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ON AN EFFECTIVE CRITERION OF SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATION OF n-TH ORDER

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Abstract. New sufficient conditions for the existence of a solution of the boundary value problem for an ordinary differential equation of n-th order with certain functional boundary conditions are constructed by a method of a priori estimates.

Introduction

In this paper we give new sufficient conditions for the existence of a solution of the ordinary differential equation

$$u^{(n)}(t) = f\Big(t, u(t), \dots, u^{(n-1)}(t)\Big) \tag{1}$$

with the boundary conditions

$$\Phi_{0i}\left(u^{(i-1)}\right) = \varphi_i(u) \qquad i = 1, \dots, n, \tag{2}$$

resp.

$$l_i\left(u, u', \dots, u^{(k_0-1)}\right) = 0 \qquad i = 1, \dots, k_0, \tag{3_1}$$

$$\Phi_{0i}\left(u^{(i-1)}\right) = \varphi_i\left(u^{(k_0)}\right) \qquad i = k_0+1, \dots, n, \tag{3_2}$$

where $f : [a,b] \times R^n \to R$ satisfies the local Carathéodory conditions, $n \geq 2$, and $1 \leq k_0 \leq n-2$.

For each index i, the functional Φ_{0i} in the conditions (2), resp. (3_2), is supposed to be linear, nondecreasing, nontrivial, continuous on $C([a,b])$, and concentrated on $[a_i, b_i] \subseteq [a,b]$ (i.e., the value of functional Φ_{0i} depends only on a function restricted to $[a_i, b_i]$ and this segment can be degenerated to a point). In general $\Phi_{0i}(1) \in R$, without loss of generality we can suppose that $\Phi_{0i}(1) = 1$, which simplifies the notation.

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In the condition (3_1), the functionals $l_i : [C([a,b])]^{k_0} \to R$ $(i = 1,\dots,k_0)$ are linear and continuous.

For each index i $(i = 1,\dots,n)$, the functional $\varphi_i : C^{n-1}([a,b]) \to R$ in the conditions (2) is continuous and satisfies

$$(4_1) \qquad \xi_i(\rho) = \frac{1}{\rho} \sup\Big\{ |\varphi_i(\rho v)| \ : \ \|v\|_{C^{n-1}_{([a,b])}} \le 1 \Big\} \to 0 \quad \text{as} \quad \rho \to +\infty\,.$$

For each index i $(i = k_0+1,\dots,n)$, the functional $\varphi_i : C^{n-1-k_0}([a,b]) \to R$ in the conditions (3_2) is continuous and satisfies

$$(4_2) \qquad \delta_i(\rho) = \frac{1}{\rho} \sup\Big\{ |\varphi_i(\rho v)| \ : \ \|v\|_{C^{n-1-k_0}_{([a,b])}} \le 1 \Big\} \to 0 \quad \text{as} \quad \rho \to +\infty\,.$$

The special cases of boundary conditions (2) are

$$(5_1) \qquad u^{(i-1)}(t_i) = \varphi_i(u) \qquad i = 1,\dots,n\,,$$

where $a \le a_i \le t_i \le b_i \le b$ $(i = 1,\dots,n)$ or

$$(5_2) \qquad \int_{a_i}^{b_i} u^{(i-1)}(t)\, d\sigma_i(t) = \varphi_i(u) \qquad i = 1,\dots,n\,.$$

The integral is understood in the Lebesgue–Stieltjes sense, where σ_i is nondecreasing in $[a_i, b_i]$ and $\sigma(b_i) - \sigma(a_i) > 0$ $(i = 1,\dots,n)$. We know that the problem (1), (5_1) was studied by B. Půža in the paper [4], so in this paper we will receive more general results than in [4].

Problem (1), (3) was studied by Nguyen Anh Tuan in the paper [5] and by Gegelia G. T. in the paper [1]. In this paper, however, we will give new sufficient conditions for the existence of a solution of the problem (1), (3).

Main results

We adopt the following notation:

$[a,b]$ – a segment, $-\infty < a \le a_i \le b_i \le b < +\infty$ $(i = 1,\dots,n)$.

R^n – n-dimensional real space with elements $x = (x_i)_{i=1}^n$ normed by $\|x\| = \sum\limits_{i=1}^n |x_i|$.

$R^n_+ = \{x \in R^n \ : \ x_i \ge 0,\ i = 1,\dots,n\}$, $(0,+\infty) = R_+ - \{0\}$.

