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PROLONGATION OF PROJECTABLE TANGENT VALUED FORMS

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Abstract. First we deduce some general properties of product preserving bundle functors on the category of fibered manifolds. Then we study the prolongation of projectable tangent valued forms with respect to these functors and describe the complete lift of the Frölicher-Nijenhuis bracket. We also present the coordinate formula for composition of semiholonomic jets.

Recently it has been clarified that the Weil functors represent a unified technique for studying a large class of geometric problems. A survey on the results concerning the product preserving bundle functors on the category $\mathcal{M}f$ of all manifolds and all smooth maps can be found in [6]. Our starting point was a paper by W. Mikulski, [11]. He deduced that the product preserving bundle functors on the category $\mathcal{FM}$ of all fibered manifolds and all fibered morphisms are in bijection with the Weil algebra homomorphisms $\mu : A \to B$. Our main aim is to study the prolongation of projectable tangent valued forms, introduced by L. Mangiarotti and M. Modugno, [10], with respect to such a functor T^μ. In particular, we are interested in the Frölicher-Nijenhuis bracket, which is a powerful tool for the theory of connections, [6], and their torsions, [8]. In the manifold case, such problems were studied in [4] and [1].

In Section 1 we discuss T^μ in the case of product fibered manifolds. Our results represent a basis for coordinate descriptions of T^μ. In Section 2 we study an important special case, the functor $T^{r,s,q}_{k,l}$ of the fibered velocities of dimension (k,l) and order (r,s,q). The coordinate formula for t he prolongation $T^{r,s,q}_{k,l} f$ of a fibered manifold morphism f is reduced to the jet composition. That is why we present a coordinate formula for the composition of jets in the appendix. We start with the semiholonomic case, which reflects the core of the problem. For the

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holonomic case, we obtain another approach to recent results by D. R. Grigore and D. Krupka, [5], M. Kureš, [9] and M. Modugno, [12]. In Section 2 we also deduce that each functor T^μ is dominated by a fibered velocities functor analogously to the manifold case.

Then we describe the natural tensor fields of type $(1,1)$ on Weil bundles. In Section 4 we study the flow prolongation of projectable vector fields in connection with the natural $(1,1)$-tensor fields. On one hand, the flow prolongation of projectable vector fields can be composed with the natural tensor fields determined by the elements of the algebra A. On the other hand, the flow prolongation of vertical vector fields admits an additional operation related to the algebra B. Hence we need three formulae for the bracket of the flow prolongations of vector fields. As the main result of the paper, we then deduce the corresponding three formulae for the Frölicher-Nijenhuis bracket of the complete lifts of projectable tangent valued forms in Proposition 6.

All manifolds and maps are assumed to be infinitely differentiable and all manifolds are paracompact. Unless otherwise specified, we use the terminology and notation from [6].

1. Product preserving bundle functors on $\mathcal{FM}$

First we present one construction of a product preserving bundle functor on $\mathcal{FM}$. Let $\mu : A \to B$ be a Weil algebra homomorphism. By the classical theory, μ induces two bundle functors T^A, T^B on $\mathcal{M}f$ and a natural transformation (denoted by the same symbol) $\mu : T^A \to T^B$, [6], Chapter VIII. For every fibered manifold $p : Y \to M$, we consider $T^B p : T^B Y \to T^B M$. Then we take into account the map $\mu_M : T^A M \to T^B M$ and construct the induced bundle $T^\mu Y = \mu_M^* T^B Y$, which will also be denoted by

$$T^\mu Y = T^A M \times_{T^B M} T^B Y\,. \tag{1}$$

In other words,

$$T^\mu Y = \{(x,y) \in T^A M \times T^B Y, \mu_M(x) = T^B p(y)\}\,. \tag{2}$$

Given another fibered manifold $q : Z \to P$ and an $\mathcal{FM}$-morphism $f : Y \to Z$ over $\underline{f} : M \to P$, we have $T^B f : T^B Y \to T^B Z$ and we construct the induced map $T^\mu f := T^A \underline{f} \times_{T^B \underline{f}} T^B f : T^\mu Y \to T^\mu Z$,

$$T^\mu f(x,y) = (T^A \underline{f}(x), T^B f(y))\,, \qquad (x,y) \in T^\mu Y\,. \tag{3}$$

This defines a bundle functor T^μ on $\mathcal{FM}$ that preserves products.

