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ON RICCI CURVATURE OF TOTALLY REAL SUBMANIFOLDS IN A QUATERNION PROJECTIVE SPACE

LIU XIMIN

Abstract. Let M^n be a Riemannian n-manifold. Denote by $S(p)$ and $\overline{Ric}(p)$ the Ricci tensor and the maximum Ricci curvature on M^n, respectively. In this paper we prove that every totally real submanifolds of a quaternion projective space $QP^m(c)$ satisfies $S \leq ((n-1)c + \frac{n^2}{4}H^2)g$, where H^2 and g are the square mean curvature function and metric tensor on M^n, respectively. The equality holds identically if and only if either M^n is totally geodesic submanifold or $n = 2$ and M^n is totally umbilical submanifold. Also we show that if a Lagrangian submanifold of $QP^m(c)$ satisfies $\overline{Ric} = (n-1)c + \frac{n^2}{4}H^2$ identically, then it is minimal.

1. Introduction

Let M^n be a Riemannian n-manifold isometrically immersed in a Riemannian m-manifold $\bar{M}^m(c)$ of constant sectional curvature c. Denote by g, R and h the metric tensor, Riemann curvature tensor and the second fundamental form of M^n, respectively. Then the mean curvature vector H of M^n is given by $H = \frac{1}{n}\, trace\ h$. The Ricci tensor S and the scalar curvature ρ at a point $p \in M^n$ are given by $S(X,Y) = \sum_{i=1}^{n} \langle R(e_i, X)Y, e_i\rangle$ and $\rho = \sum_{i=1}^{n} S(e_i, e_i)$, respectively, where $\{e_1, \dots, e_n\}$ is an orthonormal basis of the tangent space T_pM^n. A submanifold M is called totally umbilical if h, H and g satisfy $h(X,Y) = g(X,Y)H$ for X, Y tangent to M^n.

The equation of Gauss for the submanifold M^n is given by

$$\begin{aligned} g(R(X,Y)Z,W) &= c(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)) \\ &\quad + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W))\,, \end{aligned} \tag{1}$$

where $X, Y, Z, W \in TM^n$. From (1) we have

$$\rho = n(n-1)c + n^2H^2 - |h|^2\,, \tag{2}$$

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where $|h|^2$ is the squared norm of the second fundamental form. From (2) we have

$$\rho \leq n(n-1)c + n^2H^2\,,$$

with equality holding identically if and only if M^n is totally geodesic.

Let $\overline{Ric}(p)$ denote the maximum Ricci curvature function on M^n defined by

$$\overline{Ric}(p) = \max\{S(u,u)|u \in T_p^1M^n,\ p \in M^n\}\,,$$

where $T_p^1M^n = \{v \in T_pM^n|\langle v,v\rangle = 1\}$.

In [2], Chen proves that there exists a basic inequality on Ricci tensor S for any submanifold M^n in $\bar{M}^m(c)$, i.e.

$$S \leq \Big((n-1)c + \frac{n^2}{4}H^2\Big)g\,, \tag{3}$$

with the equality holding if and only if either M^n is a totally geodesic submanifold or $n = 2$ and M^n is a totally umbilical submanifold. And in [3], Chen proves that every isotropic submanifold M^n in a complex space form $\bar{M}^m(4c)$ satisfies $\overline{Ric} \leq (n-1)c + \frac{n^2}{4}H^2$, and every Lagrangian submanifold of a complex space form satisfying the equality case identically is a minimal submanifold. In the present paper, we would like to extend the above results to the totally real submanifolds of quaternion projective space, namely, we prove that every totally real submanifolds of quaternion projective space $QP^m(c)$ satisfies $S \leq ((n-1)c + \frac{n^2}{4}H^2)g$, and the equality holds identically if and only if either M^n is totally geodesic submanifold or $n = 2$ and M^n is totally umbilical submanifold. Also we show that if a Lagrangian submanifold of $QP^m(c)$ satisfies $\overline{Ric} = (n-1)c + \frac{n^2}{4}H^2$ identically, then it is minimal.

2. Preliminary

Let $\bar{M}^m$ be a $4m$-dimensional Riemannian manifold with metric g. $\bar{M}^m$ is called a quaternion Kaehlerian manifold if there exists a 3-dimensional vector space V of tensors of type (1,1) with local basis of almost Hermitian structure I, J and K such that

(a) $IJ = -JI = K$, $JK = -KJ = I$, $KI = -IK = J$, $I^2 = J^2 = K^2 = -1$,

(b) for any local cross-section ϕ of V, $\bar{\nabla}_X\phi$ is also a cross-section of V, where X is an arbitrary vector field on $\bar{M}^m$ and $\bar{\nabla}$ the Riemannian connection on $\bar{M}^m$.