$C^{n-1}([a,b])$ – the space of functions continuous together with their derivatives up to the order $(n-1)$ on $[a,b]$ with the norm

$$\|u\|_{C^{n-1}_{([a,b])}} = \max\Big\{ \sum_{i=1}^n |u^{(i-1)}(t)| \ : \ a \le t \le b \Big\}\,.$$

$AC^{n-1}([a,b])$ – the set of all functions absolutely continuous together with their derivatives up to the order $(n-1)$ on $[a,b]$.

$L^p([a,b])$ – the space of functions Lebesgue integrable on $[a,b]$ in the p-th power with the norm

$$\|u\|_{L^p_{([a,b])}} = \begin{cases} \Big(\int\limits_a^b |u(t)|^p\, dt\Big)^{\frac{1}{p}} & \text{if} \quad 1 \le p < +\infty\,, \\ \text{ess sup}\big\{|u(t)| \ : \ a \le t \le b\big\} & \text{if} \quad p = +\infty\,. \end{cases}$$

$L^p([a,b],R_+) = \{u \in L^p([a,b]) \ : \ u(t) \ge 0 \text{ for a. a. } a \le t \le b\}$.

Let $x = (x_i(t))_{i=1}^n\,, y = (y_i(t))_{i=1}^n \in [C([a,b])]^n$. We will say that $x \le y$ if $x_i(t) \le y_i(t)$ for all $t \in [a,b]$ and $i = 1,\dots,n$.

A functional $\Phi : [C([a,b])]^n \to R$ is said to be nondecreasing if $\Phi(x) \le \Phi(y)$ for all $x,y \in [C([a,b])]^n$, $x \le y$, and positively homogeneous if $\Phi(\lambda x) = \lambda\Phi(x)$ for all $\lambda \in (0,+\infty)$ and $x \in [C([a,b])]^n$.

Let us consider the problems (1), (2) and (1), (3). Under a solution of the problem (1), (2), resp. (1), (3), we understand a function $u \in AC^{n-1}([a,b])$ which satisfies the equation (1) almost everywhere on $[a,b]$ and fulfils the boundary conditions (2), resp. (3).

Theorem 1. *Let the inequalities*

$$(6_1) \qquad f(t,x_1,x_2,\dots,x_n)\,\text{sign}\,x_n \le \omega\left(|x_n|\right) \sum_{i=1}^{n-1}\sum_{j=1}^{m} g_{ij}(t) h_{ij}\left(x_i\right) |x_{i+1}|^{\frac{1}{q_{ij}}} \quad \textit{for} \quad t \in [a_n,b], (x_i)_{i=1}^n \in R^n$$

$$(6_2) \qquad f(t,x_1,x_2,\dots,x_n)\,\text{sign}\,x_n \ge -\omega\left(|x_n|\right) \sum_{i=1}^{n-1}\sum_{j=1}^{m} g_{ij}(t) h_{ij}\left(x_i\right) |x_{i+1}|^{\frac{1}{q_{ij}}} \quad \textit{for} \quad t \in [a,b_n], (x_i)_{i=1}^n \in R^n$$

hold, where $g_{ij} \in L^{p_{ij}}([a,b],R_+)$, $p_{ij}, q_{ij} \ge 1$, $1/p_{ij}+1/q_{ij} = 1$ $(i = 1,\dots,n-1; j = 1,\dots,m)$, $\omega : R_+ \to (0,+\infty)$ *and* $h_{ij} : R \to R_+$ $(i = 1,\dots,n-1; j = 1,\dots m)$ *are continuous nondecreasing functions satisfying*

$$(7) \qquad \Omega(\rho) = \int\limits_0^\rho \frac{ds}{\omega(s)} \to +\infty \quad \textit{as} \quad \rho \to +\infty$$

and

$$(8) \qquad \lim_{\rho\to+\infty} \frac{\Omega(\rho\xi_n(\rho))}{\Omega(\rho)} = 0 = \lim_{\rho\to+\infty} \frac{\|h_{ij}\|_{L^{q_{ij}}_{([-\rho,\rho])}}}{\Omega(\rho)} \qquad i = 1,\dots,n-1\,; \ j = 1,\dots,m\,.$$

