lambda)X\right)$ for $r \geq 1$ and $\Delta^1(\lambda)X = \Delta(\lambda)X = (\lambda_n x_n - \lambda_{n-1}x_{n-1})_{n\geq 1}$ with $x_0 = 0$. We obtain a similar definition for $\Delta^{+r}(\lambda)$. There is no explicit expression of the sequences $\Delta^r(\lambda)X$ or $\Delta^{+r}(\lambda)$.

In the following we will use the convention $x_n = 1$ for $n \leq 0$. So we get for instance

$$X' = (x_{n-2})_n = (1, 1, x_1, x_2, \dots),$$

and $X' \in s_{\alpha}$ if and only if $x_{n-2}/\alpha_n = O(1)$ $(n \to \infty)$.

Theorem 12. *Let α, $\lambda \in U^+$ and let $r \geq 1$ be an integer. Then*
(i) *if $\alpha/\lambda^i \in \widehat{C_1}$ for $i = 0, 1, \dots, r-1$, then*

$$s_{\alpha}(\Delta^r(\lambda)) = s_{\left(\frac{\alpha}{\lambda^r}\right)} \quad \text{and} \quad \overset{\circ}{s}_{\alpha}(\Delta^r(\lambda)) = \overset{\circ}{s}_{\left(\frac{\alpha}{\lambda^r}\right)};$$

(ii) *if* α/λ, $\left(\alpha_{n-1}/\left(\lambda_{n-1}\lambda_n\right)\right)_n, \dots, \left(\alpha_{n-r+1}/\left(\lambda_{n-r+1}, \dots, \lambda_n\right)\right)_n \in \widehat{C_1}$, *then*

$$s_\alpha\left(\Delta^{+r}\left(\lambda\right)\right) = s_{\left(\frac{\alpha_{n-r}}{\lambda_{n-r}, \dots, \lambda_{n-1}}\right)_n}.$$

Proof. (i) First $s_\alpha\left(\Delta\left(\lambda\right)\right) = D_{1/\lambda}\Sigma s_\alpha = s_{\alpha/\lambda}$ for $\alpha \in \widehat{C_1}$. Now suppose $\alpha/\lambda^i \in \widehat{C_1}$ for $i = 0, 1, \dots, j-1$ for given integer $j \geq 1$. Then $s_\alpha\left(\Delta^j\left(\lambda\right)\right) = s_{\alpha/\lambda^j}$

$$\begin{aligned} s_\alpha\left(\Delta^{j+1}\left(\lambda\right)\right) &= \left\{X \in s : \Delta^j\left(\lambda\right)\left(\Delta\left(\lambda\right)X\right) \in s_\alpha\right\} \\ &= \left\{X \in s : \Delta\left(\lambda\right)X \in s_\alpha\left(\Delta^j\left(\lambda\right)\right)\right\} = s_{\alpha/\lambda^j}\left(\Delta\left(\lambda\right)\right). \end{aligned}$$

If $\alpha/\lambda^j \in \widehat{C_1}$ then $s_{\alpha/\lambda^j}\left(\Delta\left(\lambda\right)\right) = s_{\alpha/\lambda^{j+1}}$ and we have shown

$$s_\alpha\left(\Delta^{j+1}\left(\lambda\right)\right) = s_{\left(\frac{\alpha}{\lambda^{j+1}}\right)} \quad \text{for} \quad \frac{\alpha}{\lambda^i} \in \widehat{C_1}, \quad i = 0, 1, \dots, j.$$

Similarly we obtain $\overset{\circ}{s}_\alpha\left(\Delta^r\left(\lambda\right)\right) = \overset{\circ}{s}_{\left(\frac{\alpha}{\lambda^r}\right)}$. This concludes the proof of (i).