In general, if we have an $\mathcal{FM}$-morphism $f : Y \to Z$ over $\underline{f} : M \to P$ and we need distinguish the manifold map $f : Y \to Z$ from the $\mathcal{FM}$-morphism itself, we write $(f, \underline{f})$ for the latter. In [11], W. Mikulski clarified that every product preserving bundle functor F on $\mathcal{FM}$ is of the above form. Let pt denote one element manifold and $pt_M : M \to pt$ the unique map. There are two canonical functors $i_1, i_2 : \mathcal{M}f \to \mathcal{FM}$ defined by $i_1 M = (\mathrm{id}_M : M \to M)$, $i_1 f = (f, f)$, $i_2 M = (pt_M : M \to pt)$, $i_2 f = (f, \mathrm{id}_{pt})$ and a natural transformation $t : i_1 \to i_2$, $t_M = (\mathrm{id}_M, pt_M) : i_1 M \to i_2 M$. Applying F, we obtain two product preserving bundle functors $F \circ i_1$, $F \circ i_2$ on $\mathcal{M}f$ and a natural transformation $F \circ t : F \circ i_1 \to F \circ i_2$. By the Weil theory, there exists a Weil algebra homomorphism $\mu : A \to B$ such that $F \circ i_1 = T^A$, $F \circ i_2 = T^B$, $F \circ t = \mu$. Then $F = T^\mu$, [11] (see also [2] for a simplified proof).

If we have a product fibered manifold $Y = M \times N$, it coincides with the product $Y = i_1 M \times i_2 N$ in $\mathcal{FM}$. This implies directly

$$T^\mu(M \times N) = T^A M \times T^B N\,. \tag{4}$$

In the form (2), we have

$$T^\mu(M \times N) = \{(x, v) \in T^A M \times T^B(M \times N), \mu_M(x) = pr_1(v)\}$$

where $T^B(M \times N) = T^B M \times T^B N$. If we write $v = (u, y)$, we obtain

$$T^\mu(M \times N) = \{(x, \mu_M(x), y)\} \approx T^A M \times T^B N\,. \tag{5}$$

Given another product fibered manifold $Z = P \times Q$, every $\mathcal{FM}$-morphism $f : Y \to Z$ is identified with a pair $f = (f_1, f_2)$, $f_1 : M \to P$, $f_2 : M \times N \to Q$,

$$f(x, y) = (f_1(x), f_2(x, y))\,.$$

Then $T^A f_1 : T^A M \to T^A P$ and $T^B f_2 : T^B M \times T^B N \to T^B Q$. The following assertion describes T^μ in the case of product fibered manifolds.

Proposition 1. *We have*

$$T^\mu f = (T^A f_1, T^B f_2 \circ (\mu_M \times \mathrm{id}_{T^B N}))\,. \tag{6}$$

Proof. By (3) and (5),

$$T^\mu f(x, \mu_M(x), y) = (T^A f_1(x), T^B f_1(\mu_M(x)), T^B f_2(\mu_M(x), y))\,.$$

The naturality of μ on $f_1 : M \to P$ yields $T^B f_1(\mu_M(x)) = \mu_P(T^A f_1(x))$. □

In particular, consider a function $f : Y \to \mathbb{R}$. It can be interpreted as an $\mathcal{FM}$-morphism $Y \to i_2\mathbb{R}$, so that $T^\mu f : T^\mu Y \to B$. If $Y = M \times N$, then $T^\mu f = T^B f \circ (\mu_M \times \mathrm{id}_{T^B N})$.

2. Velocities in the fibered case

Given two manifolds M, S and a smooth map $f : M \to S$, we can construct the r-jet $j^r_x f$ at $x \in M$. If we replace M by a fibered manifold $p : Y \to M$, we can require a higher order contact along the fiber Y_x passing through $y \in Y$, $x = p(y)$. Thus, for two maps $f, g : Y \to S$ and two integers $s \geqslant r$ we define $j^{r,s}_y f = j^{r,s}_y g$ by

$$j^r_y f = j^r_y g \quad \text{and} \quad j^s_y(f|Y_x) = j^s_y(f|Y_x)\,. \tag{7}$$

The space of all such (r,s)-jets is denoted by $J^{r,s}(Y,S)$.