In fact, condition (b) is equivalent to the following condition:

(b') there exist local 1-forms p, q and r such that

$$\begin{aligned} \bar{\nabla}_XI &= \phantom{-r(X)I +{}} r(X)J - q(X)K \\ \bar{\nabla}_XJ &= -r(X)I \phantom{{}+ q(X)J} + p(X)K \\ \bar{\nabla}_XK &= q(X)I - p(X)J \end{aligned} \tag{4}$$

Now let X be a unit vector on $\bar{M}^m$, then X, IX, JX and KX form an orthonormal frame on $\bar{M}^m$. We denote by $Q(X)$ the 4-plane spanned by them. For any two orthonormal vectors X, Y on $\bar{M}^m$, if $Q(X)$ and $Q(Y)$ are orthogonal, the plane $\pi(X,Y)$ spanned by X, Y is called a totally real plane. Any 2-plane in a $Q(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic plane π is called the quaternionic sectional curvature of π. A quaternion Kaehlerian manifold is a quaternion space form if its quaternionic sectional curvatures are equal to a constant. A quaternion projective space, denoted by $QP^m(4c)$, is a quaternion Kaehlerian manifold of constant quaternionic sectional curvature $4c$.

It is known that a quaternionic Kaehlerian manifold $\bar{M}^m$ is a quaternion space form if and only if its curvature tensor $\bar{R}$ is of the following form [7]:

$$\begin{aligned}\bar{R}(X,Y)Z = c\{&g(Y,Z)X - g(X,Z)Y \\ &+ g(IY,Z)IX - g(IX,Z)IY + 2g(X,IY)IZ \\ &+ g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ \\ &+ g(KY,Z)KX - g(KX,Z)KY + 2g(X,KY)KZ\end{aligned} \tag{5}$$

for vectors X, Y, Z tangent to $\bar{M}^m$.

Let M^n be an n-dimensional Riemannian manifold isometrically immersed in $QP^m(4c)$. We call M^n a totally real submanifold of $QP^m(4c)$ if each 2-plane of M^n is mapped into a totally real plane in $QP^m(4c)$. Consequently, if M^n is a totally real submanifold of $QP^m(4c)$, then $\phi(TM^n) \subset T^{\perp}M^n$ for $\phi = I$, J or K, where $T^{\perp}M^n$ is the normal bundle of M^n in $QP^m(4c)$.

An n-dimensional totally real submanifold of a quaternion projective space $QP^m(4c)$ is called a Lagrangian submanifold when $n = m$.

Assume that M^n is a totally real submanifold of $QP^m(4c)$. For any orthonormal vectors X, Y in M^n, the plane $\pi(X,Y)$ spanned by X and Y is totally real in $QP^m(4c)$, $Q(X)$ and $Q(Y)$ are orthogonal and $g(X,\phi Y) = g(\phi X, Y) = 0$ for $\phi = I, J$ or K.

By (5) we have

$$\bar{R}(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}, \qquad X,Y,Z \in TM.$$

By the Gauss formula the curvature tensor R of M^n satisfies

$$\begin{aligned}g(R(X,Y)Z,W) &= c(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)) \\ &\quad + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)).\end{aligned} \tag{6}$$

We know that when M^n is totally real in $QP^m(4c)$, then $n \leq m$. We choose a local field of orthonormal frames in $QP^m(4c)$:

$$\begin{aligned}&e_1,\dots,e_n,e_{n+1},\dots,e_m; e_{I(1)} = Ie_1,\dots,e_{I(m)} = Ie_m\,; \\ &e_{J(1)} = Je_1,\dots,e_{J(m)} = Je_m; e_{K(1)} = Ke_1,\dots,e_{K(m)} = Ke_m\,,\end{aligned} \tag{7}$$

in such a way that, restricting to M^n, $e_1,\dots,e_n$ are tangent to M^n.