(ii) First we have $s_\alpha\left(\Delta^+\left(\lambda\right)\right) = s_{\left(\alpha_{n-1}/\lambda_{n-1}\right)_n}$ if $\alpha/\lambda \in \widehat{C_1}$ so (ii) holds for $r = 1$. Let j be an integer with $1 \leq j \leq r-1$ and assume

$$s_\alpha\left(\Delta^{+j}\left(\lambda\right)\right) = s_{\left(\frac{\alpha_{n-j}}{\lambda_{n-j}\cdots\lambda_{n-1}}\right)_n}$$

for α/λ, $\left(\alpha_{n-1}/\left(\lambda_{n-1}\lambda_n\right)\right)_n, \dots, \left(\alpha_{n-j+1}/\left(\lambda_{n-j+1}\dots\lambda_n\right)\right)_n \in \widehat{C_1}$. Then

$$\begin{aligned} s_\alpha\left(\Delta^{+(j+1)}\left(\lambda\right)\right) &= \left\{X \in s : \Delta^{+j}\left(\lambda\right)\left(\Delta^+\left(\lambda\right)X\right) \in s_\alpha\right\} \\ &= \left\{X \in s : \Delta^+\left(\lambda\right)X \in s_\alpha\left(\Delta^{+j}\left(\lambda\right)\right)\right\} \\ &= s_{\left(\alpha_{n-j}/\left(\lambda_{n-j}\dots\lambda_{n-1}\right)\right)_n}\left(\Delta^+\left(\lambda\right)\right). \end{aligned}$$

Now the condition

$$\left(\frac{\alpha_{n-j}}{\lambda_{n-j}\dots\lambda_{n-1}}\frac{1}{\lambda_n}\right)_n \in \widehat{C_1} \tag{13}$$

implies

$$s_{\left(\frac{\alpha_{n-j}}{\lambda_{n-j}\cdots\lambda_{n-1}}\right)_n}\left(\Delta^+\left(\lambda\right)\right) = s_{\left(\frac{\frac{\alpha_{n-1-j}}{\lambda_{n-1-j}\cdots\lambda_{n-2}}}{\lambda_{n-1}}\right)_n} = s_{\left(\frac{\alpha_{n-(j+1)}}{\lambda_{n-(j+1)}\cdots\lambda_{n-1}}\right)_n}.$$

Since condition (13) is equivalent to

$$\left(\frac{\alpha_{n-(j+1)+1}}{\lambda_{n-(j+1)+1}\dots\lambda_n}\right)_n \in \widehat{C_1},$$

we have shown that α/λ, $\left(\alpha_{n-1}/(\lambda_{n-1}\lambda_n)\right)_n, \dots, \left(\alpha_{n-(j+1)+1}/(\lambda_{n-(j+1)+1}\dots\lambda_n)\right)_n \in \widehat{C_1}$ implies

$$s_\alpha\left(\Delta^{+(j+1)}\left(\lambda\right)\right) = s_{\left(\frac{\alpha_{n-(j+1)}}{\lambda_{n-(j+1)}\cdots\lambda_{n-1}}\right)_n}.$$

This completes the proof. □

Remark 5. We immediatly see that $\alpha \in \widehat{C_1}$ successively implies

$$s_\alpha\left(\Delta^r\right) = s_\alpha, \quad \overset{\circ}{s}_\alpha\left(\Delta^r\right) = \overset{\circ}{s}_\alpha, \quad s_\alpha\left(\Delta^{+r}\right) = s_{\left(\alpha_{n-r}\right)_n} \text{ and } \overset{\circ}{s}_\alpha\left(\Delta^{+r}\right) = \overset{\circ}{s}_{\left(\alpha_{n-r}\right)_n}.$$

4.3. **The sets** $\left[A_1^r(\lambda), A_2^{(p)}(\mu)\right]$ **for** $A_1 = \Delta(\lambda)$**, or** $\Delta^+(\lambda)$**, or** $C(\lambda)$**, or** $C^+(\lambda)$ **and** $A_2 = \Delta(\mu)$**, or** $\Delta^+(\mu)$**, or** $C(\mu)$**, or** $C^+(\mu)$**.** In this subsection we consider sets that generalize those given in Subsection 3.1.

We will put

$$\left[A_1^r(\lambda), A_2^{(p)}(\mu)\right] = \left\{X \in s \,:\, A_1^r(\lambda)\left(|A_2(\mu)X|^p\right) \in s_\alpha\right\},$$

for $A_1 = C(\lambda)$, or $C^+(\lambda)$, or $\Delta(\lambda)$, or $\Delta^+(\lambda)$, and $A_2 = \Delta(\mu)$, or $\Delta^+(\mu)$, or $C(\mu)$, or $C^+(\mu)$.