Then the problem (1), (2) *has at least one solution.*

To prove Theorem 1 we need the following

Lemma 1. *Let the functions* ω, Ω, g_{ij}, h_{ij} *and the numbers* p_{ij}, q_{ij} $(i = 1,\dots,n-1; j = 1,\dots,m)$ *be given as in Theorem 1, and let* $\eta_i : R_+ \to R_+$ $(i = 1,\dots,n)$ *be* *nondecreasing functions satisfying*

$$\lim_{\rho\to+\infty}\frac{\Omega(\eta_n(\rho))}{\Omega(\rho)}=0=\lim_{\rho\to+\infty}\frac{\eta_i(\rho)}{\rho}\qquad i=1,\dots,n. \tag{9}$$

Then there exists a constant $\rho_0>0$ such that the estimate

$$\|u\|_{C^{n-1}_{([a,b])}}\le\rho_0 \tag{10}$$

holds for each solution $u\in AC^{n-1}([a,b])$ of the differential inequalities

$$\begin{aligned}(11_1)\quad & u^{(n)}(t)\operatorname{sign}u^{(n-1)}(t)\\ &\le\omega(|u^{(n-1)}(t)|)\sum_{i=1}^{n-1}\sum_{j=1}^{m}g_{ij}(t)h_{ij}\big(u^{(i-1)}(t)\big)|u^{(i)}(t)|^{\frac{1}{q_{ij}}}\\ &\qquad\qquad\textit{for}\quad t\in[a_n,b]\end{aligned}$$

$$\begin{aligned}(11_2)\quad & u^{(n)}(t)\operatorname{sign}u^{(n-1)}(t)\\ &\ge-\omega\big(|u^{(n-1)}(t)|\big)\sum_{i=1}^{n-1}\sum_{j=1}^{m}g_{ij}(t)h_{ij}\big(u^{(i-1)}(t)\big)|u^{(i)}(t)|^{\frac{1}{q_{ij}}}\\ &\qquad\qquad\textit{for}\quad t\in[a,b_n]\end{aligned}$$

with the boundary condition

$$\min\big\{|u^{(i-1)}(t)|\ :\ a_i\le t\le b_i\big\}\le\eta_i\big(\,\|u\|_{C^{n-1}_{([a,b])}}\big)\qquad i=1,\dots,n\,. \tag{12}$$

Proof. Put

$$\mu=\sum_{i=1}^{n}(b-a)^{n-i}\quad\text{and}\quad\varepsilon=[2\mu(n-1)]^{-1}\,.$$

Then according to (9) there exists a number $r_0>0$ such that

$$\eta_i(\rho)\le\varepsilon\rho\quad\text{for}\quad\rho>r_0\qquad i=1,\dots,n\,. \tag{13}$$

We suppose that the estimate (10) does not hold. Then for arbitrary $\rho_1\ge r_0$ there exists a solution u of the problem (11), (12) such that

$$\|u\|_{C^{n-1}_{([a,b])}}>\rho_1\,. \tag{14}$$

We put

$$\rho=\max\big\{|u^{(n-1)}(t)|\ :\ a\le t\le b\big\} \tag{15}$$

and choose $\tau_i\in[a_i,b_i]$ $(i=1,\dots,n)$ such that

$$|u^{(i-1)}(\tau_i)|=\min\big\{|u^{(i-1)}(t)|\ :\ a_i\le t\le b_i\big\}\,.$$

Then from (12) we have

$$|u^{(i-1)}(\tau_i)|\le\eta_i\big(\,\|u\|_{C^{n-1}_{([a,b])}}\big)\qquad i=1,\dots,n\,. \tag{16}$$

Using (15), (16) we have

$$
\begin{aligned}
|u^{(n-2)}(t)| &\le \Big| \int\limits_{\tau_{n-1}}^{t} |u^{(n-1)}(\tau)|\, d\tau \Big| + |u^{(n-2)}(\tau_{n-1})| \\
&\le (b-a)\rho + \eta_{n-1}\big(\|u\|_{C^{n-1}_{([a,b])}} \big) \quad \text{for} \quad t \in [a,b]\,.
\end{aligned} \tag{17}
$$

