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GENERALIZATIONS OF THE FAN-BROWDER FIXED POINT THEOREM AND MINIMAX INEQUALITIES

MIRCEA BALAJ AND SORIN MURESAN

Abstract. In this paper fixed point theorems for maps with nonempty convex values and having the local intersection property are given. As applications several minimax inequalities are obtained.

1. Introduction

A *map* (or a *multifunction*) $T : X \multimap Y$ is a function from a set X into the power set 2^Y of Y, that is a function with the values $T(x) \subset Y$. For $y \in Y$, $T^{-1}(y)$ is called the *fiber* of T on y.

Using an infinite dimensional version of the Knaster-Kuratowski-Mazurkiewicz theorem, Fan [10] proved in 1961 the following:

Theorem 0. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and M be a closed subset of $X \times X$ such that*:
(i) $(x, x) \in M$ *for all* $x \in X$;
(ii) *for each* $y \in X$ *the set* $\{x \in X : (x, y) \notin X\}$ *is convex* (*or empty*).
Then $X \times \{y_0\} \subset M$ *for some* $y_0 \in X$.

Subsequently, Browder [4] obtained in 1968 the following fixed point theorem:

Theorem 1. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $T : X \multimap X$ be a map with nonempty convex values and open fibers. Then T has a fixed point.*

Browder's proof for his theorem was based on the existence of a partition of unity for open coverings of compact sets and on the Brouwer fixed point theorem. Let us observe that Browder's theorem is just Theorem 0 reformulated in a more convenient form (to see this, take $T(x) = \{y \in X : (x, y) \notin M\}$). For this reason Theorem 1 is known in the literature as the Fan-Browder fixed point theorem.

The existence of many significant applications in nonlinear functional analysis, game theory and economic theory gave rise to a number of generalizations or

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versions of Theorem 1 (see [1], [2], [3], [6], [7], [16], [17], [19]). In Section 2 we give new generalizations of Theorem 1 involving maps with the local intersection property. Two well-known applications of the Fan-Browder fixed point theorem will be considered in this paper. The first one is the following Fan's minimax inequality [12]

Theorem 2. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $f : X \times X \to \mathbb{R}$ be a function quasiconvex in y and upper semicontinuous in x. Then*

$$\inf_{x\in X} f(x,x) \leq \max_{x\in X} \inf_{y\in X} f(x,y) .$$

The second application is a two-function minimax inequality due also to Fan [11] which generalizes the celebrate Sion's minimax theorem [18]. We state this result as follows

Theorem 3. *Let X, Y be nonempty compact convex subsets of topological vector spaces and $f, g : X \times Y \to \mathbb{R}$. Suppose that f is lower semicontinuous in y and quasiconcave in x, g is upper semicontinuous in x and quasiconvex in y, and $f \leq g$ on $X \times Y$. Then*

$$\min_{y\in Y} \sup_{x\in X} f(x,y) \leq \sup_{x\in X} \inf_{y\in Y} g(x,y) .$$

Note that "quasiconvex" and further notions will be explained in the last section of the paper. In the same section, from each fixed point theorem established in Section 2 we derive a Fan type minimax inequality and a Fan-Sion type minimax theorem. Throughout this paper we assume that the topological vector spaces are separated.

2. Local intersection property and fixed point theorems

Let X be a topological space and Y be a set. A map $T : X \multimap Y$ is said to have the *local intersection property* (see[20]) if for each $x \in X$ with $T(x) \neq \emptyset$ there exists an open neighbourhood $V(x)$ of x such that $\bigcap_{z\in V(x)} T(z) \neq \emptyset$. It is not hard to see that each map with open fibers has the local intersection property but the example given in [20, p.63], shows that the converse is not true.

The following lemma is useful in what follows and can be found in [9].

Lemma 4. *Let X be a topological space, Y be a set and $T : X \multimap Y$ be a map with nonempty values. Then the following assertions are equivalent*

(i) *T has the local intersection property;*

(ii) *There exists a map $F : X \multimap Y$ such that $F(x) \subset T(x)$ for each $x \in X$, $F^{-1}(y)$ is open for each $y \in Y$ and $X = \bigcup_{y\in Y} F^{-1}(y)$.*

Theorem 5. *Let X be a topological space, Y be a convex subset of a topological vector space and $T : X \multimap Y$ be a map with nonempty convex values and having the local intersection property. Then T admits a selection G (i.e. $G(x) \subset T(x)$ for all $x \in X$) with nonempty convex values and open fibers.*

