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ASYMPTOTIC BEHAVIOUR OF NONOSCILLATORY SOLUTIONS OF THE FOURTH ORDER DIFFERENTIAL EQUATIONS

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Abstract. In the paper the fourth order nonlinear differential equation $y^{(4)} + (q(t)y')' + r(t)f(y) = 0$, where $q \in C^1([0,\infty))$, $r \in C^0([0,\infty))$, $f \in C^0(R)$, $r \geq 0$ and $f(x)x > 0$ for $x \neq 0$ is considered. We investigate the asymptotic behaviour of nonoscillatory solutions and give sufficient conditions under which all nonoscillatory solutions either are unbounded or tend to zero for $t \to \infty$.

1. Introduction

Consider the fourth-order differential equation with the middle term

$$y^{(4)} + (q(t)y')' + r(t)f(y) = 0\,, \tag{1}$$

where $q \in C^1(R_+)$, $r \in C^0(R_+)$, $R_+ = [0,\infty)$, $r \geq 0$ on R_+, $f \in C^0(R)$ and $f(x)x > 0$ for $x \neq 0$ and

$$\int_0^\infty r(t)dt = \infty, \qquad \liminf_{|x|\to\infty} |f(x)| > 0 \tag{2}$$

will be supposed.

A continuous function $y : R_+ \to R$ is said to be oscillatory if it is nontrivial in any neighbourhood of ∞ and there exists a sequence of its zeros tending to ∞.

A continuous function $y : R_+ \to R$ is called solution of Eq. (1) if it has derivatives up to the fourth order and fulfills (1).

A solution is called nonoscillatory if it is different from zero in a neighbourhood of ∞.

This paper is concerned with asymptotic behaviour of nonoscillatory solutions of Eq. (1). Sufficient conditions will be given under which all nonoscillatory solutions either are unbounded or tend to zero for $t \to \infty$.

For $q \equiv 0$ the similar problems are hardly studied mainly under the concept of Property A, see e.g. [4]. Recall, that (1) has Property A if every proper solution

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y is oscillatory. It is necessary to say that the structure of solutions of Eq. (1) is more complex, by the reason, that for $q \equiv 0$ there is no nonoscillatory solution y with an oscillatory derivatives $y^{(j)}$, $j \in \{1,2,3\}$ (so called a weakly oscillatory solution), see e.g. the set N^0, defined below.

For $q \equiv 1$, Kiguradze proved that under certain assumptions nonoscillatory solutions of Eq. (1) do not exist.

Proposition 1. ([3], Cor. 1.1, 1.2, 1.3). *Let $q \equiv 1$ and $|f(x)| \geq |x|^\lambda$ on R and $\int^\infty t^{\mu(\lambda)} r(t)\,dt = +\infty$, where*

$$\mu(\lambda) = \begin{cases} 1 & \text{for} \quad \lambda > 1\,, \\ \lambda & \text{for} \quad 0 < \lambda < 1\,, \\ 0 & \text{for} \quad \lambda = 1\,, \end{cases}$$

then every solution of Eq. (1) *is oscillatory.*

The following example shows that nonoscillatory solution of Eq. (1) may exist.

Example 1. Let $\alpha \in (0,1)$. The equation $y^{(4)} + (q(t)y')' + y = 0$, with $q(t) = \frac{(t+1)^{\alpha+1}}{\alpha}(t + 1 - \frac{1}{1-\alpha} + \frac{1}{(1-\alpha)(t+1)^{\alpha-1}} - \frac{\alpha(\alpha+1)(\alpha+2)}{(t+1)^{\alpha+3}}) > 0$, $t \geq \frac{1}{1-\alpha} - 1$ has the solution of the form $y = 1 + \frac{1}{(t+1)^\alpha} \to_{t\to\infty} 1$.

Another impulse is that the similar problems are studied for the third order differential equation with the middle term

$$y''' + q(t)y' + r(t)f(y) = 0\,,$$

see e.g. [5], [2].