First we will deal with the sets $\left[A_1^r(\lambda), A_2^{(p)}(\mu)\right]$, for $A_1(\lambda) \in \{C(\lambda), C^+(\lambda)\}$ and $A_2(\mu) \in \{\Delta(\mu), \Delta^+(\mu), C(\mu), C^+(\mu)\}$. We get the following

Theorem 13. *Let* α, λ, $\mu \in U^+$.

(a) We assume $\alpha\lambda^i \in \widehat{C_1}$ *for* $i = 1, 2, \dots, r$. *Then*

(i) if $(\alpha\lambda^r)^{1/p} \in \widehat{C_1}$, *then*

$$\left[C^r(\lambda), \Delta^{(p)}(\mu)\right] = s_{\frac{(\alpha\lambda^r)^{1/p}}{\mu}};$$

(ii) if $(\alpha\lambda^r)^{1/p}/\mu \in \widehat{C_1}$, *then*

$$\left[C^r(\lambda), \Delta^{+(p)}(\mu)\right] = s_{\left(\frac{\left(\alpha_{n-1}\lambda_{n-1}^r\right)^{1/p}}{\mu_{n-1}}\right)_n};$$

(iii) if $(\alpha\lambda^r)^{1/p}\mu \in \widehat{C_1}$, *then*

$$\left[C^r(\lambda), C^{(p)}(\mu)\right] = s_{(\alpha\lambda^r)^{1/p}\mu}.$$

(b) We assume $\alpha\lambda^i \in \widehat{C_1^+}$ *for* $i = 0, 1, \dots, r-1$. *Then*

(iv) if $(\alpha\lambda^r)^{1/p} \in \widehat{C_1}$, *then*

$$\left[C^{+r}(\lambda), \Delta^{(p)}(\mu)\right] = s_{\frac{(\alpha\lambda^r)^{1/p}}{\mu}};$$

(v) if $(\alpha\lambda^r)^{1/p}/\mu \in \widehat{C_1}$, *then*

$$\left[C^{+r}(\lambda), \Delta^{+(p)}(\mu)\right] = s_{\left(\frac{\left(\alpha_{n-1}\lambda_{n-1}^r\right)^{1/p}}{\mu_{n-1}}\right)_n}.$$

(vi) if $(\alpha\lambda^r)^{1/p}\mu \in \widehat{C_1}$, *then*

$$\left[C^{+r}(\lambda), C^{(p)}(\mu)\right] = s_{(\alpha\lambda^r)^{1/p}\mu};$$

(vii) if $(\alpha\lambda^r)^{1/p} \in \widehat{C_1^+}$, *then*

$$\left[C^{+r}(\lambda), C^{+(p)}(\mu)\right] = s_{(\alpha\lambda^r)^{1/p}\mu}.$$

Proof. We will write $[A_1, A_2]$ instead of $[A_1(\lambda), A_2(\mu)]$ for short. So we will write $\left[C^r, \Delta^{(p)}\right]$ instead of $\left[C^r(\lambda), \Delta^{(p)}(\mu)\right]$ for instance. (i) If $\alpha\lambda^i \in \widehat{C_1}$ for $i = 1, 2, \ldots, r$, then

$$\begin{aligned}\left[C^r, \Delta^{(p)}\right] &= \left\{X \in s \,:\, \Delta(\mu) X \in w_\alpha^{r,p}(\lambda) = s_{(\alpha\lambda^r)^{1/p}}\right\} \\ &= s_{(\alpha\lambda^r)^{1/p}}(\Delta(\mu)) \ .\end{aligned}$$

Since $(\alpha\lambda^r)^{1/p} \in \widehat{C_1}$ we conclude $\left[C^r, \Delta^{(p)}\right] = s_{(\alpha\lambda^r)^{1/p}/\mu}$.

(ii) Assume $\alpha\lambda^i \in \widehat{C_1}$ for $i = 1, 2, \ldots, r$. Then

$$\begin{aligned}\left[C^r, \Delta^{+(p)}\right] &= \left\{X \in s \,:\, \Delta^+(\mu) X \in w_\alpha^{r,p}(\lambda) = s_{(\alpha\lambda^r)^{1/p}}\right\} \\ &= s_{(\alpha\lambda^r)^{1/p}}\left(\Delta^+(\mu)\right) \ .\end{aligned}$$

Then $(\alpha\lambda^r)^{1/p}/\mu \in \widehat{C_1}$ implies $s_{(\alpha\lambda^r)^{1/p}}(\Delta^+(\mu)) = s_{\left(\left(\alpha_{n-1}\lambda_{n-1}^r\right)^{1/p}/\mu_{n-1}\right)_n}$. This shows (ii).

