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ON MODIFIED MEYER-KÖNIG AND ZELLER OPERATORS OF FUNCTIONS OF TWO VARIABLES

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Abstract. This paper is motivated by Kirov results on generalized Bernstein polynomials given in [11]. We introduce certain modified Meyer-König and Zeller operators in the space of differentiable functions of two variables and we study approximation properties for them.

Some approximation properties of the Meyer-König and Zeller operators of differentiable functions of one variable are given in [15] and [16].

1. Introduction

1.1. Let $C(I)$ be the space of real-valued functions f continuous on the interval $I := [0,1]$ with the norm $\|f\| = \max_{x\in I}|f(x)|$ and let $C^r(I)$, $r \in N_0 := \{0,1,2,\dots\}$, be the set of all $f \in C(I)$ having the derivative $f^{(r)} \in C(I)$ $(C^0(I) \equiv C(I))$.

In [14] were introduced the Meyer-König and Zeller operators

$$M_n(f;x) := \begin{cases} \sum\limits_{k=0}^{\infty} p_{nk}(x) f\big(\frac{k}{n+k}\big) & \text{if} \quad 0 \le x < 1\,, \\ f(1) & \text{if} \quad x = 1\,, \end{cases} \tag{1}$$

for $n \in N$ and f defined and bounded on I, where

$$p_{nk}(x) := \binom{n+k}{k} x^k (1-x)^{n+1} \qquad k \in N_0,\ n \in N\,. \tag{2}$$

The approximation properties of the Meyer-König and Zeller operators of functions of one variable vere examined in many papers, for example [1-6, 8-10, 13-14].

It is known ([1-6, 10]) that $M_n(f)$ is a positive linear operator from the space $C(I)$ into $C(I)$. Moreover, for every $f \in C(I)$ there holds the following inequality

$$\|M_n(f) - f\| \ \le\ \frac{37 - 16\sqrt{3}}{9}\, \omega\Big(f; \frac{1}{\sqrt{n}}\Big)\,, \qquad n \in N\,, \tag{3}$$

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where $\omega(f;\cdot)$ is the modulus of continuity of f, i.e.

$$\omega(f;t) := \sup\left\{|f(x)-f(y)| \,:\, x,y \in I,\ |x-y| \le t\right\}, \qquad t \in I. \tag{4}$$

In [15] were introduced the following modified Meyer-König and Zeller operators

$$M_{n;r}(f;x) := \begin{cases} \sum\limits_{k=0}^{\infty} p_{nk}(x) \sum\limits_{s=0}^{r} \frac{f^{(s)}\left(\frac{k}{n+k}\right)}{s!}\left(x-\frac{k}{n+k}\right)^s & \text{if} \quad 0 \le x < 1, \\ f(1) & \text{if} \quad x = 1, \end{cases} \tag{5}$$

for $f \in C^r(I)$, $r \in N_0$, and $n \in N$, where $p_{nk}(\cdot)$ is defined by (2). It is obvious that if $r = 0$, then $M_{n;0}(f;x) = M_n(f;x)$ for every $f \in C(I)$, $x \in I$ and $n \in N$. Moreover from (1), (2) and (5) we deduce that

$$M_{n;r}(1;x) = \sum_{k=0}^{\infty} p_{nk}(x) = 1 \quad \text{for} \quad x \in I,\ n \in N,\ r \in N_0. \tag{6}$$

In [15] it is proved that if n, $r \in N$, then $M_{n;r}$ is a linear operator from the space $C^r(I)$ into $C(I)$. Moreover in [15] it is proved that for every $r \in N$ there exists a positive constant $K_1(r)$ depending only on r such that

$$\|M_{n;r}(f) - f\| \ \le\ K_1(r)\, n^{-\frac{r}{2}}\, \omega\left(f^{(r)}; \frac{1}{\sqrt{n}}\right), \tag{7}$$

for every $f \in C^r(I)$ and $n \in N$, where $\omega\left(f^{(r)};\cdot\right)$ is the modulus of continuity of $f^{(r)}$ defined by (4).

From (3) and (7) we can deduce that if $r \ge 2$, then operators $M_{n;r}$ defined by (5) have better approximation properties for $f \in C^r(I)$ than operators M_n defined by (1).