Write $\mathbb{R}^{k,l} = (p_{k,l} : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k)$ for the product fibered manifold. Analogously to the classical functor T^r_k of (k,r)-velocities, we introduce

$$T^{r,s}_{k,l} S = J^{r,s}_{0,0}(\mathbb{R}^{k,l}, S)\,,\ T^{r,s}_{k,l} f(j^{r,s}_{0,0} g) = j^{r,s}_{0,0}(f \circ g)$$

for every manifold S and every map $f : S \to \overline{S}$. Hence $T^{r,s}_{k,l}$ is a bundle functor on $\mathcal{M}f$ that preserves products.

In general, we have a natural transformation $\varrho^l : T^r_h \to T^r_l$, $h \geqslant l$, defined as follows. Consider the injection $\mathbb{R}^l \hookrightarrow \mathbb{R}^h$, $(x^1, \dots, x^l) \mapsto (0, \dots, 0, x^1, \dots, x^l)$. Then we define

$$\varrho^l_M(j^r_0 \varphi) = j^r_0(\varphi|\mathbb{R}^l)\,, \quad \varphi : \mathbb{R}^h \to S\,.$$

On the other hand, we have the jet projection $T^s_l S \to T^r_l S$, $s \geqslant r$. Clearly,

$$T^{r,s}_{k,l} S = T^r_{k+l} S \times_{T^r_l S} T^s_l S \tag{8}$$

and $T^{r,s}_{k,l} f = T^r_{k+l} f \times_{T^r_l f} T^s_l f$. Write α for a multiindex of range $1, \dots, k$ and β for a multiindex of range $k+1, \dots, k+l$. Thus, if y^p are some local coordinates on S, the induced coordinates on $T^{r,s}_{k,l} S$ are

$$y^p_{\alpha\beta},\ |\alpha| > 0,\ |\alpha| + |\beta| \le r \quad \text{and} \quad y^p_\beta,\ |\beta| \le s\,. \tag{9}$$

Having two $\mathcal{FM}$-morphisms $f, g : Y \to Z$, we can require a higher order contact of the base maps in addition to (7). Hence for $s \ge r \le q$ we define

$$j^{r,s,q}_y f = j^{r,s,q}_y g$$

by (7) and $j^q_x \underline{f} = j^q_x \underline{g}$. We write $J^{r,s,q}(Y,Z)$ for the space of all (r,s,q)-jets from Y to Z. Then we introduce the space of fibered velocities of dimension (k,l) and order (r,s,q) by

$$T^{r,s,q}_{k,l} Y = J^{r,s,q}_{0,0}(\mathbb{R}^{k,l}, Y)\,.$$

Clearly, $T^{r,s,q}_{k,l}$ is a product preserving bundle functor on $\mathcal{FM}$.

We are going to describe $T^{r,s,q}_{k,l}$ in the product form of (1) and (6). Clearly, $T^{r,s,q}_{k,l} \circ i_1 = T^q_k$. The Weil algebra of T^q_k is

$$\mathbb{D}^q_k = \mathbb{R}[x_1, \dots, x_k]/\mathfrak{m}(k)^{q+1}\,, \tag{10}$$

where $\mathfrak{m}(k) = \langle x_1, \dots, x_k \rangle$ is the maximal ideal in the algebra $\mathbb{R}[x_1, \dots, x_k]$. On the other hand, $T^{r,s,q}_{k,l} \circ i_2 = T^{r,s}_{k,l}$. By [2], the Weil algebra of $T^{r,s}_{k,l}$ is

$$\mathbb{D}^{r,s}_{k,l} = \mathbb{R}[x_1, \dots, x_{k+l}] \Big/ \langle \mathfrak{m}(k+l)^{s+1}, \mathfrak{m}(k)\mathfrak{m}(k+l)^r \rangle . \tag{11}$$