We shall use the following convention in the range of indices unless mentioned otherwise:

$$A,B,C,D,\ldots = 1,\dots,m,I(1),\dots,I(m),J(1),\dots,J(m),K(1),\dots,K(m);$$

$$i,j,k,l,\ldots = 1,\dots,n; \qquad r,s,t,\ldots = n+1,\dots,m,I(1),\dots,K(m);$$

$$u,v,\ldots = n+1,\dots,m; \qquad \phi,\psi,\ldots = I,J,K.$$

Let $A_r = A_{e_r}$ denote the shape operator on M^n in $QP^m(4c)$. Then A_r is related to the second fundamental form h by

$$g(h(X,Y), e_r) = g(A_r X, Y)\,. \tag{8}$$

Let M^n be a totally real submanifold in $QP^m(4c)$, $\{\phi_r, \phi_s, \phi_t\}$ be the set $\{I, J, K\}$ or a set of the circular permutation of the three elements I, J and K. Then we have

Lemma 2.1 [6]. *For any X, Y, Z, W in TM^n, we have*

(i) $\bar{R}(Z, W, \phi_r X, \phi_r Y) = \bar{R}(Z, W, X, Y)$,

(ii) $g(h(X,Y), \phi_r Z) = g(h(Z,Y), \phi_r X)$, $r = 1, 2, 3$.

3. Ricci tensor of totally real submanifolds

We will need the following algebraic lemma due to Chen [1].

Lemma 3.1. *Let $a_1, \dots, a_n, c$ be $n+1$ $(n \geq 2)$ real numbers such that*

$$\Big(\sum_{i=1}^n a_i\Big)^2 = (n-1)\Big(\sum_{i=1}^n a_i^2 + c\Big)\,.$$

Then $2a_1a_2 \geq c$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

For a totally real submanifold M^n in a quaternion projective space $QP^m(4c)$, we have the following.

Theorem 3.1. *If M^n is a totally real submanifold in a quaternion projective space $QP^m(4c)$, then the Ricci tensor of M^n satisfies*

$$S \leq \Big((n-1)c + \frac{n^2}{4}H^2\Big)g\,, \tag{9}$$

and the equality holds identically if and only if either M^n is totally geodesic or $n = 2$ and M^n is totally umbilical.

Proof. Let M^n be a totally real submanifold of a quaternion projective space $QP^m(4c)$, from Gauss equation (6), we have

$$\rho = n(n-1)c + n^2H^2 - |h|^2\,. \tag{10}$$

Put $\delta = \rho - n(n-1)c - \frac{n^2}{2}H^2$. Then from (10) we obtain

$$n^2H^2 = 2(\delta + |h|^2)\,. \tag{11}$$

Let

$$\omega^1, \dots, \omega^n, \omega^{n+1}, \dots, \omega^m, \omega^{I(1)}, \dots, \omega^{I(m)}, \omega^{J(1)}, \dots, \omega^{J(m)}, \omega^{K(1)}, \dots, \omega^{K(m)}$$

be the dual frame of the frame given by (7).

Since M^n is totally real, $Q(e_i)$ and $Q(e_j)$, $i \neq j$, are orthogonal. Thus $g(\phi(e_i), \psi(e_j)) = 0$ when $i \neq j$. From the structure equations:

$$d\omega^A = -\sum \omega^A_B \wedge \omega^B, \quad \omega^A_B + \omega^B_A = 0$$

and from (4), we obtain

$$\begin{aligned} \omega_j^i &= \omega_{\phi(j)}^{\phi(i)}, & \omega_j^{\phi(i)} &= \omega_i^{\phi(j)}, & \omega_u^i &= \omega_{\phi(u)}^{\phi(i)}, \\ \omega_u^{\phi(i)} &= \omega_i^{\phi(u)}, & \omega_v^u &= \omega_{\phi(v)}^{\phi(u)}, & \omega_v^{\phi(u)} &= \omega_u^{\phi(v)}. \end{aligned} \tag{12}$$

If we write $\omega_i^r = \sum_j h_{ij}^r \omega^j$ then $h_{ij}^r = h_{ji}^r$ and the mean curvature vector of M^n is $H = \frac{1}{n}\sum_{i,r} h_{ii}^r e_r$.

From (12) we have

$$h_{jk}^{\phi(i)} = h_{ik}^{\phi(j)} = h_{ji}^{\phi(k)}. \tag{13}$$

Let L be a linear $(n-1)$-subspace of T_pM^n, $p \in M^n$, such that $e_1, \dots, e_{n-1} \in L$ and if $H(p) \neq 0$, e_{n+1} is in the direction of the mean curvature vector at p.