1.2. In this paper we shall introduce modified Meyer-König and Zeller operators in the space of differentiable functions of two variables and we shall give an approximation theorem for them. We shall show that these operators have better approximation properties than classical Meyer-König and Zeller operators.

Let $I^2 := \{(x,y) : x,y \in I\}$ and let $C(I^2)$ be the space of all real-valued functions f of two variables continuous on I^2 with the norm

$$\|f\| \ := \max_{(x,y)\in I^2} |f(x,y)|. \tag{8}$$

For $f \in C(I^2)$ we define the modulus of continuity

$$\begin{aligned} \omega(f;s,t) := \sup\Big\{ \big|f(u,v) - f(x,y)\big| : (u,v),(x,y) \in I^2, \\ |u-x| \le s,\ |v-y| \le t \Big\}, \quad s,t \in I. \end{aligned} \tag{9}$$

It is known ([17], p.124) that if $f \in C(I^2)$, then $\omega(f;s,t)$ is nondecreasing function of variables s, t and

$$\begin{aligned} \omega\left(f;\lambda_1 s,\lambda_2 t\right) &\le \omega\left(f;\lambda_1 s,0\right) + \omega\left(f;0,\lambda_2 t\right) \\ &\le (\lambda_1+1)\,\omega(f;s,0) + (\lambda_2+1)\,\omega(f;0,t) \\ &\le (\lambda_1+\lambda_2+2)\,\omega(f;s,t) \end{aligned}$$

for $\lambda_1, \lambda_2 = const \geq 0$ and $\lambda_1 s, \lambda_2 t \in I$. Moreover for every $f \in C(I^2)$ we have

$$\lim_{s,t \to 0+} \omega(f; s, t) = 0.$$

Similarly to §1.1 we define the set $C^r(I^2)$, $r \in N_0$, of all $f \in (I^2)$ having all partial derivatives $f^{(m)}_{x^{m-i}y^i} \in C(I^2)$, $0 \leq i \leq m \leq r$. Clearly $C^0(I^2) = C(I^2)$.

In the space $C^r(I^2)$, we introduce analogues of operators M_n and $M_{n;r}$ given by formulas (1) and (5).

Definition 1. Let $m, n \in N$. The Meyer-König and Zeller operator of $f \in C(I^2)$ is defined by the formula

$$(10)\quad M_{m,n}(f; x, y) := \begin{cases} \sum\limits_{j=0}^{\infty} \sum\limits_{k=0}^{\infty} p_{mj}(x) p_{nk}(y) f\left(\xi_{mj}, \xi_{nk}\right) & \text{if} \quad 0 \leq x,\ y < 1\,, \\ \sum\limits_{j=0}^{\infty} p_{mj}(x) f\left(\xi_{mj}, 1\right) & \text{if} \quad 0 \leq x < 1,\ y = 1\,, \\ \sum\limits_{k=0}^{\infty} p_{nk}(y) f\left(1, \xi_{nk}\right) & \text{if} \quad x = 1,\ 0 \leq y < 1\,, \\ f(1,1) & \text{if} \quad x = y = 1\,, \end{cases}$$

where $p_{mj}(\cdot)$ is given by (2) and

$$(11)\qquad \xi_{\alpha\beta} := \frac{\beta}{\alpha + \beta} \quad \text{for} \quad \alpha \in N,\ \beta \in N_0\,.$$

In Section 2 we shall prove that $M_{m,n}(f) \in C(I^2)$ if $f \in C(I^2)$.

From (10), (11) and (1) we deduce that

$$(12)\qquad M_{m,n}(f(t,z); x, 1) = M_m\left(f_1(t); x\right) \text{ for } x \in I,$$

$$(13)\qquad M_{m,n}(f(t,z); 1, y) = M_n\left(f_2(z); y\right) \text{ for } y \in I,$$

for $f \in C(I^2)$, where

$$(14)\qquad f_1(x) := f(x,1),\ f_2(y) := f(1,y) \text{ for } x, y \in I.$$

Definition 2. Let $n, r \in N$ be fixed numbers. The n-th modified Meyer-König and Zeller operator of functions $f \in C^r(I^2)$ we define by the formula