Integrating $u^{(n-2)}$ from τ_{n-2} to t and using (16) and (17) again we get

$$
\begin{aligned}
|u^{(n-3)}(t)| &\le \Big| \int\limits_{\tau_{n-2}}^{t} |u^{(n-2)}(\tau)|\, d\tau \Big| + |u^{(n-3)}(\tau_{n-2})| \\
&\le (b-a)^2\rho + (b-a)\eta_{n-1}\big(\|u\|_{C^{n-1}_{([a,b])}} \big) + \eta_{n-2}\big(\|u\|_{C^{n-1}_{([a,b])}} \big)
\end{aligned}
$$

for $t \in [a,b]$. Applying this procedure $(n-1)$-times we obtain

$$
\|u\|_{C^{n-1}_{([a,b])}} \le \mu\Big(\rho + \sum_{i=1}^{n-1} \eta_i\big(\|u\|_{C^{n-1}_{([a,b])}} \big) \Big)\,.
$$

Using (13) and (14) we get

$$
\|u\|_{C^{n-1}_{([a,b])}} \le \mu\Big(\rho + (n-1)\varepsilon\, \|u\|_{C^{n-1}_{([a,b])}} \Big) = \mu\rho + \frac{1}{2}\, \|u\|_{C^{n-1}_{([a,b])}}\ .
$$

Therefore we have

$$
\|u\|_{C^{n-1}_{([a,b])}} \le 2\mu\rho. \tag{18}
$$

We choose a point $\tau^* \in [a,b]$ such that $\tau^* \ne \tau_n$ and

$$
|u^{(n-1)}(\tau^*)| = \max\big\{ |u^{(n-1)}(t)| \ :\ a \le t \le b \big\}\,.
$$

Then either $\tau_n < \tau^*$ or $\tau^* < \tau_n$.

If $\tau_n < \tau^*$, then the integration of (11_1) from τ_n to τ^*, in view of (18) and using Hölder's inequality, we get

$$
\begin{aligned}
\int\limits_{\tau_n}^{\tau^*} \frac{u^{(n)}(t)\ \operatorname{sign} u^{(n-1)}(t)\ dt}{\omega\big(|u^{(n-1)}(t)|\big)} &\le \int\limits_{\tau_n}^{\tau^*} \sum_{i=1}^{n-1}\sum_{j=1}^{m} g_{ij}(t) h_{ij}\big(u^{(i-1)}(t)\big) |u^{(i)}(t)|^{\frac{1}{q_{ij}}}\, dt \\
&\le \sum_{i=1}^{n-1}\sum_{j=1}^{m} \|g_{ij}\|_{L^{p_{ij}}_{([a,b])}}\ \|h_{ij}\|_{L^{q_{ij}}_{([-2\mu\rho,2\mu\rho])}}\ .
\end{aligned} \tag{19}
$$

Applying (15), (16), (18), and the definition of Ω in (19), we get

$$
\Omega(\rho) \le \Omega(\eta_n(2\mu\rho)) + \sum_{i=1}^{n-1}\sum_{j=1}^{m} \|g_{ij}\|_{L^{p_{ij}}_{([a,b])}}\ \|h_{ij}\|_{L^{q_{ij}}_{([-2\mu\rho,2\mu\rho])}}\ .
$$

Now, in view of (8), (9), (14), and (18), since ρ_1 was chosen arbitrarily, we get

$$
\lim_{\rho\to+\infty} \frac{\Omega(\rho)}{\Omega(2\mu\rho)} = 0\,.
$$

On the other hand, in view of (7) and the facts that $2\mu > 1$ and ω is a nondecreasing function, we have

$$\liminf_{\rho \to +\infty} \frac{\Omega(\rho)}{\Omega(2\mu\rho)} > 0\,,$$

a contradiction.

If $\tau^* < \tau_n$, then the integration of (11_2) from τ^* to τ_n yields the same contradiction in analogous way. $\square$

Proof of Theorem 1. Let ρ_0 be the constant from Lemma 1. Put

$$\chi(s) = \begin{cases} 1 & \text{if} \quad |s| \le \rho_0 \\ 2 - \dfrac{|s|}{\rho_0} & \text{if} \quad \rho_0 < |s| < 2\rho_0, \\ 0 & \text{if} \quad |s| \ge 2\rho_0 \end{cases}$$

$$(20) \qquad \begin{aligned} \widetilde{f}(t, x_1, \dots, x_n) &= \chi\left(\|x\|\right) f(t, x_1, \dots, x_n) \quad \text{for} \quad a \le t \le b, \quad (x_i)_{i=1}^n \in R^n\,, \\ \widetilde{\varphi}_i(u) &= \chi\big(\|u\|_{C^{n-1}_{([a,b])}}\big)\varphi_i(u) \quad \text{for} \quad u \in C^{n-1}([a,b]) \quad i = 1, \dots, n \end{aligned}$$

and consider the problem

$$(21) \qquad u^{(n)}(t) = \widetilde{f}\big(t, u(t), \dots, u^{(n-1)}(t)\big)\,,$$

$$(22) \qquad \Phi_{0i}\big(u^{(i-1)}\big) = \widetilde{\varphi}_i(u) \qquad i = 1, \dots, n\,.$$