Put $a_i = h_{ii}^{n+1}$, $i = 1, \dots, n$. Then from (11) we get

$$\Big(\sum_{i=1}^n a_i\Big)^2 = 2\Big\{\delta + \sum_{i=1}^n a_i^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^n (h_{ij}^r)^2\Big\}. \tag{14}$$

Equation (14) is equivalent to

$$\Big(\sum_{i=1}^3 \bar{a}_i\Big)^2 = 2\Big\{\delta + \sum_{i=1}^3 \bar{a}_i^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^n (h_{ij}^r)^2 - \sum_{2\leq i\neq j\leq n-1} a_i a_j\Big\}, \tag{15}$$

where $\bar{a}_1 = a_1$, $\bar{a}_2 = a_2 + \cdots + a_{n-1}$, $\bar{a}_3 = a_n$.

By Lemma 3.1 we know that if $(\sum_{i=1}^3 \bar{a}_i)^2 = 2(c + \sum_{i=1}^3 \bar{a}_i^2)$, then $2\bar{a}_1\bar{a}_2 \geq c$ with equality holding if and only if $\bar{a}_1 + \bar{a}_2 = \bar{a}_3$. Hence from (15) we can get

$$\sum_{1\leq i\neq j\leq n-1} a_i a_j \geq \delta + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^n (h_{ij}^r)^2, \tag{16}$$

which gives

$$n(n-1)c + \frac{n^2}{2}H^2 \geq \rho - \sum_{1\leq i\neq j\leq n-1} a_i a_j + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^n (h_{ij}^r)^2. \tag{17}$$

Using Gauss equation we have

$$\begin{aligned} \rho - \sum_{1\leq i\neq j\leq n-1} a_i a_j &+ 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^n (h_{ij}^r)^2 \\ &= 2S(e_n, e_n) + (n-1)(n-2)c + 2\sum_{i<n}(h_{in}^{n+1})^2 \\ &\quad + \sum_{r=n+2}^{4m}\Big[(h_{nn}^r)^2 + 2\sum_{i=1}^{n-1}(h_{in}^r)^2 + (\sum_{j=1}^{n-1} h_{jj}^r)^2\Big]. \end{aligned} \tag{18}$$

From (17) and (18) we have

$$(n-1)c+\frac{n^2}{4}H^2 \ge S(e_n,e_n)+2\sum_{i<n}(h^{n+1}_{in})^2$$
$$+\sum_{r=n+2}^{4m}\Big[\sum_{i=1}^{n}(h^r_{in})^2+\Big(\sum_{j=1}^{n-1}h^r_{jj}\Big)^2\Big]. \tag{19}$$

So we have

$$(n-1)c+\frac{n^2}{4}H^2 \ge S(e_n,e_n) \tag{20}$$

with equality holding if and only if

$$h^s_{jn}=0\,,\qquad h^r_{in}=0\qquad \sum_{j=1}^{n-1}h^s_{jj}=h^s_{nn} \tag{21}$$

for $1\le j\le n-1$, $1\le i\le n$ and $n+2\le r\le 4m$ and, since Lemma 3.1 states that $2\bar a_1\bar a_2=c$ if and only if $\bar a_1+\bar a_2=\bar a_3$, we also have $h^{n+1}_{nn}=\sum_{j=1}^{n-1}h^{n+1}_{jj}$. Since e_n can be any unit tangent vector of M^n, then (20) implies inequality (9).

If the equality sign case of (9) holds identically. Then we have

$$\begin{aligned}
&h^{n+1}_{ij}=0 && (1\le i\ne j\le n)\,,\\
&h^r_{ij}\ =0 && (1\le i,j\le n;\ n+2\le r\le 4m)\,,\\
&h^{n+1}_{ii}=\sum_{k\ne i}h^{n+1}_{kk}\,,\quad \sum_{k\ne i}h^r_{kk}=0\,, && (n+2\le r\le 4m)\,.
\end{aligned} \tag{22}$$

If $\lambda_i=h^{n+1}_{ii}(1\le i\le n)$, we find $\sum_{k\ne i}\lambda_k=\lambda_i(1\le i\le n)$ and, since the matrix $A^{(n)}=(a^{(n)}_{ij})$ with $a^{(n)}_{ij}=1-2\delta_{ij}$ is regular for $n\ne 2$ and has kernel $R(1,1)$ for $n=2$, we conclude that M^n is either totally geodesic or $n=2$ and M^n is totally umbilical.