(15)

$$\widetilde{M}_{n;r}(f; x, y) := \begin{cases} \sum\limits_{j=0}^{\infty} \sum\limits_{k=0}^{\infty} p_{nj}(x) p_{nk}(y) \sum\limits_{s=0}^{r} \frac{d^s f(\xi_{nj}, \xi_{nk})}{s!} & \text{if} \quad 0 \leq x,\ y < 1\,, \\ \sum\limits_{j=0}^{\infty} p_{nj}(x) \sum\limits_{s=0}^{r} \frac{d^s f_1(\xi_{nj})}{s!} & \text{if} \quad 0 \leq x < 1,\ y = 1\,, \\ \sum\limits_{k=0}^{\infty} p_{nk}(y) \sum\limits_{s=0}^{r} \frac{d^s f_2(\xi_{nk})}{s!} & \text{if} \quad x = 1,\ 0 \leq y < 1\,, \\ f(1,1) & \text{if} \quad x = y = 1\,, \end{cases}$$

where p_{nj} and ξ_{nj} are given by (2) and (11), f_1 and f_2 are defined by (14) and $d^s f(x_0, y_0)$, $d^s f_1(x_0)$ and $d^s f_2(y_0)$ are the s-th differentials:

$$d^s f(x_0, y_0) = \sum_{i=0}^{s} \binom{s}{i} f^{(s)}_{x^{s-i} y^i}(x_0, y_0)(x - x_0)^{s-i} (y - y_0)^i, \tag{16}$$

$$d^s f_1(x_0) = f_1^{(s)}(x_0)(x - x_0)^s, \quad d^s f_2(y_0) = f_2^{(s)}(y_0)(y - y_0)^s. \tag{17}$$

From (14)–(17), (11) and (5) we deduce that

$$\widetilde{M}_{n;r}(f; x, 1) = M_{n;r}(f_1; x), \quad \widetilde{M}_{n;r}(f; 1, y) = M_{n;r}(f_2; y) \tag{18}$$

and (similarly to (6))

$$\widetilde{M}_{n;r}(1; x, y) = 1 \tag{19}$$

for all $x, y \in I$, $n \in N$ and $r \in N_0$.

In Section 2 we shall give some auxiliary results. The approximation theorems will be given in Section 3.

In this paper we shall denote by $K_i(r)$, $i \in N$, suitable positive constants depending only on indicated parameter r.

2. Lemmas

2.1. First we shall give some elementary properties of operators $M_{m,n}$ defined by (10).

Lemma 1. *Let $m, n \in N$ be fixed numbers. Then for every $f \in C(I^2)$ we have*

$$\lim_{x,y \to 1-} M_{m,n}(f; x, y) = f(1, 1). \tag{20}$$

Proof. Fix $m, n \in N$ and $f \in C(I^2)$. From (10), (8), (2) and (6) we deduce that $M_{m,n}(f)$ is continuous function on $D = \{(x, y) : 0 \leq x, y < 1\}$ and

$$|M_{m,n}(f; x, y)| \leq \|f\| \sum_{j=0}^{\infty} p_{mj}(x) \sum_{k=0}^{\infty} p_{nk}(y) = \|f\|.$$

Obviously,

$$M_{m,n}(f; x, y) - f(1, 1) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} p_{mj}(x)\, p_{nk}(y) \left(f(\xi_{mj}, \xi_{nk}) - f(1, 1)\right), \tag{21}$$

for $(x, y) \in D$. Next by (11) we have

$$\lim_{j,k \to \infty} \left(f(\xi_{mj}, \xi_{nk}) - f(1, 1)\right) = 0 \tag{22}$$

and by (8)

$$|f(\xi_{mj}, \xi_{nk}) - f(1, 1)| \leq 2\|f\|, \qquad j, k \in N_0. \tag{23}$$

Choose $\varepsilon > 0$. Then by (22) there exist natural numbers $j_0 = j_0(\varepsilon)$ and $k_0 = k_0(\varepsilon)$ such that

$$|f(\xi_{mj}, \xi_{nk}) - f(1, 1)| < \varepsilon \qquad \text{for} \qquad j > j_0,\ k > k_0. \tag{24}$$

Moreover from (21) we get

$$\begin{aligned}\big|M_{m,n}(f;x,y)-f(x,y)\big| \le \Big(&\sum_{j=0}^{j_0}\sum_{k=0}^{k_0}+\sum_{j=j_0+1}^{\infty}\sum_{k=0}^{k_0}\\&+\sum_{j=0}^{j_0}\sum_{k=k_0+1}^{\infty}+\sum_{j=j_0+1}^{\infty}\sum_{k=k_0+1}^{\infty}\Big)p_{mj}(x)\,p_{nk}(y)\\&\times\big|f\big(\xi_{mj},\xi_{nk}\big)-f(1,1)\big|\\&:=\sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3+\sum\nolimits_4\,,\qquad (x,y)\in D\,.\end{aligned}$$