From (20) it immediately follows that $\widetilde{f} : [a,b] \times R^n \to R$ satisfies the local Carathéodory conditions, $\widetilde{\varphi}_i : C^{n-1}([a,b]) \to R$ $(i = 1, \dots, n)$ are continuous functionals and

$$(23_1) \qquad \sup\big\{|\widetilde{f}(\cdot, x_1, \dots, x_n)| \ :\ (x_i)_{i=1}^n \in R^n\big\} \in L([a,b])\,,$$

$$(23_2) \qquad \sup\big\{|\widetilde{\varphi}_i(u)| \ :\ u \in C^{n-1}([a,b])\big\} < +\infty \qquad i = 1, \dots, n\,.$$

Now we will show that the homogeneous problem

$$(21_0) \qquad v^{(n)}(t) = 0\,,$$

$$(22_0) \qquad \Phi_{0i}\big(v^{(i-1)}\big) = 0 \qquad i = 1, \dots, n$$

has only the trivial solution.

Let v be an arbitrary solution of this problem. Integrating (21_0) we get

$$v^{(n-1)}(t) = \text{const} \quad \text{for} \quad a \le t \le b\,.$$

According to (22_0) we have

$$v^{(n-1)}(a)\Phi_{0n}(1) = 0\,.$$

However, since $\Phi_{0n}(1) = 1$, we have $v^{(n-1)}(t) = 0$ for $a \le t \le b$. Referring to (22_0) and $\Phi_{0i}(1) = 1$ $(i = 1, \dots, n-1)$, we come to the conclusion that $v(t) \equiv 0$. Using Theorem 2.1 from [3], in view of (23) and the uniqueness of the trivial solution of the problem (21_0), (22_0), we get the existence of a solution of the problem (21), (22).

Let u be a solution of the problem (21), (22). Then, using (6), we get

$$\begin{aligned}u^{(n)}(t)\operatorname{sign}u^{(n-1)}(t) &= \widetilde{f}(t,u(t),\dots,u^{(n-1)}(t))\operatorname{sign}u^{(n-1)}(t)\\ &= \chi\Big(\sum_{j=1}^{n}\big|u^{(j-1)}(t)\big|\Big)f(t,u(t),\dots,u^{(n-1)}(t))\operatorname{sign}u^{(n-1)}(t)\\ &\le \omega\big(\big|u^{(n-1)}(t)\big|\big)\sum_{i=1}^{n-1}\sum_{j=1}^{m}g_{ij}(t)h_{ij}(u^{(i-1)}(t))|u^{(i)}(t)|^{\frac{1}{q_{ij}}}\end{aligned}$$

for $t\in[a_n,b]$, and

$$\begin{aligned}u^{(n)}(t)\operatorname{sign}u^{(n-1)}(t) &= \widetilde{f}(t,u(t),\dots,u^{(n-1)}(t))\operatorname{sign}u^{(n-1)}(t)\\ &= \chi\Big(\sum_{j=1}^{n}\big|u^{(j-1)}(t)\big|\Big)f(t,u(t),\dots,u^{(n-1)}(t))\operatorname{sign}u^{(n-1)}(t)\\ &\ge -\omega\big(\big|u^{(n-1)}(t)\big|\big)\sum_{i=1}^{n-1}\sum_{j=1}^{m}g_{ij}(t)h_{ij}(u^{(i-1)}(t))|u^{(i)}(t)|^{\frac{1}{q_{ij}}}\end{aligned}$$

for $t\in[a,b_n]$. Put

$$\eta_i(\rho)=\sup\big\{\big|\widetilde{\varphi}_i(v)\big| \;:\; \|v\|_{C^{n-1}_{([a,b])}}\le\rho\big\}\qquad i=1,\dots,n\,.$$