Write $\nu = T^{r,s,q}_{k,l} \circ t$ for the natural transformation in question. An $\mathcal{FM}$-morphism $\varphi : \mathbb{R}^{k,l} \to i_1 S$ is of the form $(\varphi \circ p_{k,l}, \varphi)$, $\varphi : \mathbb{R}^k \to S$. Then $t_S \circ \varphi = (\varphi \circ p_{k,l}, pt_{\mathbb{R}^{k,l}})$. This implies

$$\nu_S(j^q_0 \varphi) = j^{r,s}_{0,0}(\varphi \circ p_{k,l}) . \tag{12}$$

Since $\varphi \circ p_{k,l}$ is constant along each fiber of $\mathbb{R}^{k,l}$, this construction is independent of $s \geqslant r$. One verifies directly that the algebra form of ν is determined by the canonical injection

$$\mathbb{R}[x_1, \dots, x_k] \hookrightarrow \mathbb{R}[x_1, \dots, x_k, x_{k+1}, \dots, x_{k+l}] .$$

For the product fibered manifold $M \times N$, we have $T^{r,s,q}_{k,l}(M \times N) = T^q_k M \times T^{r,s}_{k,l} N$. If x^i are some local coordinates on N, then the induced coordinates on $T^{r,s,q}_{k,l}(M \times N)$ are

$$x^i_\alpha \,, \ |\alpha| \le q \,, \ y^p_{\alpha\beta} \,, \ |\alpha| > 0 \,, \ |\alpha| + |\beta| \le r \,, \ y^p_\beta \,, |\beta| \le s . \tag{13}$$

For every $\mathcal{FM}$-morphism $f : Y \to Z$, we have $T^{r,s,q}_{k,l} f = T^q_k \underline{f} \times_{T^{r,s}_{k,l} \underline{f}} T^{r,s}_{k,l} f$ and $T^{r,s}_{k,l} f = T^r_{k+l} f \times_{T^r_l f} T^s_l f$. Hence $T^{r,s,q}_{k,l} f$ is expressed in terms of the jet composition. We present a coordinate expression for the composition of jets in the appendix.

In the manifold case, every Weil bundle T^A is dominated by a velocities bundle, i.e. there exists a (k,q)-velocities bundle T^q_k and a surjective natural transformation $\tau : T^q_k \to T^A$. Indeed, let N_A be the nilpotent ideal of A. The number $w(A) = \dim(N_A / N_A^2)$ is called the width of A and the minimum $\operatorname{ord} A$ of the integers satisfying $N_A^{p+1} = 0$ is called the order of A. If we take k elements $e_1, \dots, e_k \in N_A$ such that their projections into N_A/N_A^2 form a basis of this vector space, then $e_1, \dots, e_k$ determine a surjective algebra homomorphism $\mathbb{D}^q_k \to A$, where q is the order of A.

We are going to deduce a similar result for the fibered case. Let B be another Weil algebra with nilpotent ideal N_B, $\operatorname{ord} B = s$. Every algebra homomorphism $\mu : A \to B$ induces a linear map

$$\mu_1 : N_A/N_A^2 \to N_B/N_B^2 . \tag{14}$$

Define $w(\mu) := w(B) + \dim \operatorname{Ker} \mu_1$.

Our problem requires the following general concept.

Definition 1. The smallest integer r satisfying

$$\mu(N_A)N_B^r = 0 \tag{15}$$

is called the order of μ.

In other words, $r = \operatorname{ord}\mu$ is characterized by

$$\mu(a)b_1 \dots b_r = 0 \qquad \text{for all} \;\; a \in N_A, b_1, \dots, b_r \in N_B\,.$$

Since $\mu(a) \in N_B$, we have $\operatorname{ord}\mu \leq \operatorname{ord} B$.

Proposition 2. *For every product preserving bundle functor T^μ on $\mathcal{FM}$ there exists a velocities functor $T^{r,s,q}_{k,l}$ and a surjective natural transformation $\tau : T^{r,s,q}_{k,l} \to T^\mu$, where*

$$k = w(A),\;\; k+l = w(\mu),\;\; s = \operatorname{ord} B,\;\; r = \operatorname{ord}\mu,\;\; q = \max(\operatorname{ord}\mu, \operatorname{ord} A)\,. \tag{16}$$