Proof. By Lemma 4, T admits a selection F with open fibers such that

$$X = \bigcup_{y \in Y} F^{-1}(y) \,. \tag{1}$$

From (1) we infer that $F(x) \neq \emptyset$ for all $x \in X$. Define the map $G : X \multimap Y$, by $G(x) = coF(x)$. Since T has convex values, $G(x) \subset T(x)$ and $G(x)$ is convex for each $x \in X$. Since F has open fibers, by Lemma 5.1 in [21], it follows that G has also open fibers. □

The first generalization of the Fan-Browder fixed point theorem is the following

Theorem 6. *Let X be a compact convex subset of a topological vector space and $T : X \multimap X$ be a map with nonempty convex values having the local intersection property. Then T has a fixed point.*

Proof. By Theorem 5, T has a selection G with nonempty convex values and open fibers, and Theorem 1 guarantees the existence of a point $x_0 \in X$ such that $x_0 \in G(x_0) \subset T(x_0)$. □

Theorem 7. *Let X be a compact convex subset of a topological vector space and Y a nonempty set. Suppose that $F : X \multimap Y, T : X \multimap X$ are two maps satisfying the following conditions*

(i) *T takes convex values;*

(ii) *F has nonempty values and open fibers;*

(iii) *for each $y \in Y$ there exists $z \in X$ such that $F^{-1}(y) \subset T^{-1}(z)$.*

Then T has a fixed point.

Proof. Since F has nonempty values, $\bigcup_{y \in Y} F^{-1}(y) = X$, and from (iii) we get $\bigcup_{z \in X} T^{-1}(z) = X$, hence T has also nonempty values. According to Theorem 6 it suffices to show that T has the local intersection property. Let $x \in X$. Since $F(x) \neq \emptyset$ there exist $y \in Y$ and $z \in X$ such that

$$x \in F^{-1}(y) \subset T^{-1}(z) \,. \tag{2}$$

Then $F^{-1}(y)$ is an open neighbourhood of x and, by (2), it follows that $z \in \bigcap_{x' \in F^{-1}(y)} T(x')$. Thus the proof is complete. □

The following result extends the Fan-Browder fixed point theorem to the case when the convex set X is not compact.

Theorem 8. *Let X be a convex subset of a topological vector space and $T : X \multimap X$ be a map with nonempty convex values, having the local intersection property. Suppose that there exist a nonempty compact convex subset X_0 of X and a compact subset K of X satisfying the following condition*

for each $x \in X \backslash K$ there exists an open neighbourhood $V(x)$ of x such that

$$\bigcap_{z \in V(x)} T(z) \cap X_0 \neq \emptyset \,. \tag{3}$$

Then T has a fixed point.

Proof. Define the maps $H, G : X \multimap X$ by

$$H(y) = \operatorname{int}\left(T^{-1}(y)\right) \quad \text{for} \quad y \in X$$

and

$$G(x) = \operatorname{co} H^{-1}(x) \quad \text{for} \quad x \in X .$$

We see that H takes open values and $H(y) \subset T^{-1}(y)$ for each $y \in X$. Since the values of T are convex, $G(x) \subset T(x)$ for all $x \in X$. Using once again Lemma 5.1 in [21] we infer that G has open fibers. For an arbitrary $x \in X$, since T has the local intersection property, there exist a neighbourhood $V(x)$ of x and a point y such that

$$x \in V(x) \subset T^{-1}(y) \quad \text{whence} \quad x \in H(y) \subset G^{-1}(y) .$$

Consequently, G has nonempty values and

$$X = G^{-1}(X) . \tag{4}$$

For each $x \in X \backslash K$, by (3), there exists $y \in X_0$ such that $x \in H(y) \subset G^{-1}(y)$, hence

$$X \backslash K = G^{-1}(X_0) . \tag{5}$$

On the other hand, by (4), $K \subset G^{-1}(X)$ and, since K is compact, there exists a finite set $A \subset X$ such that

$$K \subset G^{-1}(A) . \tag{6}$$

Thus, by (5) and (6), we have $X = G^{-1}(X_0 \cup A)$.
Let $C = \operatorname{co}(X_0 \cup A)$. Then C is a compact, convex subset of X and

$$C \subset G^{-1}(X_0 \cup A) \subset G^{-1}(C) . \tag{7}$$

Define the map $\widetilde{G} : C \to C$ by $\widetilde{G}(x) = G(x) \bigcap C$. Then the values of $\widetilde{G}$ are nonempty (by (7)) and convex. Since $\widetilde{G}^{-1}(y) = G^{-1}(y) \bigcap C$ for each $y \in C$, the fibers of $\widetilde{G}$ are open in C. Applying Theorem 1 to the map $\widetilde{G}$ we find a point $x_0 \in C$ such that $x_0 \in \widetilde{G}(x_0) \subset T(x_0)$. □

Remark. The local intersection property imposed on T and condition (3) can be unified in the following condition

$$\text{the map } \widetilde{T} : X \multimap X, \text{ defined by } \widetilde{T}(x) = \begin{cases} T(x) & \text{for} \quad x \in K \\ T(x) \cap X_0 & \text{for} \quad x \in X \backslash K \end{cases}$$

has the local intersection property.