From (4_1) and (8), it immediately follows that the functions η_i $(i=1,\dots,n)$ satisfy (9) and

$$\begin{aligned}\min\big\{\big|u^{(i-1)}(t)\big| \;:\; a_i\le t\le b_i\big\} &= \Phi_{0i}\big(\min\big\{\big|u^{(i-1)}(t)\big| \;:\; a_i\le t\le b_i\big\}\big)\\ &\le \big|\Phi_{0i}\big(u^{(i-1)}\big)\big| = \big|\widetilde{\varphi}_i(u)\big| \le \eta_i\big(\|u\|_{C^{n-1}_{([a,b])}}\big)\\ &\qquad i=1,\dots,n\,.\end{aligned}$$

Therefore, by Lemma 1 we get

$$\|u\|_{C^{n-1}_{([a,b])}}\le\rho_0\,.$$

Consequently,

$$\chi\Big(\sum_{i=1}^{n}|u^{(i-1)}(t)|\Big)=1\qquad\text{for}\;\; a\le t\le b$$

and

$$\chi\big(\|u\|_{C^{n-1}_{([a,b])}}\big)=1\,.$$

Using these equalities in (20), we obtain that u is a solution of the problem (1), (2). □

Remark 1. If $\Phi_{0i}(u^{(i-1)})=u^{(i-1)}(t_i)$, $a\le a_i\le t_i\le b_i\le b$ $(i=1,\dots,n)$, then Theorem 1 is Theorem in [4].

Now we give new sufficient conditions guaranteeing the existence of a solution of the problem (1), (3) provided that the equation

$$u^{(k_0)} = 0 \tag{24}$$

with the boundary conditions (3_1) has only the trivial solution.

Theorem 2. *Let the problem* (24), (3_1) *have only the trivial solution and let the inequalities*

$$f(t,x_1,\dots,x_n)\operatorname{sign} x_n \le \omega\left(|x_n|\right)\sum_{i=k_0+1}^{n-1}\sum_{j=1}^{m} g_{ij}(t)h_{ij}(x_i)|x_{i+1}|^{\frac{1}{q_{ij}}} \quad \text{for} \quad t\in[a_n,b],\ (x_i)_{i=1}^n\in R^n \tag{25_1}$$

$$f(t,x_1,\dots,x_n)\operatorname{sign} x_n \ge -\omega\left(|x_n|\right)\sum_{i=k_0+1}^{n-1}\sum_{j=1}^{m} g_{ij}(t)h_{ij}(x_i)|x_{i+1}|^{\frac{1}{q_{ij}}} \quad \text{for} \quad t\in[a,b_n],\ (x_i)_{i=1}^n\in R^n \tag{25_2}$$

hold, where $g_{ij}\in L^{p_{ij}}([a,b],R_+)$, $p_{ij},q_{ij}\ge 1$, $1/p_{ij}+1/q_{ij}=1$ $(i=k_0+1,\dots,n-1;j=1,\dots,m)$, $\omega : R_+\to(0,+\infty)$ *and* $h_{ij} : R\to R_+$ $(i=k_0+1,\dots,n-1;j=1,\dots,m)$ *are continuous nondecreasing functions satisfying* (7) *and*

$$\begin{aligned}&\lim_{\rho\to+\infty}\frac{\Omega(\rho\delta_n(\rho))}{\Omega(\rho)}=0\\ &\lim_{\rho\to+\infty}\frac{\|h_{ij}\|_{L^{q_{ij}}_{([-\rho,\rho])}}}{\Omega(\rho)}=0 \qquad i=k_0+1,\dots,n-1;j=1,\dots,m\,.\end{aligned}\tag{26}$$

Then the problem (1), (3) *has at least one solution.*

To prove Theorem 2 we need the following

Lemma 2. *Let the problem* (24), (3_1) *have only the trivial solution and let the functions* ω, Ω, g_{ij}, h_{ij} *and the numbers* p_{ij}, q_{ij} $(i=k_0+1,\dots,n-1;$ $j=1,\dots,m)$ *be given as in Theorem 2, and let* $\eta_i : R_+\to R_+$ $(i=k_0+1,\dots,n)$ *be nondecreasing functions satisfying*

$$\lim_{\rho\to+\infty}\frac{\Omega(\eta_n(\rho))}{\Omega(\rho)}=0=\lim_{\rho\to+\infty}\frac{\eta_i(\rho)}{\rho} \qquad i=k_0+1,\dots,n\,.$$