The set of all nonoscillatory solutions of Eq. (1) we will denote N. The structure of N was studied e.g. by Bartušek and Sobalová, see [1]. For our purpose N can be devided in the following way:
$N = N^0 \cup N^+ \cup N^-$, where

$$\begin{aligned} N^0 =& \{y \in N : y(t) \neq 0 \text{ for } t \geq T_y \in R_+, y' \text{ is an oscillatory function}\}\,, \\ N^+ =& \{y \in N : y(t)y'(t) > 0 \text{ for } t \geq T_y \in R_+\}\,, \\ N^- =& \{y \in N : y(t)y'(t) < 0 \text{ for } t \geq T_y \in R_+\}\,. \end{aligned}$$

Following lemmas deal with solutions of N^+ and N^- with regard to signs or oscillatory character of $y^{(i)}$ for $i = 2,3$, see [1].

Lemma 1. *Let $y \in N^+$. Then for large t one of the following statements holds:*

(i) *either $y(t)y''(t) > 0$ or y'' is an oscillatory function;*

(ii) *or $y(t)y''(t) < 0$ and either $y(t)y'''(t) > 0$ or y''' is an oscillatory function.*

Lemma 2. *Let $y \in N^-$. Then for large t*

(i) *either y'' is an oscillatory function*

(ii) *either $y(t)y''(t) > 0$, y''' is an oscillatory function*

(iii) *or $y(t)y''(t) > 0$, $y(t)y'''(t) < 0$.*

Our study will be differentiated to two situations $q \le 0$, $q \ge 0$ in view of different results in these cases, see [1].

Investigation of asymptotic properties of N will use reducing of order of Eq. (1) obtaining by the double integration of Eq. (1).

Let y be a solution of Eq. (1) and $\tau \in R_+$. Then

$$y'''(t) + q(t)y'(t) = K - \int_\tau^t r(s)f(y(s))\,ds\,, \quad t \ge \tau\,, \tag{3}$$

$$y''(t) = y''(\tau) - \int_\tau^t q(s)y'(s)\,ds + K(t-\tau) - \int_\tau^t \int_\tau^s r(\xi)f(y(\xi))\,d\xi\,ds\,, \tag{4}$$

where $K = y'''(\tau) + q(\tau)y'(\tau)$.

2. Case $q \ge 0$

In view of following lemma we will concentrate attention only to N^0 and N^-.

Lemma 3. [1]. *Let $q \ge 0$ be valid. Then every nonoscillatory solution of Eq. (1) belongs to $N^0 \cup N^-$.*

Examples of solutions of N^0 are given by Kiguradze [3] for $q \equiv 1$. It is necessary to say that these examples do not satisfy the first condition of (2). The question of the existence of an equation satisfying condition (2), for which $N^0 \ne \emptyset$, is open to this time.

The following proposition shows that $y \in N^0$ can not have one-side oscillatory y' such that $y(t)y'(t) \ge 0$ for large t.

Proposition 2. *Let $y \in N^0$. Then there exists the sequence $\{t_k\}_{k=1}^\infty$ such that $\lim\limits_{k\to\infty} t_k = \infty$ and $y(t_k)y'(t_k) < 0$, $k = 1, 2, \dots$.*

Proof. Let on the contrary $y(t) > 0$, $y'(t) \ge 0$, $t \ge T \ge T_y$. Then y is positive and nondecreasing on $[T, \infty)$. We will note $M_1 = \min\limits_{T \le t < \infty} f(y(t))$. According to (2), $M_1 > 0$. Further, using (3),

$$\begin{aligned} y'''(t) \le y'''(t) + q(t)y'(t) &= K - \int_T^t r(s)f(y(s))\,ds \\ &\le K - M_1 \int_T^t r(s)\,ds \to_{t\to\infty} -\infty\,, \end{aligned}$$

which contradicts to the oscillation of y'''. □

Theorem 1. *Let $q \ge 0$, $q' \le 0$ for $t \ge T \in R_+$ and $y \in N^0$ be valid. Then $\liminf\limits_{t\to\infty} |y(t)| = 0$.*