By (24) and (6) we have

$$\sum\nolimits_4<\varepsilon\sum_{j=0}^{\infty}p_{mj}(x)\sum_{k=0}^{\infty}p_{nk}(y)=\varepsilon\quad\text{for}\quad(x,y)\in D\,.$$

From (2) we deduce that

$$\lim_{x\to1-}p_{nk}(x)=0\quad\text{for fixed}\quad k\in N_0\quad\text{and}\quad n\in N\,.\tag{25}$$

Applying (23) and (25), we get

$$\sum\nolimits_1\le 2\,\|f\|\sum_{j=0}^{j_0}p_{mj}(x)\sum_{k=0}^{k_0}p_{nk}(y)=o(1)\,,\quad\text{as}\quad x,y\to1-\ .$$

Analogously, by (23), (25) and (6) we get

$$\begin{aligned}\sum\nolimits_2&\le 2\,\|f\|\sum_{j=0}^{j_0}p_{mj}(x)\sum_{k=0}^{\infty}p_{nk}(y)=2\,\|f\|\sum_{j=0}^{j_0}p_{mj}(x)\\&=o(1)\quad\text{as}\quad x\to1-,\ 0\le y<1\,,\\\sum\nolimits_3&\le 2\,\|f\|\sum_{k=0}^{k_0}p_{nk}(y)=o(1)\quad\text{as}\quad y\to1-,\ 0\le x<1\,.\end{aligned}$$

Combining the above, we obtain

$$M_{m,n}(f;x,y)-f(x,y)=o(1)\text{ as }x\to1-,\ y\to1-\,.$$

Thus the proof of (20) is completed. □

Arguing similarly as in the proof of Lemma 1, we can prove the following

Lemma 2. *Let $m,n\in N$ and let $f\in C(I^2)$. Then*

$$\lim_{y\to1-}\sum_{j=0}^{\infty}p_{mj}(x)\sum_{k=0}^{\infty}p_{nk}(y)\,(\,f\,(\xi_{mj},\xi_{nk})-f\,(\xi_{mj},1))=0$$

for every $0\le x<1$, and

$$\lim_{x\to1-}\sum_{k=0}^{\infty}p_{nk}(y)\sum_{j=0}^{\infty}p_{mj}(x)\,(f\ (\xi_{mj},\xi_{nk})-f\,(1,\xi_{nk}))=0$$

for every $0 \le y < 1$. *Moreover we have*

$$\lim_{x\to 1-} \sum_{j=0}^{\infty} p_{mj}(x)\, f\left(\xi_{mj}, 1\right) = f(1,1),$$

$$\lim_{y\to 1-} \sum_{k=0}^{\infty} p_{nk}(y)\, f\left(1, \xi_{nk}\right) = f(1,1).$$

Applying (10)–(15), Lemma 1 and Lemma 2, we easily obtain

Lemma 3. *The Meyer-König and Zeller operator* $M_{m,n}$, $m, n \in N$, *defined by* (10) *is a positive linear operator from the space* $C(I^2)$ *into* $C(I^2)$. *Moreover for every* $f \in C(I^2)$ *we have*

$$\|M_{m,n}(f)\| \le \|f\|\,, \qquad m, n \in N\,.$$

In [1, 4] is proved the following

Lemma 4. *For every* $x \in I$ *and* $n \in N$ *we have*

$$M_n(1;x) = 1\,, \qquad M_n(t;x) = x,$$

$$M_n\left(t^2;x\right) = \begin{cases} x^2 + x \sum\limits_{k=0}^{\infty} p_{nk}(x)\left(\frac{k+1}{n+k+1} - \frac{k}{n+k}\right) & \text{if} \quad 0 \le x < 1\,, \\ 1 & \text{if} \quad x = 1\,, \end{cases}$$

which imply that

$$\begin{aligned} M_n\left((t-x)^2;x\right) &= M_n\left(t^2;x\right) - 2xM_n(t;x) + x^2 = M_n\left(t^2;x\right) - x^2 \\ &= \begin{cases} x \sum\limits_{k=0}^{\infty} p_{nk}(x)\frac{n}{(n+k+1)(n+k)} & \text{if} \quad 0 \le x < 1\,, \\ 0 & \text{if} \quad x = 1\,, \end{cases} \\ &\le \frac{1}{n} \quad \text{for} \quad x \in I,\ n \in N\,. \end{aligned}$$