Proof. Take $e_1, \dots, e_k \in N_A$ such that their images in N_A/N_A^2 form a basis. This determines a surjective homomorphism $\tau_1 : \mathbb{D}^q_k \to A$. Further, take some elements $e_{k+1}, \dots, e_{k+l} \in N_B$ with the property that their images in N_B/N_B^2 together with the images of $\mu(e_1), \dots, \mu(e_k)$ generate N_B/N_B^2 as a vector space. The elements $e_1, \dots, e_{k+l}$ determine a surjective homomorphism $\tau_2 : \mathbb{D}^s_{k+l} \to B$. Since μ has order r and $\mathbb{D}^{r,s}_{k,l}$ is of the form (11), τ_2 factorizes through a map (denoted by the same symbol) $\mathbb{D}^{r,s}_{k,l} \to B$. By the construction of τ_1 and τ_2 the following diagram is commutative in the case $q \geqslant \operatorname{ord}\mu$

$$\begin{tikzcd}
\mathbb{D}^q_k \arrow[r, "\nu"] \arrow[d, "\tau_1"'] & \mathbb{D}^{r,s}_{k,l} \arrow[d, "\tau_2"] \\
A \arrow[r, "\mu"] & B
\end{tikzcd} \tag{17}$$

By the general result of W. Mikulski, [11], the pair τ_1, τ_2 determines a surjective natural transformation $\tau : T^{r,s,q}_{k,l} \to T^\mu$. □

Remark 1. We recall that a bundle functor F on $\mathcal{FM}$ is said to be of order (r,s,q), $s \geq r \leq q$, [7], if $j^{r,s,q}_y f = j^{r,s,q}_y g$ implies $Ff|F_yY = Fg|F_yY$. (We do not assume the values of r,s,q are minimal.) Clearly, if G is another bundle functor on $\mathcal{FM}$ and $\tau : F \to G$ is a surjective natural transformation, then G has also the order (r,s,q). Hence Proposition 2 characterizes the order of T^μ in an algebraic way.

3. Natural tensor fields of type (1,1)

In the case of one Weil algebra A, we have a canonical isomorphism $\varkappa_M : T^A TM \to TT^A M$, [6]. Every $a \in A$ determines a $(1,1)$-tensor field $L(a)_M : TT^A M \to TT^A M$ as follows. The multiplication of the tangent vectors by reals is a map $m_M : \mathbb{R} \times TM \to TM$. Applying the functor T^A, we obtain $T^A m_M : A \times T^A TM \to T^A TM$. Then

$$\mathcal{T}^A m_M := \varkappa_M \circ T^A m_M \circ (\mathrm{id}_A \times \varkappa_M^{-1}) : A \times TT^A M \to TT^A M \tag{18}$$

and we define $L(a)_M = \mathcal{T}^A m_M(a, -)$. Since the multiplication in A is deduced from the multiplication of reals, we have

$$L(a_1)_M \circ L(a_2)_M = L(a_1 a_2)_M \,, \qquad a_1, a_2 \in A \,. \tag{19}$$

In the case of $\mu : A \to B$, the tangent bundle of $T^\mu Y = T^A M \times_{T^B M} T^B Y$ is

$$TT^\mu Y = TT^A M \times_{TT^B M} TT^B Y \,.$$

For every $a \in A$, we have a natural $(1,1)$-tensor field $\lambda(a)_Y$ on $T^\mu Y$ defined by

$$\lambda(a)_Y(U_1, U_2) = (L(a)_M(U_1), L(\mu(a))_Y(U_2)), \qquad (U_1, U_2) \in TT^\mu Y \,, \tag{20}$$

see [13]. By (19) we obtain

$$\lambda(a_1 a_2)_Y = \lambda(a_1)_Y \circ \lambda(a_2)_Y \,, \qquad a_1, a_2 \in A \,. \tag{21}$$

(We remark that Tomáš deduced in [13] that all natural $(1,1)$-tensor fields on $T^\mu Y$ are of the form (20).)

The vertical tangent bundle $V(T^\mu Y \to T^A M)$ is the space of all pairs $(U_1, U_2) \in TT^A M \times_{TT^B M} TT^B Y$, where U_1 is the zero vector. Hence the elements of $V(T^\mu Y \to T^A M)$ are of the form (x, U), $x \in T^A M$, $U \in T_y T^B Y$, $\mu_M(x) = T^B p(y)$, $TT^B p(U) = 0$. By construction, the $(1,1)$-tensor fields $L(b)_Y$ and $L(b)_M$ are $T^B p$-related for all $b \in B$. In particular, $TT^B p(U) = 0$ implies $TT^B p(L(b)_Y(U)) = 0$. Hence the rule

$$\widetilde{L}(b)_Y(x, U) = (x, L(b)_Y(U)) \tag{22}$$

defines a natural map $\widetilde{L}(b)_Y : V(T^\mu Y \to T^A M) \to V(T^\mu Y \to T^A M)$ over the identity of $V(T^\mu Y \to T^A M)$. By (19), we obtain directly

$$\widetilde{L}(b_1)_Y \circ \widetilde{L}(b_2)_Y = \widetilde{L}(b_1 b_2)_Y \,. \tag{23}$$