Proof. Let $y \in N^0$ and $y(t) > 0$ for $t \ge T_y$ for certainty. We use Eq. (4), where $\tau \ge \bar{T} = \max\{T, T_y\}$. Using integration per partes and putting $k = y''(\tau) + q(\tau)y(\tau)$

it follows from (4), that

$$\begin{aligned} y''(t) =& k - q(t)y(t) + \int_\tau^t q'(s)y(s)\,ds + K(t-\tau) \\ (5)\qquad & - \int_\tau^t \int_\tau^s r(\xi)f(y(\xi))\,d\xi\,ds \le k + K(t-\tau)\,, \quad t \ge \tau\,. \end{aligned}$$

Only two cases are possible.

a) There exists $\tau \ge \bar{T}$ such that $y'(\tau) = 0$ and $y'''(\tau) < 0$.

b) For arbitrary zero τ of y' such that $\tau \ge \bar{T}$, $y'''(\tau) \ge 0$ is valid.

Let (a) be valid. Then $K < 0$ and it follows from (5), that $\lim_{t\to\infty} y''(t) = -\infty$. It contradicts to the fact that y'' is oscillatory function.

Then (b) is valid. Let $\tau_1 \le \tau$ be arbitrary zero of y'. But it follows from (3) that $\int_{\tau_1}^{\tau} r(s)f(y(s))\,ds \le K$. As y' is oscillatory and $r(t)$ and $f(y(t))$ are positive we receive

$$(6)\qquad \int_\tau^\infty r(s)f(y(s))\,ds \le K\,.$$

On the contrary suppose that $\liminf_{t\to\infty} y(t) = C > 0$. Then according to the second inequality of (2) we have $\inf_{t\in[\tau,\infty)} f(y(t)) > 0$ but this together with (6) contradicts to $\int_0^\infty r(s)\,ds = \infty$. □

Remark 1. It follows from (6), that we obtain the following integral estimation for $y \in N^0$

$$\int_0^\infty r(s)|f(y(s)|\,ds < \infty\,.$$

According to the definition of N^-, $|y(t)|$ is decreasing function for $t \ge T_y$ but not always to zero as we can see in Example 1. The following theorem gives a sufficient condition on functions q and r under which $y \in N^-$ tends to zero.

Notation. $\bar{q}(t) = \max_{0\le s\le t} q(s)$.

Theorem 2. *Let $q \ge 0$ and*

$$(7)\qquad \lim_{t\to\infty} \frac{\int_0^t \int_0^s r(\xi)\,d\xi\,ds}{\bar{q}(t)} = \infty$$

be valid. Then $\lim_{t\to\infty} y(t) = 0$ for $y \in N^-$.

Proof. Let $y \in N^-$ and $y(t) > 0$ for $t \ge T_y$ for simplicity. Thus y is decreasing for $t \ge T_y$. Let $\tau \ge T_y$ be such that $K \le 0$, where K is given by (4). It is possible due to Lemma 2. On the contrary we will suppose $\lim_{t\to\infty} y(t) = C > 0$.

Put $c_1 = \int_0^\tau \int_\tau^s r(\xi)\,d\xi\,ds$, $c_2 = \int\limits_0^\tau r(\xi)\,d\xi$, $\bar{q}_\tau(t) = \max\limits_{\tau \le s \le t} q(s)$. As, with respect to (2)

$$\frac{\int_0^t \int_0^s r(\xi)\,d\xi ds}{\bar{q}(t)} \le \frac{c_1 + c_2(t-\tau) + \int_\tau^t \int_\tau^s r(\xi)\,d\xi ds}{\bar{q}_\tau(t)} \le \frac{2\int_\tau^t \int_\tau^s r(\xi)\,d\xi\,ds}{\bar{q}_\tau(t)}$$

for large t, (7) yields

$$\lim_{t\to\infty} \frac{\int_\tau^t \int_\tau^s r(\xi)\,d\xi\,ds}{\bar{q}_\tau(t)} = \infty\,. \tag{8}$$