From results given in [1] and [10] we obtain the following

Lemma 5. *For every* $s \in N$ *there exists* $K_2(s) = \text{const.} > 0$ *such that*

$$M_n\big((t-x)^{2s};x\big) \le K_2(s)\, n^{-s} \qquad \text{for} \quad x \in I,\ n \in N\,.$$

Consequently,

$$M_n\big(|t-x|^{s};x\big) \le \left(K_2(s)\, n^{-s}\right)^{\frac{1}{2}} \qquad \text{for} \quad x \in I,\ n \in N\,.$$

2.2. Reasoning similarly as in the proof of Lemma 1 and applying (11)–(14), we can derive the following properties of operators $\widetilde{M}_{n;r}$ defined by (15).

Lemma 6. *Let $n, r \in N$. Then $\widetilde{M}_{n;r}$ is a linear operator from the space $C^r(I^2)$ into $C(I^2)$. Moreover there exists $K_3(r) = \text{const.} > 0$ such that*

$$\left\|\widetilde{M}_{n;r}(f;\cdot,\cdot)\right\| \le K_3(r)\sum_{s=0}^{r}\sum_{i=0}^{s}\left\|f^{(s)}_{x^{s-i}y^i}\right\|, \tag{26}$$

for every $f \in C^r(I^2)$.

Proof. We shall prove only (26). By (11) we have

$$|\,x-\xi_{nj}| \le 1, \quad |\,y-\xi_{nk}| \le 1 \quad \text{for} \quad x, y \in I,\ j,k \in N_0,\ n \in N\,.$$

Using these inequalities and (16) and (8) to (15), we can write

$$\begin{aligned}\left|\widetilde{M}_{n;r}(f;x,y)\right| &\le \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}p_{nj}(x)p_{nk}(y)\sum_{s=0}^{r}\frac{1}{s!}\sum_{i=0}^{s}\binom{s}{i}\left\|f^{(s)}_{x^{s-i}y^i}\right\| \\ &\le \sum_{s=0}^{r}\frac{1}{s!}\sum_{i=0}^{s}\binom{s}{i}\left\|f^{(s)}_{x^{s-i}y^i}\right\|\end{aligned}$$

for $0 \le x,\ y < 1$ and $n, r \in N$. Similarly, by (17), we obtain

$$\left|\widetilde{M}_{n;r}(f;x,1)\right| \le \sum_{s=0}^{r}\frac{1}{s!}\left\|f_1^{(s)}(\cdot)\right\| \le \sum_{s=0}^{r}\frac{1}{s!}\left\|f^{(s)}_{x^s}\right\|,$$

$$\left|\widetilde{M}_{n;r}(f;1,y)\right| \le \sum_{s=0}^{r}\frac{1}{s!}\left\|f_2^{(s)}(\cdot)\right\| \le \sum_{s=0}^{r}\frac{1}{s!}\left\|f^{(s)}_{y^s}\right\|,$$

for $0 \le x,\ y < 1$ and $n, r \in N$. Applying the above inequalities and (15), we immediately derive (26). □

3. Theorems

3.1. First we shall prove approximation theorem for $f \in C(I^2)$ and $M_{m,n}(f)$.

Theorem 1. *For every $f \in C(I^2)$ and $m, n \in N$ we have*

$$\|M_{m,n}(f) - f\| \le 4\,\omega\Big(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\Big), \tag{27}$$

where $\omega(f;\cdot,\cdot)$ is the modulus of continuity of f defined by (9).