(iii) The condition $\alpha\lambda^i \in \widehat{C_1}$ for $i = 1, 2, \ldots, r$ implies

$$\begin{aligned}\left[C^r, C^{(p)}\right] &= \left\{X \in s \,:\, C(\mu) X \in w_\alpha^{r,p}(\lambda) = s_{(\alpha\lambda^r)^{1/p}}\right\} \\ &= s_{(\alpha\lambda^r)^{1/p}}(C(\mu)) \ ;\end{aligned}$$

and the condition $(\alpha\lambda^r)^{1/p}\mu \in \widehat{C_1}$ implies $\left[C^r, C^{(p)}\right] = s_{(\alpha\lambda^r)^{1/p}\mu}$.

(iv) Assume $\alpha\lambda^i \in \widehat{C_1^+}$ for $i = 0, 1, \ldots, r-1$ and $(\alpha\lambda^r)^{1/p} \in \widehat{C_1}$. Reasoning as above we easily see that if $\alpha\lambda^i \in \widehat{C_1^+}$ for $i = 0, 1, \ldots, r-1$, then $w_\alpha^{+r,p}(\lambda) = s_{(\alpha\lambda^r)^{1/p}}$ and

$$\begin{aligned}\left[C^{+r}, \Delta^{(p)}\right] &= \left\{X \in s \,:\, \Delta(\mu) X \in w_\alpha^{r,p}(\lambda) = s_{(\alpha\lambda^r)^{1/p}}\right\} \\ &= s_{(\alpha\lambda^r)^{1/p}}(\Delta(\mu)) \ .\end{aligned}$$

Now since $(\alpha\lambda^r)^{1/p} \in \widehat{C_1}$ we conclude $s_{(\alpha\lambda^r)^{1/p}}(\Delta(\mu)) = s_{(\alpha\lambda^r)^{1/p}/\mu}$. Thus we have shown $\left[C^{+r}, \Delta^{(p)}\right] = s_{(\alpha\lambda^r)^{1/p}/\mu}$.

(v) Here we have

$$\begin{aligned}\left[C^{+r}, \Delta^{+(p)}\right] &= \left\{X \in s \,:\, \Delta^+(\mu) X \in w_\alpha^{+r,p}(\lambda) = s_{(\alpha\lambda^r)^{1/p}}\right\} \\ &= s_{(\alpha\lambda^r)^{1/p}}\left(\Delta^+(\mu)\right) ,\end{aligned}$$

since $\alpha\lambda^i \in \widehat{C_1^+}$ for $i = 0, 1, \ldots, r-1$. Now we have

$$s_{(\alpha\lambda^r)^{1/p}}\left(\Delta^+(\mu)\right) = s_{\left(\left(\alpha_{n-1}\lambda_{n-1}^r\right)^{1/p}/\mu_{n-1}\right)_n}$$

since $(\alpha\lambda^r)^{1/p}/\mu \in \widehat{C_1}$. We conclude

$$\left[C^{+r}, \Delta^{+(p)}\right] = s_{\left(\left(\alpha_{n-1}\lambda_{n-1}^r\right)^{1/p}/\mu_{n-1}\right)_n} .$$

The statements (vi) and (vii) can be shown similarly. $\square$

Remark 6. We deduce from the preceding theorem and Lemma 8 that the identity $\left[C^r, \Delta^{(p)}\right] = s_{(\lambda/\mu)^{r/p}}$ holds in the next cases:

(i) $r \geq p$ and $\lambda \in \widehat{C_1}$;

(ii) $1 \leq r \leq p$ and $\lambda^{r/p} \in \widehat{C_1}$.