We can restrict to (4). Using Lemma 2 there exists the sequence $\{t_k\}_{k=1}^\infty$, $t_k \ge \tau$ such that $\lim\limits_{k\to\infty} y''(t_k) \ge 0$. Hence (4) yields

$$\begin{aligned} &-y''(\tau) - \bar{q}_\tau(t_k)y(\tau) \le y''(t_k) - y''(\tau) + \bar{q}_\tau(t_k)[y(t_k) - y(\tau)] \\ &\le y''(t_k) - y''(\tau) + \int_\tau^{t_k} q(s)y'(s)\,ds \le -M_2 \int_\tau^{t_k}\int_\tau^s r(\xi)\,d\xi\,ds\,, \end{aligned}$$

where $M_2 = \min\limits_{C \le s \le y(\tau)} f(s) > 0$ from (2) and $f(x)x > 0$. Further we divide this inequality by $\bar{q}_\tau(t_k)$ and use (8) and Lemma 2. We obtain a contradiction

$$M \le -\frac{y''(\tau)}{\bar{q}_\tau(t_k)} - y(\tau) \le \frac{-M_2 \int_\tau^{t_k}\int_\tau^s r(\xi)\,d\xi\,ds}{\bar{q}_\tau(t_k)} \to_{k\to\infty} -\infty\,,$$

where $M = \frac{-y''(\tau)}{q(\tau)} - y(\tau)$ for $y''(\tau) > 0$ and $M = -y(\tau)$ for $y''(\tau) \le 0$. ☐

Remark 2. In Example 1 condition (7) is not satisfied, $\lim\limits_{t\to\infty} \frac{\int_0^t\int_0^s r(\xi)\,d\xi\,ds}{\bar{q}(t)} = 0$.

Consequence 1. *Let $q \ge 0$ and $q' \le 0$. Then every nonoscillatory solution of Eq. (1) is of the type N^0 and $\liminf\limits_{t\to\infty} |y(t)| = 0$.*

Proof. As according to [1] Th. 3 the set $N^- = \emptyset$, the conclusion follows from Th. 1. ☐

Remark 3. Conseq. 1 extends, in some sence, the results in Prop. 1.

3. Case $q \le 0$.

For $q \le 0$ we receive $N^0 = \emptyset$ without any further assumptions on Eq. (1), putting certain conditions on functions q and r we can eliminate also some subclasses of N^- and N^+, see [1]. Now we will study asymtotic behaviour of N^- and N^+. Solutions $y \in N^-$ are decreasing in their absolute values and there are two possibilities of their behaviour, either $\lim\limits_{t\to\infty} |y(t)| = C > 0$ or $\lim\limits_{t\to\infty} y(t) = 0$. Following theorem eliminates the case $\lim\limits_{t\to\infty} |y(t)| = C$.

Theorem 3. *Let $q \le 0$ be valid. Then $\lim\limits_{t\to\infty} y(t) = 0$ for $y \in N^-$.*

Proof. Let $y \in N^-$ and $y(t) > 0$ for $t \geq T_y$ for simplicity. On the contrary we will suppose $\lim\limits_{t\to\infty} y(t) = C > 0$. Using the structure of N^- there exists $\tau \geq T_y$ such that y is decreasing, $y'(t) < 0$ on $[\tau, \infty)$. Further there exists the sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \geq \tau$, $\lim\limits_{k\to\infty} t_k = \infty$ and either $y'''(t_k) \geq 0$ (Lemma 2 (i)-(ii)) or $y'''(t_k) \to 0$ as $k \to \infty$ (Lemma 2 (iii)). Hence (2) and (3) yield

$$y'''(t_k) \leq y'''(t_k) + q(t_k)y'(t_k) \leq K - M_2 \int_{\tau}^{t_k} r(s)\, ds \to_{k\to\infty} -\infty\,,$$

where M_2 is given in the proof of Th. 2. The contradiction proves the conclusion. □