The converse is easy to prove. This completes the proof of Theorem 3.1. □

4. Minimality of Lagrangian submanifolds

Theorem 4.1. *If M^n is a Lagrangian submanifold in a quaternion projective space $QP^m(4c)$, then*

$$\overline{Ric}\le (n-1)c+\frac{n^2}{4}H^2\,. \tag{23}$$

If M^n satisfies the equality case of (23) identically, then M^n is minimal submanifold.

Clearly Theorem 4.1 follows immediately from the following Lemma.

Lemma 4.1. *If M^n is a n-dimensional totally real submanifold in a quaternion projective space $QP^m(4c)$, then we have (23). If a totally real submanifold M^n in $QP^m(4c)$ satisfies the equality case of (23) at a point p, then the mean curvature vector H at p is perpendicular to $\phi(T_pM^n)$.*

Proof. Inequality (23) is an immediate consequence of inequality (9).

Now let us assume that M^n is a totally real submanifold of $QP^m(4c)$ which satisfies the equality sign of (23) at a point $p \in M^n$. Without loss of the generality we may choose an orthonormal basis $\{\bar{e}_1, \dots, \bar{e}_n\}$ of T_pM^n such that $\overline{Ric}(p) = S(\bar{e}_n, \bar{e}_n)$. From the proof of Theorem 3.1, we get

$$h^s_{in} = 0\,, \quad \sum_{i=1}^{n-1} h^s_{ii} = h^s_{nn}\,, \qquad i = 1, \dots, n-1;\ s = n+1, \dots, 4m\,, \tag{24}$$

where h^s_{ij} denote the coefficients of the second fundamental form with respect to the orthonormal basis $\{\bar{e}_1, \dots, \bar{e}_n\}$ and $\{\bar{e}_{n+1}, \dots, \bar{e}_{4m}\}$.

If for all tangent vectors u, v and w at p, $g(h(u,v), \phi w) = 0$, there is nothing to prove. So we assume that this is not the case. We define a function f_p by

$$f_p\,:\ T^1_pM^n \to R:\ v \mapsto f_p(v) = g\big(h(v,v), \phi v\big)\,. \tag{25}$$

Since $T^1_pM^n$ is a compact set, there exists a vector $v \in T^1_pM^n$ such that f_p attains an absolute maximum at v. Then $f_p(v) > 0$ and $g(h(v,v), \phi w) = 0$ for all w perpendicular to v. So from (8), we know that v is an eigenvector of $A_{\phi v}$. Choose a frame $\{e_1, e_2, \dots, e_n\}$ of T_pM^n such that $e_1 = v$ and e_i be an eigenvector of $A_{\phi e_1}$ with eigenvalue λ_i. The function f_i, $i \geq 2$, defined by $f_i(t) = f_p(\cos\, t\, e_1 + \sin\, t\, e_2)$ has relative maximum at $t = 0$, so $f_i''(0) \leq 0$. This will lead to the inequality $\lambda_1 \geq 2\lambda_i$. Since $\lambda_1 > 0$, we have

$$\lambda_i \neq \lambda_1\,, \quad \lambda_1 \geq 2\lambda_i\,, \quad i \geq 2\,. \tag{26}$$

Thus, the eigenspace of $A_{\phi e_1}$ with eigenvalue λ_1 is 1-dimensional.

From (24) we know that $\bar{e}_n$ is a common eigenvector for all shape operators at p. On the other hand, we have $e_1 \neq \pm\bar{e}_n$ since otherwise, from (24) and $A_{\phi e_i}\bar{e}_n = \pm A_{\phi e_i} e_1 = \pm A_{\phi e_1} e_i = \pm \lambda_i e_i \perp \bar{e}_n$ $(i = 2, \dots, n)$, we obtain $\lambda_i = 0$, $i = 2, \dots, n$; and hence $\lambda_1 = 0$ by (24), which is a contradiction. Consequently, without loss of generality we may assume $e_1 = \bar{e}_1, \dots, e_n = \bar{e}_n$.

By Lemma 2.1, $A_{\phi e_n} e_1 = A_{\phi e_1} e_n = \lambda_n e_n$. Comparing this with (24) we obtain $\lambda_n = 0$. Thus, by applying (24) once more, we get $\lambda_1 + \cdots + \lambda_{n-1} = \lambda_n = 0$. Therefore, *trace* $A_{\phi e_1} = 0$.