In our opinion it is worth comparing Theorem 8 with other noncompact generalizations of the Fan-Browder fixed point theorem due to Browder [4], Lassonde [15], Mehta [16] and Park [17].

3. Minimax inequalities

Let X, Y nonempty convex subsets of topological vector spaces. Recall that a function $f : X \times Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is said to be:

(i) *quasiconcave* (resp. *upper semicontinuous*) *in* x if for each $y \in Y$ and $\lambda \in \mathbb{R}$ the set $\{x \in X : f(x,y) \geq \lambda\}$ is convex (resp. closed);

(ii) *quasiconvex* (resp. *lower semicontinuous*) *in* y if for each $x \in X$ and $\lambda \in \mathbb{R}$ the set $\{y \in Y : f(x,y) \leq \lambda\}$ is convex (resp. closed).

A function $f : X \times Y \to \overline{\mathbb{R}}$ (X, Y topological spaces) is said to be:

(iii) *transfer upper semicontinuous* in x (see [8]) if, for each $\lambda \in \mathbb{R}$ and all $x \in X$, $y \in Y$ with $f(x,y) < \lambda$, there exist a neighbourhood $V(x)$ of x and a point $y' \in Y$ such that $f(z,y') < \lambda$, for all $z \in V(x)$;

(iv) *transfer lower semicontinous* in y (see [8]) if, for each $\lambda \in \mathbb{R}$ and all $x \in X$, $y \in Y$ with $f(x,y) > \lambda$, there exist a neighbourhood $V(y)$ of y and a point $x' \in X$ such that $f(x',u) > \lambda$, for all $u \in V(y)$.

It is clear that every function which is upper semicontinuous in x (resp. lower semicontinuous in y) is transfer upper semicontinous in x (resp. transfer lower semicontinuous in y) but the converse is not true (see [8]).
From each fixed point theorem obtained in the previous section we shall derive a Fan type minimax inequality and a Fan-Sion type minimax theorem.

Theorem 9. *Let X be a nonempty compact convex subset of a topological vector space and $f : X \times X \to \overline{\mathbb{R}}$ be a function quasiconvex in y and transfer upper semicontinuous in x. Then*

$$\inf_{x \in X} f(x,x) \leq \sup_{x \in X} \inf_{y \in X} f(x,y) .$$

Proof. We may assume that $\sup\limits_{x \in X} \inf\limits_{y \in X} f(x,y) < \infty$. Let $\lambda > \sup\limits_{x \in X} \inf\limits_{y \in X} f(x,y)$ be arbitrarily fixed; we define the map $T : X \multimap X$ by

$$T(x) = \{y \in X : f(x,y) < \lambda\} .$$

From $\lambda > \sup\limits_{x \in X} \inf\limits_{y \in X} f(x,y)$ it follows that $T(x)$ is nonempty for each $x \in X$. Since f is quasiconvex in y, the values of T are convex; since f is transfer upper semicontinuous in x, T has the local intersection property. By Theorem 6 there exists a point $x_0 \in X$ such that $x_0 \in T(x_0)$. Hence $\inf\limits_{x \in X} f(x,x) \leq f(x_0,x_0) \leq \lambda$, which proves the theorem. □

Theorem 10. *Let X and Y be nonempty compact convex subsets of topological vector spaces and $f, g : X \times Y \to \overline{\mathbb{R}}$ be two functions satisfying the following conditions:*

(i) $f \leq g$;

(ii) *f is quasiconcave in x;*

(iii) *f is transfer lower semicontinuous in y*

(iv) *g is quasiconvex in y;*

(v) *g is transfer upper semicontinuous in x.*

Then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \leq \sup_{x\in X}\inf_{y\in Y} g(x,y) .$$

Proof. Suppose that there exists a real λ such that

$$\sup_{x\in X}\inf_{y\in Y} g(x,y) < \lambda < \inf_{y\in Y}\sup_{x\in X} f(x,y) .$$

Define the map $T : X \times Y \to X \times Y$ by

$$T(x,y) = \{x' \in X : f(x',y) > \lambda\} \times \{y' \in Y : g(x,y') < \lambda\} .$$