Then there exists a constant $\rho_0>0$ *such that the estimate* (10) *holds for each solution* $u\in AC^{n-1}([a,b])$ *of the differential inequalities*

$$u^{(n)}(t)\operatorname{sign} u^{(n-1)}(t)\le\omega\big(|u^{(n-1)}(t)|\big)\sum_{i=k_0+1}^{n-1}\sum_{j=1}^{m}g_{ij}(t)h_{ij}\big(u^{(i-1)}(t)\big)|u^{(i)}(t)|^{\frac{1}{q_{ij}}} \quad \text{for} \quad t\in[a_n,b] \tag{27_1}$$

$$u^{(n)}(t)\operatorname{sign}u^{(n-1)}(t)\ge -\omega\big(|u^{(n-1)}(t)|\big)\sum_{i=k_0+1}^{n-1}\sum_{j=1}^{m}g_{ij}(t)h_{ij}\big(u^{(i-1)}(t)\big)\Big|u^{(i)}(t)\Big|^{\frac{1}{q_{ij}}}$$

$$\text{(27}_2\text{)}\qquad \textit{for}\quad t\in[a,b_n]$$

with the boundary conditions (3_1) *and*

$$\text{(28)}\quad \min\big\{|u^{(i-1)}(t)| : a_i\le t\le b_i\big\}\le \eta_i\big(\|u^{(k_0)}\|_{C^{n-k_0-1}_{([a,b])}}\big)\qquad i=k_0+1,\dots,n\,.$$

Proof. Let u be an arbitrary solution of the problem (27), (3_1), (28). Put

$$\text{(29)}\qquad v(t)=u^{(k_0)}(t)\,.$$

Then the formulas (27) and (28) imply that

$$\begin{aligned}
v^{(n-k_0)}(t)\operatorname{sign}v^{(n-k_0-1)}(t)&\le \omega\big(|v^{(n-k_0-1)}(t)|\big)\\
&\quad\times\sum_{i=1}^{n-k_0-1}\sum_{j=1}^{m}g_{ij}(t)h_{ij}\big(v^{(i-1)}(t)\big)|v^{(i)}(t)|^{\frac{1}{q_{ij}}}\\
&\quad\text{for}\quad t\in[a_n,b]\,,\\
v^{(n-k_0)}(t)\operatorname{sign}v^{(n-k_0-1)}(t)&\ge -\omega\big(|v^{(n-k_0-1)}(t)|\big)\\
&\quad\times\sum_{i=1}^{n-k_0-1}\sum_{j=1}^{m}g_{ij}(t)h_{ij}\big(v^{(i-1)}(t)\big)|v^{(i)}(t)|^{\frac{1}{q_{ij}}}\\
&\quad\text{for}\quad t\in[a,b_n]\,,
\end{aligned}$$

and

$$\min\big\{|v^{(i-1)}(t)| : a_i\le t\le b_i\big\}\le \eta_i\big(\|v\|_{C^{n-k_0-1}_{([a,b])}}\big)\qquad i=1,\dots,n-k_0\,.$$

Consequently, according to Lemma 1 there exists $\rho_1>0$ such that

$$\text{(30)}\qquad \|v\|_{C^{n-k_0-1}_{([a,b])}}\le\rho_1\,.$$

By virtue of the assumption that the problem (24), (3_1) has only the trivial solution, there exists a Green function $G(t,s)$ such that

$$\text{(31)}\qquad u^{(i-1)}(t)=\int_a^b\frac{\partial^{i-1}G(t,s)}{\partial t^{i-1}}v(s)\,ds\quad\text{for}\quad t\in[a,b]\qquad i=1,\dots,k_0$$

(see e.g., [2]).

Put

$$\rho_2=\max_{a\le t\le b}\int_a^b\sum_{i=1}^{k_0}\Big|\frac{\partial^{i-1}G(t,s)}{\partial t^{i-1}}\Big|\,ds\,.$$

According to (30) and (31) we have

$$\|u\|_{C^{k_0-1}_{([a,b])}}\le\rho_1\rho_2\,.$$

Therefore we obtain (10), where $\rho_0=\rho_1+\rho_2\rho_1$. □

Theorem 2 can be proved analogously to Theorem 1 using Lemma 2.

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