Proof. From (10) and (6) we deduce that

$$\begin{aligned} &M_{m,n}(f;x,y) - f(x,y) = \\ &= \begin{cases} \sum\limits_{j=0}^{\infty}\sum\limits_{k=0}^{\infty}p_{mj}(x)p_{nk}(y)\left(f\left(\xi_{mj},\xi_{nk}\right)-f(x,y)\right) & \text{if} \quad 0 \le x,\ y < 1\,, \\ \sum\limits_{j=0}^{\infty}p_{mj}(x)\left(f\left(\xi_{mj},1\right)-f(x,1)\right) & \text{if} \quad 0 \le x < 1,\ y = 1\,, \\ \sum\limits_{k=0}^{\infty}p_{nk}(y)\left(f\left(1,\xi_{nk}\right)-f(1,y)\right) & \text{if} \quad x = 1,\ 0 \le y < 1\,, \\ 0 & \text{if} \quad x = y = 1\,, \end{cases}\end{aligned} \tag{28}$$

for $f \in C(I^2)$ and $m, n \in N$. If $0 \leq x, y < 1$, then by (11) and (9) and properties of the modulus of continuity we have

$$\begin{aligned}|f(\xi_{mj}, \xi_{nk}) - f(x,y)| &\leq \omega\left(f; |\xi_{mj} - x|\ , |\xi_{nk} - y|\right)\\ &\leq \left(\sqrt{m}\ |\xi_{mj} - x| + \sqrt{n}\ |\xi_{nk} - y| + 2\right)\ \omega\Big(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\Big)\end{aligned}$$

and next by (28), (6) and (1) we can write

$$\begin{aligned}|M_{m,n}(f; x, y) - f(x,y)| \leq \omega\Big(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\Big) \big\{\sqrt{m}\, M_m(|t-x|; x)\\ + \sqrt{n}\, M_n(|z-y|; y) + 2\big\}\ .\end{aligned}$$

Using the Hölder inequality and Lemma 4, we get

$$M_m(|t-x|; x) \leq \left\{M_m\left((t-x)^2; x\right)\right\}^{\frac{1}{2}} \left\{M_m(1; x)\right\}^{\frac{1}{2}} \leq \frac{1}{\sqrt{m}}\,,$$

and analogously

$$M_n(|z-y|; y) \leq \frac{1}{\sqrt{n}}\,,$$

for $0 \leq x,\ y < 1$. From the above we obtain

$$|M_{m,n}(f; x, y) - f(x,y)| \leq 4\,\omega\Big(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\Big) \tag{29}$$

for $0 \leq x,\ y < 1$ and $m, n \in N$. Analogously we deduce that

$$|M_{m,n}(f; x, 1) - f(x,1)| \leq 4\,\omega\Big(f; \frac{1}{\sqrt{m}}, 0\Big)\,, \tag{30}$$

$$|M_{m,n}(f; 1, y) - f(1,y)| \leq 4\,\omega\Big(f; 0, \frac{1}{\sqrt{n}}\Big)\,, \tag{31}$$

for all $0 \leq x < 1$, $0 \leq y < 1$ and $m, n \in N$. Now from (28)–(31) and (8) and by properties of $\omega(f; \cdot, \cdot)$ immediately results (27). □

From Theorem 1 we can derive

Corollary 1. *If $f \in C(I^2)$, then*

$$\lim_{m,n \to \infty} \|M_{m,n}(f) - f\| = 0\,.$$

Corollary 2. *If $f \in C^1(I^2)$, then*

$$\|M_{n,n}(f) - f\| \leq 4\left(\|f'_x\| + \|f'_y\|\right) n^{-\frac{1}{2}}\,, \qquad n \in N\,. \tag{32}$$

Indeed, by (27), (9) and properties of $\omega(f; \cdot, \cdot)$ ([17], p.124) we can write

$$\|M_{n,n}(f) - f\| \leq 4\Big(\omega\Big(f; \frac{1}{\sqrt{n}}, 0\Big) + \omega\Big(f; 0, \frac{1}{\sqrt{n}}\Big)\Big)\,,$$

for $f \in C(I^2)$ and $n \in N$. Moreover, if $f \in C^1(I^2)$, then we have

$$\begin{aligned}\omega\Big(f;\frac{1}{\sqrt{n}},0\Big) &= \sup\Big\{|f(u,y)-f(x,y)| : (u,y),(x,y) \in I^2, |u-x| \le \frac{1}{\sqrt{n}}\Big\}\\ &\le \|f'_x\|\, n^{-1/2}\end{aligned}$$

and analogously

$$\omega\Big(f;0,\frac{1}{\sqrt{n}}\Big) \le \|f'_y\|\, n^{-1/2}\,.$$

From the above follows (32).