Then $T(x,y)$ is nonempty and convex (by (ii) and (iv)) for each $(x,y) \in X \times Y$. By (iii) and (v) one can easily prove that T has the local intersection property. Applying Theorem 6 we get a fixed point $(x_0,y_0) \in T(x_0,y_0)$. Therefore $\lambda < f(x_0,y_0) \leq g(x_0,y_0) < \lambda$, a contradiction. □

Theorem 11. *Let X be a compact convex subset of a topological vector space and Y be a nonempty set. Suppose that $f : X \times X \to \overline{\mathbb{R}}$, $g : X \times Y \to \overline{\mathbb{R}}$ are two functions satisfying the following conditions:*

(i) *f is quasiconvex in the second variable;*

(ii) *g is upper semicontinuous in x;*

(iii) *for each $y \in Y$ there exists $z \in X$ such that $f(\cdot,z) \leq g(\cdot,y)$.*

Then

$$\inf_{x\in X} f(x,x) \leq \sup_{x\in X}\inf_{y\in Y} g(x,y) .$$

Proof. We may assume that $\sup_{x\in X}\inf_{y\in Y} g(x,y) < \infty$. Let $\lambda > \sup_{x\in X}\inf_{y\in Y} g(x,y)$ be arbitrarily fixed; we define the maps $T : X \multimap X$, $F : X \multimap Y$, by

$$T(x) = \{z \in X : f(x,z) < \lambda\}$$

and

$$F(x) = \{y \in Y : g(x,y) < \lambda\} .$$

Since $\lambda > \sup_{x\in X}\inf_{y\in Y} g(x,y)$, $F(x)$ is nonempty for each $x \in X$. It is easy to prove that conditions (i), (ii), (iii) in our theorem imply the conditions similarly denoted in Theorem 7. By Theorem 7, T has a fixed point x_0. It follows that $\inf_{x\in X} f(x,x) \leq f(x_0,x_0) < \lambda$ and the proof is complete. □

Theorem 12. *Let X_1, Y_1 be nonempty compact convex subsets of topological vector spaces and X_2, Y_2 be nonempty sets. Let $f : X_2 \times Y_1 \to \overline{\mathbb{R}}$, $g : X_1 \times Y_2 \to \overline{\mathbb{R}}$, $h,k : X_1 \times Y_1 \to \overline{\mathbb{R}}$ be four functions satisfying:*

(i) *$h \leq k$;*

(ii) *f is lower semicontinuous on Y_1;*

(iii) *g is upper semicontinous on X_1;*

(iv) *h is quasiconcave on X_1;*

(v) *k is quasiconvex on Y_1;*

(vi) *for each* $x_2 \in X_2$ *there exists* $x_1 \in X_1$ *such that* $f(x_2, \cdot) \leq h(x_1, \cdot)$;
(vii) *for each* $y_2 \in Y_2$ *there exists* $y_1 \in Y_1$ *such that* $k(\cdot, y_1) \leq g(\cdot, y_2)$.

Then

$$\inf_{y_1 \in Y_1} \sup_{x_2 \in X_2} f(x_2, y_1) \leq \sup_{x_1 \in X_1} \inf_{y_2 \in Y_2} g(x_1, y_2) .$$

Proof. Suppose that there exists a real λ such that

$$\sup_{x_1 \in X_1} \inf_{y_2 \in Y_2} g(x_1, y_2) < \lambda < \inf_{y_1 \in Y_1} \sup_{x_2 \in X_2} f(x_2, y_1) . \tag{8}$$

Define the maps $T : X_1 \times Y_1 \to X_1 \times Y_1$, $F : X_1 \times Y_1 \to X_2 \times Y_2$ by

$$T(x_1, y_1) = \{x_1' \in X_1 : h(x_1', y_1) > \lambda\} \times \{y_1' \in Y_1 : k(x_1, y_1') < \lambda\}$$

and

$$F(x_1, y_1) = \{x_2' \in X_2 : f(x_2', y_1) > \lambda\} \times \{y_2' \in Y_2 : g(x_1, y_2') < \lambda\} .$$

By (8), F has nonempty values. In view of conditions (iv) and (v) the values of T are convex and by (ii) and (iii), F has open fibers. From (vi) and (vii) it follows readily that for each $(x_2, y_2) \in X_2 \times Y_2$ there exists $(x_1, y_1) \in X_1 \times Y_1$ such that $F^{-1}(x_2, y_2) \subset T^{-1}(x_1, y_1)$. Therefore all hypotheses of Theorem 7 are verified. Applying Theorem 7 we get a point $(\overline{x_1}, \overline{y_1}) \in X_1 \times Y_1$ such that $(\overline{x_1}, \overline{y_1}) \in T(\overline{x_1}, \overline{y_1})$. Taking into account condition (i) we obtain the following contradiction

$$\lambda < h(\overline{x_1}, \overline{y_1}) \leq k(\overline{x_1}, \overline{y_1}) < \lambda .$$

□

When $X_1 = Y_1$, $X_2 = Y_2$ and conditions (vi), (vii) are replaced by a unique stronger condition one can get at once the following known result (see [3]).