For each $i = 2, \dots, n$, we have

$$h^{n+i}_{nn} = g(A_{\phi e_i} e_n, e_n) = g(A_{\phi e_n} e_i, e_n) = h^{2n}_{in}\,.$$

Hence, by applying (24) again, we get $h^{n+i}_{nn} = 0$. Combining this with (24) yields *trace* $A_{\phi e_i} = 0$. So we have *trace* $A_{\phi X} = 0$ for any $X \in T_pM^n$. Therefore, by using the definition of the mean curvature vector, we conclude that the mean curvature vector at p is perpendicular to $\phi(T_pM^n)$. This completes the proof of Lemma 4.1. $\square$

Remark 4.1. From the proof of Lemma 4.1 we know that if M^n is a Lagrangian submanifold of a quaternion projective space $QP^m(4c)$ satisfying

$$\overline{Ric} = (n-1)c + \frac{n^2}{4}H^2\,, \tag{27}$$

then M^n is minimal and $A_{\phi v} = 0$ for any unit tangent vector satisfying $S(v,v) = \overline{Ric}$. Thus, for any X tangent to M^n, by Lemma 2.1, we have $A_{\phi X} v = 0$. Hence, we obtain $h(v,X) = 0$ for any X tangent to M^n and any v satisfying $S(v,v) = \overline{Ric}$. Conversely, if M^n is a Lagrangian minimal submanifold of $QP^m(4c)$ such that for each $p \in M^n$ there exists a unit vector $v \in T_pM^n$ such that $h(v,X) = 0$ for all $X \in T_pM^n$, then it satisfies (27) indentically.

For each $p \in M^n$, the kernel of the second fundamental form is defined by

$$\mathcal{D}(p) = \left\{ Y \in T_pM^n | h(X,Y) = 0, \quad \forall\ X \in T_pM^n \right\}. \tag{28}$$

From the above discussion, we conclude that M^n is a Lagrangian minimal submanifold of $QP^m(4c)$ satisfying (27) at p if and only if $\dim \mathcal{D}(p)$ is at least 1-dimensional.

When the $\dim \mathcal{D}(p)$ is constant, we have the following result which describes the geometry of Lagrangian submanifold satisfying (27).

Theorem 4.2. *Let M^n be a Lagrangian minimal submanifold of $QP^m(4c)$. Then*

(1) *M^n satisfies (27) at a point p if and only if $\dim \mathcal{D}(p) \geq 1$.*

(2) *If the dimension of $\mathcal{D}(p)$ is positive constant d, then $\mathcal{D}$ is a completely integral distribution and M^n is d-ruled, i.e., for each point $p \in M^n$, M^n contains a d-dimensional totally geodesic submanifold N of $QP^m(4c)$ passing through p.*

(3) *A ruled Lagrangian minimal submanifold of $QP^m(4c)$ satisfies (26) identically if and only if, for each ruling N in M^n, the normal bundle $T^\perp M^n$ restricted to N is a parallel normal subbundle of the normal bundle $T^\perp N$ along N.*

Proof. (1) is done already.

(2) Assume that the dimension of $\mathcal{D}(p)$ is constant d. We denote by $\mathcal{D}^\perp$ the orthogonal complementary distribution of $\mathcal{D}$ in TM^n. Then for any vector fields Y, Z in $\mathcal{D}$ and $X \in \mathcal{D}^\perp$, we have $h(Y,Z) = h(X,Z) = 0$. Thus, by applying the equation of Codazzi, we get $g(X, \nabla_Y Z) = 0$ for any $X \in \mathcal{D}^\perp$. Hence $\nabla_Y Z \in \mathcal{D}$, which implies that $\mathcal{D}$ is completely integrable and each leaf N of $\mathcal{D}$ is totally geodesic submanifold of M^n. Therefore, by (28) we conclude that each leaf N is totally geodesic in $QP^m(4c)$ too. This implies that M^n is d-ruled.

(3) Assume M^n is a ruled Lagrangian minimal submanifold in $QP^m(4c)$. Then we have $h(Y,Z) = 0$ for any vector fields Y, Z tangent to a ruling N. Hence (3) follows from Lemma 3.5 of [1] and Remark 4.1. This completes the proof of Theorem 4.2.

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Department of Applied Mathematics, Dalian University of Technology
Dalian 116024, China
E-mail: xmliu@dlut.edu.cn