3.2. Now we shall prove an analogue of (7) for $f \in C^r(I^2)$ and $\widetilde{M}_{n;r}(f)$.

Theorem 2. *Let $r \in N$ be a fixed number. Then there exists $K_4(r) = const. > 0$ such that for every $f \in C^r(I^2)$ and $n \in N$ we have*

$$\big\|\widetilde{M}_{n;r}(f)-f\big\| \le K_4(r)\, n^{-\frac{r}{2}} \sum_{i=0}^{r} \omega\Big(f^{(r)}_{x^{r-i}y^i};\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\Big)\,, \tag{33}$$

where $\omega\big(f^{(r)}_{x^{r-i}y^i};\cdot,\cdot\big)$ is the modulus of continuity defined by (9).

Proof. The formulas (15), (18), (19) and (6) imply that

$$\begin{aligned}&\widetilde{M}_{n;r}(f;x,y)-f(x,y) = \\ &= \begin{cases} \sum\limits_{j=0}^{\infty}\sum\limits_{k=0}^{\infty} p_{nj}(x)p_{nk}(y)\Big(\sum\limits_{s=0}^{r}\frac{d^s f(\xi_{nj},\xi_{nk})}{s!}-f(x,y)\Big) & \text{if } \ 0 \le x,\ y<1\,,\\ \sum\limits_{j=0}^{\infty} p_{nj}(x)\Big(\sum\limits_{s=0}^{r}\frac{d^s f_1(\xi_{nj})}{s!}-f_1(x)\Big) & \text{if } \ 0\le x<1,\ y=1\,,\\ \sum\limits_{k=0}^{\infty} p_{nk}(x)\Big(\sum\limits_{s=0}^{r}\frac{d^s f_2(\xi_{nk})}{s!}-f_2(y)\Big) & \text{if } \ x=1,\ 0\le y<1\,,\\ 0 & \text{if } \ x=y=1\,,\end{cases}\end{aligned} \tag{34}$$

for every $f \in C^r(I^2)$ and $n \in N$, where $f_1(x)$ and $f_2(y)$ are defined by (14).

a) First let $0 \le x, y < 1$. Then we apply the following Taylor formula ([7]) of $f \in C^r(I^2)$ at a fixed point $(x_0,y_0) \in I^2$:

$$\begin{aligned}f(x,y) = \sum_{s=0}^{r}\frac{d^s f(x_0,y_0)}{s!} &+ \frac{1}{(r-1)!}\\ &\times \int_0^1 (1-t)^{r-1}\,(d^r f(\widetilde{x},\widetilde{y})-d^r f(x_0,y_0))\,dt\,, \quad (x,y)\in I^2\,,\end{aligned} \tag{35}$$

where $(\widetilde{x},\widetilde{y}) = (x_0+t(x-x_0), y_0+t(y-y_0))$ and differentials $d^s f(x_0,y_0)$, $0 \le s \le r$, and $d^r f(\widetilde{x},\widetilde{y})$ are defined for $\Delta x = x-x_0$ and $\Delta y = y-y_0$.

Using (35) with $(x_0,y_0) = (\xi_{nj},\xi_{nk})$ to (34), we get

$$\big|\widetilde{M}_{n;r}(f;x,y)-f(x,y)\big| \le \frac{1}{(r-1)!}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_{nj}(x)p_{nk}(y)A_{j,k;r}(f;x,y)\,, \tag{36}$$

with

$$A_{j,k;r}(f;x,y) := \int_0^1 (1-t)^{r-1} |\, d^r f(\xi_{nj} + t(x-\xi_{nj}), \xi_{nk} + t(y-\xi_{nk})) - d^r f(\xi_{nj}, \xi_{nk})|\, dt\,.$$

By (16) and (9) and properties of modulus of continuity, we have

$$\begin{aligned}
A_{j,k;r}(f;x,y) &\le \int_0^1 (1-t)^{r-1} \sum_{i=0}^{r} \binom{r}{i} \\
&\quad \times \omega\left(f^{(r)}_{x^{r-i}y^i}; t\,|x-\xi_{nj}|\,, t\,|y-\xi_{nk}|\right) |x-\xi_{nj}|^{r-i}\,|y-\xi_{nk}|^i \; dt \\
&\le \frac{1}{r}\sum_{i=0}^{r}\binom{r}{i}\omega\left(f^{(r)}_{x^{r-i}y^i}; |x-\xi_{nj}|\,, |y-\xi_{nk}|\right) |x-\xi_{nj}|^{r-i}\,|y-\xi_{nk}|^i \\
&\le \frac{1}{r}\sum_{i=0}^{r}\binom{r}{i}\omega\Big(f^{(r)}_{x^{r-i}y^i}; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\Big) \\
&\quad \times \left(\sqrt{n}\,|x-\xi_{nj}| + \sqrt{n}\,|y-\xi_{nk}| + 2\right) |x-\xi_{nj}|^{r-i}\,|y-\xi_{nk}|^i \;.
\end{aligned} \tag{37}$$