4. Prolongation of vector fields

In general, let $p : Y \to M$ be a fibered manifold and $\varphi : Q \to M$ be a map. Then

$$Q \times_\varphi Y = \{q \in Q, y \in Y, p(y) = \varphi(q)\}$$

is a fibered manifold $\pi : Q \times_\varphi Y \to M$, $\pi(y, q) = p(y)$. We have $Tp : TY \to TM$, $T\varphi : TQ \to TM$ and the tangent bundle of $Q \times_\varphi Y$ is of the form

$$T(Q \times_\varphi Y) = TQ \times_{T\varphi} TY\,.$$

Consider a projectable vector field X on Y over $\underline{X}$ on M and a vector field U on Q that is φ-related with $\underline{X}$. Then the product vector field $U \times X$ on $Q \times Y$ is restrictible to $Q \times_\varphi Y$. The restriction is denoted by $U \times_{\underline{X}} X$ and is called the fibered product of U and X. Clearly, $U \times_{\underline{X}} X$ is a projectable vector field on $\pi : Q \times_\varphi Y \to M$ over $\underline{X}$.

Consider now the functor T^μ and a projectable vector field X on Y over $\underline{X}$ on M. Then the flow prolongation $\mathcal{T}^A\underline{X}$ is a vector field on T^AM that is μ_M-related with the flow prolongation $\mathcal{T}^B\underline{X}$ on T^BM. On the other hand, the flow prolongation $\mathcal{T}^B X$ is a projectable vector field on T^BY over $\mathcal{T}^B\underline{X}$ on T^BM. Hence we have defined the vector field $\mathcal{T}^A\underline{X} \times_{\mathcal{T}^B\underline{X}} \mathcal{T}^B X$. By construction, this vector field coincides with the flow prolongation $\mathcal{T}^\mu X$, i.e.

$$\mathcal{T}^\mu X = \mathcal{T}^A\underline{X} \times_{\mathcal{T}^B\underline{X}} \mathcal{T}^B X\,.$$

We recall that the flow prolongation preserves bracket of vector fields, [6].

The canonical isomorphisms $\varkappa^A_M : T^ATM \to TT^AM$ and $\varkappa^B_Y : T^BTY \to TT^BY$ induce a canonical isomorphism $\varkappa^\mu_Y : T^\mu TY \to TT^\mu Y$. The above construction implies that the functorial prolongation $T^\mu X : T^\mu Y \to T^\mu TY$ of the $\mathcal{FM}$-morphism $X : (Y \to M) \to (TY \to TM)$ satisfies

$$\mathcal{T}^\mu X = \varkappa^\mu_Y \circ T^\mu X\,.$$

For every $a \in A$, the composition $\lambda(a)\mathcal{T}^\mu X$ of $\lambda(a)_Y$ and $\mathcal{T}^\mu X$ is also a projectable vector field on $T^\mu Y$. From the manifold case, [1], we obtain directly

Proposition 3. *For every pair X_1, X_2 of projectable vector fields on Y and every a_1, $a_2 \in A$, we have*

$$[\lambda(a_1)\mathcal{T}^\mu X_1, \lambda(a_2)\mathcal{T}^\mu X_2] = \lambda(a_1a_2)\mathcal{T}^\mu([X_1, X_2])\,. \qquad \square$$

If $W : Y \to VY$ is a vertical field on Y, then $\mathcal{T}^\mu W$ is a vertical vector field on $T^\mu Y \to T^AM$ and the composition $\widetilde{L}(b)\mathcal{T}^\mu W$ of $\widetilde{L}(b)_Y$ and $\mathcal{T}^\mu W$ is defined. The manifold result implies directly