For $y \in N^+$ we receive again two situations. As $|y(t)|$ is increasing then either $\lim\limits_{t\to\infty} |y(t)| = C > 0$ or $\lim\limits_{t\to\infty} |y(t)| = \infty$.

Next example is an illustration of increasing, bounded solution $y \in N^+$.

Example 2. The equation $y^{(4)} + (q(t)y')' + \frac{1}{t+1}y = 0$, with $q(t) = \frac{(t+1)^{\alpha+1}}{\alpha}(-\ln(t+1) - \frac{1}{\alpha}((t+1)^{-\alpha} - 1) - \frac{\alpha(\alpha+1)(\alpha+2)}{(t+1)^{\alpha+3}}) < 0$, $t \in (0,\infty)$, $\alpha > 0$ is small enough has the solution $y = 1 - \frac{1}{(t+1)^{\alpha}} \in N^+$.

Now we are interested in a condition on functions q and r under which $\lim\limits_{t\to\infty} |y(t)| = \infty$.

Theorem 4. *Let $q \leq 0$,*

$$\liminf_{t\to\infty} \frac{\int_0^t r(s)\, ds}{|q(t)|} > 0 \tag{9}$$

and $y \in N^+$ be valid. Then $\lim\limits_{t\to\infty} |y(t)| = \infty$.

Proof. Let $y \in N^+$. Without loss of generality we can suppose $y(t) > 0$, $t \geq T_y$. According to the structure of nonoscillatory solutions given by Lemma 1 (i), it is evident that $\lim\limits_{t\to\infty} y(t) = \infty$ immediately. Contrarily we assume $\lim\limits_{t\to\infty} y(t) = C < \infty$ for nonoscillatory solutions given by Lemma 1 (ii). In view of this lemma there exists $\tau \geq T_y$ such that $y''(t) < 0$ for $t \geq \tau$, hence

$$\lim_{t\to\infty} y'(t) = 0\,, \tag{10}$$

in opposite case y could change its sign. Further there exists the sequence $\{t_k\}_{k=1}^{\infty}$, $t_k \geq \tau$ such that $\lim\limits_{k\to\infty} t_k = \infty$ and $y'''(t_k) \geq 0$. Then using (3)

$$\begin{aligned} q(t_k)y'(t_k) &\leq K - \textstyle\int_{\tau}^{t_k} r(s)f(y(s))\, ds \\ &\leq K - M_3 \textstyle\int_{\tau}^{t_k} r(s)\, ds \leq -\frac{M_3}{2} \int_{\tau}^{t_k} r(s)\, ds \end{aligned}$$

for large k, where $M_3 = \min\limits_{y(\tau)\leq s\leq C} f(s) > 0$. Hence for $k \to \infty$

$$y'(t_k) \geq \frac{M_3}{2} \frac{\int_{\tau}^{t_k} r(s)\, ds}{|q(t_k)|} \geq \frac{M_3}{4} \liminf_{k\to\infty} \frac{\int_0^{t_k} r(s)\, ds}{|q(t_k)|} > 0\,.$$

From (10) and assumptions we have the contradiction. $\square$

Remark 4. In Example 2 we received $\liminf_{k\to\infty} \frac{\int_0^t r(s)\,ds}{|q(t)|} = 0$.

Consequence 2. *Let $q \le 0$ and (9) be valid. Then for all nonoscillatory solutions either $\lim_{t\to\infty} y(t) = 0$ or $\lim_{t\to\infty} |y(t)| = \infty$.*

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