It remains to deal with the sets $\left[A_1^r(\lambda), A_2^{(p)}(\mu)\right]$, for $A_1(\lambda) \in \{\Delta(\lambda), \Delta^+(\lambda)\}$ and $A_2(\mu) \in \{\Delta(\mu), \Delta^+(\mu), C(\mu), C^+(\mu)\}$

Theorem 14. *Let α, λ, $\mu \in U^+$.*

(a) *We assume $\alpha/\lambda^i \in \widehat{C_1}$ for $i = 0, 1, \ldots, r-1$. Then*

(i) *if $(\alpha/\lambda^r)^{1/p} \in \widehat{C_1}$, then*

$$\left[\Delta^r(\lambda), \Delta^{(p)}(\mu)\right] = s_{\left(\frac{\alpha}{\lambda^r}\right)^{1/p}\frac{1}{\mu}};$$

(ii) *if $\left(\alpha\lambda^{-r}\right)^{1/p}\mu^{-1} \in \widehat{C_1}$, then*

$$\left[\Delta^r(\lambda), \Delta^{+(p)}(\mu)\right] = s_{\left(\left(\frac{\alpha_{n-1}}{\lambda_{n-1}^r}\right)^{1/p}\frac{1}{\mu_{n-1}}\right)_n};$$

(iii) *if $(\alpha/\lambda^r)^{1/p}\mu \in \widehat{C_1}$, then*

$$\left[\Delta^r(\lambda), C^{(p)}(\mu)\right] = s_{\left(\frac{\alpha}{\lambda^r}\right)^{1/p}\frac{1}{\mu}};$$

(iv) *if $(\alpha/\lambda^r)^{1/p} \in \widehat{C_1^+}$, then*

$$\left[\Delta^r(\lambda), C^{+(p)}(\mu)\right] = s_{\mu\left(\frac{\alpha}{\lambda^r}\right)^{1/p}}.$$

(b) *We assume $\left(\alpha_{n-j}/\left(\lambda_{n-j}, \ldots, \lambda_n\right)\right)_n \in \widehat{C_1}$ for $j = 0, 1, \ldots, r-1$. Then*

(v) *if*

$$\left(\left[\alpha_{n-r}/\left(\lambda_{n-r}, \ldots, \lambda_{n-1}\right)\right]^{1/p}\right)_n \in \widehat{C_1},$$

then

$$\left[\Delta^{+r}(\lambda), \Delta^{(p)}(\mu)\right] = s_{\left(\left(\frac{\alpha_{n-r}}{\lambda_{n-r}\cdots,\lambda_{n-1}}\right)^{1/p}\frac{1}{\mu_n}\right)_n};$$

(vi) *if*

$$\left(\left[\alpha_{n-r}/\left(\lambda_{n-r}, \ldots, \lambda_n\right)\right]^{1/p}\mu_n^{-1}\right)_n \in \widehat{C_1},$$

then

$$\left[\Delta^{+r}(\lambda), \Delta^{+(p)}(\mu)\right] = s_{\left(\left(\frac{\alpha_{n-r-1}}{\lambda_{n-r-1},\ldots,\lambda_{n-2}}\right)^{1/p}\frac{1}{\mu_{n-1}}\right)_n};$$

(vii) *if*

$$\left(\left[\alpha_{n-r}/\left(\lambda_{n-r}, \ldots, \lambda_n\right)\right]^{1/p}\mu_n\right)_n \in \widehat{C_1},$$

then

$$\left[\Delta^{+r}(\lambda), C^{(p)}(\mu)\right] = s_{\left(\mu_n\left(\frac{\alpha_{n-r}}{\lambda_{n-r},\ldots,\lambda_{n-1}}\right)^{1/p}\right)_n};$$

(viii) *if*

$$\left(\left[\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right]^{1/p}\right)_n \in \widehat{C_1^+},$$

then

$$\left[\Delta^{+r}(\lambda), C^{+(p)}(\mu)\right] = s_{\left(\mu_n\left(\frac{\alpha_{n-r}}{\lambda_{n-r}\cdots\lambda_{n-1}}\right)^{1/p}\right)_n}.$$

Proof. (i) We have

$$\left[\Delta^r, \Delta^{(p)}\right] = \left\{X \in s \,:\, |\Delta(\mu) X|^p \in s_\alpha(\Delta^r(\lambda))\right\}.$$