Corollary 13. *Let* X *and* Y *be nonempty compact convex subsets of topological vector spaces and* f, g, h, $k : X \times Y \to \overline{\mathbb{R}}$, *be four functions satisfying:*

(i) $f \leq h \leq k \leq g$;
(ii) f *is lower semicontinuous in* y;
(iii) g *is upper semicontinous in* x;
(iv) h *is quasiconcave in* x;
(v) k *is quasiconvex in* y.

Then

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y) .$$

Theorem 14. *Let* X *be a nonempty convex subset of a topological vector space and* $f : X \times X \to \overline{\mathbb{R}}$ *be a function quasiconvex in* y *and transfer upper semicontinuous in* x. *Suppose that there exists a nonempty compact convex subset* X_0 *of* X *and a compact subset* K *of* X *satisfying the following condition*

for each $x \in X \backslash K$ *and any* $y' \in X$ *there exists a neighbourhood* $V(x)$ *of*

x *and a point* $y_0 \in X_0$ *such that* $f(z, y_0) \leq f(z, y')$ *for all* $z \in V(x)$. (9)

Then

$$\inf_{x\in X} f(x,x) \leq \sup_{x\in X} \inf_{y\in Y} f(x,y) .$$

Proof. As in previous proof we assume $\sup_{x\in X} \inf_{y\in Y} f(x,y) < \infty$ and fix a real $\lambda > \sup_{x\in X} \inf_{y\in Y} f(x,y)$. The map $T : X \multimap X$ defined by

$$T(x) = \{y \in X : f(x,y) < \lambda\}$$

takes nonempty convex values and has the local intersection property. We show that it satisfies condition (3) from Theorem 8. Let $x \in X\backslash K$. Since $T(x) \neq \emptyset$ and f is transfer upper semicontinous in x, there exists a neighbourhood $V'(x)$ of x and a point $y' \in X$ such that $f(z,y') < \lambda$ for each $z \in V'(x)$. By (9) there exist a neighbourhood $V''(x)$ of x and a point $y_0 \in K$ such that $f(z,y_0) \leq f(z,y')$ for all $z \in V''(x)$. Then for each $z \in V(x) = V'(x) \cap V''(x)$ we have $f(z,y_0) \leq f(z,y') < \lambda$, hence

$$y_0 \in \bigcap_{z\in V(x)} T(z) \cap X_0 .$$

Theorem 8 implies that $x_0 \in T(x_0)$ for some $x_0 \in X$. Hence

$$\inf_{x\in X} f(x,x) \leq f(x_0,x_0) < \lambda$$

and the proof is complete. □

Combining the lines of the proofs of Theorems 10 and 14 one can easily prove the following result

Theorem 15. *Let X and Y be nonempty compact convex subsets of topological vector spaces and $f, g : X \times Y \to \overline{\mathbb{R}}$ be two functions satisfying the following conditions*

(i) *$f \leq g$;*
(ii) *f is quasiconcave in x;*
(iii) *f is transfer lower semicontinuous in y;*
(iv) *there exist a nonempty compact convex subset Y_0 of Y and a compact subset K of X satisfying the following condition:*

for each $x \in X\backslash K$ and any $y' \in Y$ there exists a neighbourhood $V(x)$ of x and a point $y_0 \in Y_0$ such that $f(z,y_0) \leq f(z,y')$ for all $z \in V(x)$;

(v) *g is quasiconvex in y;*
(vi) *g is transfer upper semicontinuous in x;*
(vii) *there exist a nonempty compact convex subset X_0 of X and a compact subset L of Y satisfying the following condition:*

for each $y \in Y\backslash L$ and any $x' \in X$ there exists a neighbourhood $V(y)$ of y and a point $x_0 \in X_0$ such that $g(x_0,u) \geq g(x',u)$ for all $u \in V(y)$.

Then

$$\inf_{y\in Y} \sup_{x\in X} f(x,y) \leq \sup_{x\in X} \inf_{y\in Y} g(x,y) .$$

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Department of Mathematics, University of Oradea
3700 Oradea, Romania
E-mail: mbalaj@uoradea.ro