Proposition 4. *For every pair W_1, W_2 of vertical vector fields on Y and every b_1, $b_2 \in B$, we have*

$$[\widetilde{L}(b_1)\mathcal{T}^\mu W_1, \widetilde{L}(b_2)\mathcal{T}^\mu W_2] = \widetilde{L}(b_1 b_2)\mathcal{T}^\mu([W_1, W_2]) . \qquad \square$$

If X is projectable and W is vertical, the bracket $[X, W]$ is a vertical vector field. To deduce a result analogous to Propositions 3 and 4, we need some lemmas. By Section 1, given a function $f : Y \to \mathbb{R}$, $T^\mu f : \mathcal{T}^\mu Y \to B$ is a vector valued function. So its derivative with respect to any vector field on $T^\mu Y$ is also a B-valued function on $T^\mu Y$.

Lemma 1. *For every projectable vector field X on Y and every function $f : Y \to \mathbb{R}$, we have $\mathcal{T}^\mu X(T^\mu f) = T^\mu(Xf)$.*

Proof. Let $\pi_\mathbb{R} : T\mathbb{R} = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the second projection. The derivative Xf can be expressed as $Xf = \pi_\mathbb{R} \circ Tf \circ X$. By functoriality,

$$T^\mu(Xf) = T^\mu \pi_\mathbb{R} \circ T^\mu T f \circ T^\mu X .$$

But $\mathcal{T}^\mu X = \varkappa_Y^\mu \circ T^\mu X$ and $T^\mu T f \circ (\varkappa_Y^\mu)^{-1} = (\varkappa_\mathbb{R}^B)^{-1} \circ T T^\mu f$ by naturality of $\varkappa$. We have $\varkappa_\mathbb{R}^B : T^B T\mathbb{R} \to T T^B \mathbb{R}$ and $T^\mu \pi_\mathbb{R} \circ \varkappa_\mathbb{R}^B$ is the second projection of $TB = B \times B$. $\square$

Given $b \in B$ we define $bT^\mu f : T^\mu Y \to B$ by using the multiplication in B.

Lemma 2. *We have $\mathcal{T}^\mu X(bT^\mu f) = bT^\mu(Xf)$ for all $b \in B$.*

Proof. Obviously, $X(kf) = k(Xf)$ for all $k \in i_2\mathbb{R}$. Applying T^μ and using Lemma 1, we obtain our claim. $\square$

Lemma 3. *For every vertical field W on Y, we have*

$$(\widetilde{L}(b)\mathcal{T}^\mu W)T^\mu f = \mathcal{T}^\mu W(bT^\mu f) \quad \textit{for all} \quad b \in B .$$

Proof. Clearly, $(kW)f = W(kf)$ for all $k \in i_2\mathbb{R}$. Applying T^μ and using Lemma 1, we obtain our claim. $\square$

Lemma 4. *For every projectable vector field X on Y, we have*

$$(\lambda(a)\mathcal{T}^\mu X)T^\mu f = \mu(a)T^\mu(Xf) .$$

Proof. Obviously, $(kX)f = k(Xf)$, where $k \in i_1\mathbb{R}$ on the left hand side and $k \in i_2\mathbb{R}$ on the right hand side. Then we apply $\mathcal{T}^\mu$ and use the fact that $\mu = T^\mu t_\mathbb{R} : A \to B$. $\square$

Lemma 5. *If two vertical vector fields U and $\overline{U}$ on $T^\mu Y \to T^A M$ satisfy $U(bT^\mu f) = \overline{U}(bT^\mu f)$ for all $f : Y \to \mathbb{R}$ and all $b \in B$, then $U = \overline{U}$.*

Proof. It suffices to consider the case $Y = M \times N$ and such functions $\widetilde{f} : M \times N \to \mathbb{R}$ that are of the form $f \circ p_2$, where $f : N \to \mathbb{R}$ and $p_2 : M \times N \to N$ is the second product projection. Then $T^\mu \widetilde{f} = T^B f \circ \pi_2$, where $\pi_2 : T^A M \times T^B N \to T^B N$ is the second product projection. Since $U(\mathcal{T}^\mu \widetilde{f})$ is constructed fiberwise, we can apply Lemma 2 from [1] to each fiber of the product fibered manifold $T^A M \times T^B N$. This proves our claim. □