Using (37) to (36) and by (10) and (1), we can write

$$\begin{aligned}
\left|\widetilde{M}_{n;r}(f;x,y) - f(x,y)\right| \le \frac{1}{r!}\sum_{i=0}^{r}\binom{r}{i}\omega\Big(f^{(r)}_{x^{r-i}y^i}; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\Big) \\
\times \Big\{\sqrt{n}\, M_n\left(|t-x|^{r+1-i}; x\right) M_n\left(|z-y|^i; y\right) \\
+ \sqrt{n}\, M_n\left(|t-x|^{r-i}; x\right) M_n\left(|z-y|^{i+1}; y\right) \\
+\; 2M_n\left(|t-x|^{r-i}; x\right) M_n\left(|z-y|^i; y\right)\Big\}\,,
\end{aligned}$$

which by Lemma 5 yields

$$|\widetilde{M}_{n;r}(f;x,y) - f(x,y)| \le K_5(r)\, n^{-\frac{r}{2}} \sum_{i=0}^{r}\binom{r}{i}\omega\Big(f^{(r)}_{x^{r-i}y^i}; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\Big) \tag{38}$$

for $0 \le x, y < 1$ and $n \in N$.

b) Now let $0 \le x < 1$, $y = 1$ and $n \in N$. By (17)-(19) and (5) and (7) we have

$$\begin{aligned}
\left|\widetilde{M}_{n;r}(f;x,1) - f(x,1)\right| &= \left|M_{n;r}(f_1;x) - f_1(x)\right| \\
&\le K_1(r)\, n^{-\frac{r}{2}}\omega\Big(f_1^{(r)}; \frac{1}{\sqrt{n}}\Big)\,.
\end{aligned} \tag{39}$$

Analogously we obtain

$$\begin{aligned}
\left|M_{n;r}(f;1,y) - f(1,y)\right| &= \left|M_{n;r}(f_2;y) - f_2(y)\right| \\
&\le K_1(r)\, n^{-\frac{r}{2}}\omega\Big(f_2^{(r)}; \frac{1}{\sqrt{n}}\Big)
\end{aligned} \tag{40}$$

for $0 \le y < 1$, $n \in N$.

From (14) results that $f_1^{(r)}(x) = f_{x^r}^{(r)}(x,1)$ and $f_2^{(r)}(y) = f_{y^r}^{(r)}(1,y)$. Next by (4) and (9) we have

$$\omega\Big(f_1^{(r)};\frac{1}{\sqrt{n}}\Big) \le \omega\Big(f_{x^r}^{(r)};\frac{1}{\sqrt{n}},0\Big) \le \omega\Big(f_{x^r}^{(r)};\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\Big)\,, \tag{41}$$

$$\omega\Big(f_2^{(r)};\frac{1}{\sqrt{n}}\Big) \le \omega\Big(f_{y^r}^{(r)};0,\frac{1}{\sqrt{n}},\Big) \le \omega\Big(f_{y^r}^{(r)};\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\Big)\,, \tag{42}$$

for $n \in N$. Collecting (38)–(42) and by (34) and (8), we obtain the desired inequality (33). □

From Theorem 2 and Theorem 1 we derive the following

Corollary 3. *For every $f \in C^r(I^2)$, $r \in N$, we have*

$$\lim_{n\to\infty} n^{\frac{r}{2}}\left\|\widetilde{M}_{n;r}(f) - f\right\| = 0\,.$$

Finally we remark that if $2 \le r \in N$, then the order of approximation of $f \in C^r(I^2)$ by $\widetilde{M}_{n;r}(f)$ defined by (15) is better than the order of approximation of this function f by classical Meyer-König and Zeller operators $M_{n,n}(f)$ defined by (10).

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