Now the condition $\alpha/\lambda^i \in \widehat{C_1}$ for $i = 0, 1 \ldots, r-1$, implies $s_\alpha(\Delta^r(\lambda)) = s_{\alpha/\lambda^r}$. Thus we have

$$\left[\Delta^r, \Delta^{(p)}\right] = s_{(\alpha/\lambda^r)^{1/p}}(\Delta(\mu)).$$

Finally since

$$\frac{\left(\frac{\alpha}{\lambda^r}\right)^{1/p}}{\mu^0} = \left(\frac{\alpha}{\lambda^r}\right)^{1/p} \in \widehat{C_1},$$

we conclude

$$\left[\Delta^r, \Delta^{(p)}\right] = s_{\left(\frac{\alpha}{\lambda^r}\right)^{1/p}\frac{1}{\mu}}.$$

(ii) Since $\alpha/\lambda^i \in \widehat{C_1}$ for $i = 0, 1, \ldots, r-1$, then $s_\alpha\big(\Delta^r(\lambda)\big) = s_{\alpha/\lambda^r}$, and

$$\begin{aligned}\left[\Delta^r, \Delta^{+(p)}\right] &= \left\{X \in s \,:\, \left|\Delta^+(\mu)X\right|^p \in s_{\alpha/\lambda^r}\right\} \\ &= \left\{X \in s \,:\, \Delta^+(\mu) X \in s_{(\alpha/\lambda^r)^{1/p}}\right\} \\ &= s_{(\alpha/\lambda^r)^{1/p}}\left(\Delta^+(\mu)\right).\end{aligned}$$

The condition $\left(\alpha\lambda^{-r}\right)^{1/p}\mu^{-1} \in \widehat{C_1}$, implies

$$s_{(\alpha/\lambda^r)^{1/p}}\left(\Delta^+(\mu)\right) = s_{\left(\left(\alpha_{n-1}/\lambda_{n-1}^r\right)^{1/p}\mu_{n-1}^{-1}\right)_n}.$$

We conclude $\left[\Delta^r, \Delta^{+(p)}\right] = s_{\left(\left(\alpha_{n-1}/\lambda_{n-1}^r\right)^{1/p}\mu_{n-1}^{-1}\right)_n}$.

(iii) Since $\alpha/\lambda^i \in \widehat{C_1}$ for $i = 0, 1, \ldots, r-1$, then $s_\alpha(\Delta^r(\lambda)) = s_{\alpha/\lambda^r}$ and $\left[\Delta^r, C^{(p)}\right] = s_{\left((\alpha/\lambda^r)^{1/p}\right)}(C(\mu))$. Then the condition $(\alpha/\lambda^r)^{1/p}\mu \in \widehat{C_1}$ implies $s_{\left((\alpha/\lambda^r)^{1/p}\right)}(C(\mu)) = s_{\left((\alpha/\lambda^r)^{1/p}\mu\right)}$ and $\left[\Delta^r, C^{(p)}\right] = s_{\left((\alpha/\lambda^r)^{1/p}\mu\right)}$.

(iv) The condition $\alpha/\lambda^i \in \widehat{C_1}$ for $i = 0, 1, \ldots, r-1$, implies $s_\alpha(\Delta^r(\lambda)) = s_{\alpha/\lambda^r}$. Then $\left[\Delta^r, C^{+(p)}\right] = s_{(\alpha/\lambda^r)^{1/p}}(C^+(\mu))$, and since $(\alpha/\lambda^r)^{1/p} \in \widehat{C_1^+}$ we conclude $\left[\Delta^r, C^{+(p)}\right] = s_{\left((\alpha/\lambda^r)^{1/p}\mu\right)}$.

(v) We have

$$\left[\Delta^{+r}, \Delta^{(p)}\right] = \left\{X \in s : \; |\Delta(\mu) X|^p \in s_\alpha\left(\Delta^{+r}(\lambda)\right)\right\}.$$

Since $\left(\alpha_{n-j}/\left(\lambda_{n-j}\ldots\lambda_n\right)\right)_n \in \widehat{C_1}$ for $j=0,1,\ldots,r-1$, then $s_\alpha\left(\Delta^{+r}(\lambda)\right) = s_{\left(\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right)_n}$ and $\left[\Delta^{+r},\Delta^{(p)}\right] = s_{\left(\left[\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right]^{1/p}\right)_n}\left(\Delta(\mu)\right)$, and since $\left(\left[\,\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right]^{1/p}\right)_n \in \widehat{C_1}$, then

$$\left[\Delta^{+r},\Delta^{(p)}\right] = s_{\left(\left[\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right]^{1/p}\mu_n^{-1}\right)_n}.$$

(vi) Here we have

$$\left[\Delta^{+r},\Delta^{+(p)}\right] = \left\{X\in s : \left|\Delta^{+}(\mu)X\right|^p \in s_\alpha\left(\Delta^{+r}(\lambda)\right)\right\}.$$

Since α/λ, $\left(\alpha_{n-1}/\left(\lambda_{n-1}\lambda_n\right)\right)_n,\ldots,\left(\alpha_{n-r+1}/\left(\lambda_{n-r+1}\ldots\lambda_n\right)\right)_n \in \widehat{C_1}$, then

$$s_\alpha\left(\Delta^{+r}(\lambda)\right) = s_{\left(\left(\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right)\right)_n},$$

and

$$\left[\Delta^{+r},\Delta^{+(p)}\right] = s_{\left(\left[\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right]^{1/p}\right)_n}\left(\Delta^{+}(\mu)\right).$$

Finally the condition $\left(\left[\,\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right]^{1/p}\mu_n^{-1}\right)_n \in \widehat{C_1}$ implies

$$s_{\left(\left[\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right]^{1/p}\right)_n}\left(\Delta^{+}(\mu)\right) = s_{\left(\left[\alpha_{n-r-1}/\left(\lambda_{n-r-1}\ldots\lambda_{n-2}\right)\right]^{1/p}/\mu_{n-1}\right)_n}$$

and $\left[\Delta^{+r},\Delta^{+(p)}\right] = s_{\left(\left[\alpha_{n-r-1}/\left(\lambda_{n-r-1}\ldots\lambda_{n-2}\right)\right]^{1/p}\mu_{n-1}^{-1}\right)_n}$.

(vii) Now

$$\begin{aligned}\left[\Delta^{+r},C^{(p)}\right] &= \left\{X\in s : \left|C(\mu)X\right|^p \in s_\alpha\left(\Delta^{+r}(\lambda)\right)\right\}\\ &= s_{\left(\left(\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right)^{1/p}\right)_n}\left(C(\mu)\right)\end{aligned}$$

for $\left(\alpha_{n-j}/\left(\lambda_{n-j}\ldots\lambda_n\right)\right)_n \in \widehat{C_1}$ for $j=0,1,\ldots,r-1$. Then the condition $\left(\mu_n\left[\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right]^{1/p}\right)_n \in \widehat{C_1}$ implies

$$s_{\left(\left[\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right]^{1/p}\right)_n}\left(C(\mu)\right) = \left[\Delta^{+r},C^{(p)}\right] = s_{\left(\mu_n\left[\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right]^{1/p}\right)_n}.$$

(viii) The condition α/λ, $\left(\alpha_{n-1}/\left(\lambda_{n-1}\lambda_n\right)\right)_n,\ldots,\left(\alpha_{n-r+1}/\left(\lambda_{n-r+1}\ldots\lambda_n\right)\right)_n \in \widehat{C_1}$ implies

$$s_\alpha\left(\Delta^{+r}(\lambda)\right) = s_{\left(\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right)_n},$$

so

$$\begin{aligned}\left[\Delta^{+r},C^{+(p)}\right] &= \left\{X\in s : |C^{+}(\mu)X|^p \in s_{\left(\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right)_n}\right\}\\ &= \left\{X\in s : C^{+}(\mu)X \in s_{\left(\left[\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right]^{1/p}\right)_n}\right\}.\end{aligned}$$

Finally

$$\left[\Delta^{+r},C^{+(p)}\right] = s_{\left(\left[\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right]^{1/p}\right)_n}\left(C^{+}(\mu)\right) = s_{\left(\mu_n\left[\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right]^{1/p}\right)_n},$$

since $\left(\left[\,\alpha_{n-r}/\left(\lambda_{n-r}\ldots\lambda_{n-1}\right)\right]^{1/p}\right)_n \in \widehat{C_1^{+}}$. This concludes the proof. $\square$

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LMAH Université du Havre
BP 4006 I.U.T Le Havre, 76610 Le Havre, France
E-mail bdemalaf@wanadoo.fr