Proposition 5. *For every projectable vector field X on Y and every vertical vector field W on Y, we have*

$$[\lambda(a)\mathcal{T}^\mu X, \widetilde{L}(b)\mathcal{T}^\mu W] = \widetilde{L}(\mu(a)b)\mathcal{T}^\mu([X,W]) \quad \textit{for all} \quad a \in A, b \in B\,.$$

Proof. Take any $f : Y \to \mathbb{R}$ and $c \in B$. Using Lemmas 2–4, we find

$$\begin{aligned}
&[\lambda(a)\ \mathcal{T}^\mu X, \widetilde{L}(b)\mathcal{T}^\mu W](cT^\mu f) = \lambda(a)\mathcal{T}^\mu X(\widetilde{L}(b)\mathcal{T}^\mu W)(cT^\mu f) \\
&\quad - \widetilde{L}(b)\mathcal{T}^\mu W(\lambda(a)\mathcal{T}^\mu X)(cT^\mu f) = \lambda(a)T^\mu X(bcT^\mu(Wf)) \\
&\quad - \widetilde{L}(b)\mathcal{T}^\mu W(\mu(a)cT^\mu(Xf)) = \mu(a)bc(T^\mu(XWf) - T^\mu(WXf)) \\
&= \mu(a)bcT^\mu([X,W]f) = \widetilde{L}(\mu(a)b)\mathcal{T}^\mu([X,W])(cT^\mu f)\,.
\end{aligned}$$

Then our claim follows from Lemma 5. □

5. Projectable tangent valued forms

A tensor field D of type $(1,k)$ on Y can be interpreted as a map

$$D : TY \times_Y \cdots \times_Y TY \to TY\,.$$

We say that D is projectable, if there is a tensor field $\underline{D}$ of type $(1,k)$ on M such that the following diagram commutes

$$\begin{array}{ccc}
TY \times_Y \cdots \times_Y TY & \xrightarrow{\ D\ } & TY \\
\Big\downarrow Tp \qquad\qquad \Big\downarrow Tp & & \Big\downarrow Tp \\
TM \times_M \cdots \times_M TM & \xrightarrow{\ \underline{D}\ } & TM
\end{array} \tag{24}$$

A projectable D is said to be vertical valued, if the values of (24) lie in the vertical tangent bundle VY, i.e. $\underline{D}$ is the zero tensor field.

An antisymmetric projectable $(1,k)$-tensor field is called a projectable tangent valued k-form on Y, [10].

To construct the induced $(1,k)$-tensor field on $T^\mu Y$, we proceed analogously to the manifold case, [1], [4]. Applying T^μ to (24), we obtain

$$\begin{array}{ccc} T^\mu TY \times_{T^\mu Y} \cdots \times_{T^\mu Y} T^\mu TY & \xrightarrow{T^\mu D} & T^\mu TY \\ \Big\downarrow T^\mu Tp \quad\quad T^\mu Tp \Big\downarrow & & \Big\downarrow T^\mu Tp \\ T^A TM \times_{T^A M} \cdots \times_{T^A M} T^A TM & \xrightarrow{T^A \underline{D}} & T^A TM \end{array} \tag{25}$$

If we add the canonical isomorphisms $\varkappa_M^A$ and $\varkappa_Y^\mu$, we obtain a projectable $(1,k)$-tensor field $\mathcal{T}^\mu D$ on $T^\mu Y$ over the $(1,k)$-tensor field $\mathcal{T}^A \underline{D}$, cf. [1], [4].

Definition 2. The $(1,k)$-tensor field $\mathcal{T}^\mu D$ is called the complete lift of D.

In the case $k=0$, D is a vector field and $\mathcal{T}^\mu D$ coincides with its flow prolongation.

Our main aim is to describe the Frölicher-Nijenhuis bracket of two projectable tangent valued forms on Y in a way similar to the manifold case, [1]. We need some lemmas.

Lemma 6. *Let C and $\overline{C}$ be two projectable $(1,k)$-tensor fields on $T^\mu Y$. If they coincide on all vector fields of the form $\lambda(a)\mathcal{T}^\mu X$ and $\widetilde{L}(b)\mathcal{T}^\mu W$, where X is a projectable vector field on Y, W is a vertical vector field on Y and $a \in A$, $b \in B$, then $C = \overline{C}$.*

Proof. It suffices to consider the case $Y = \mathbb{R}^m \times \mathbb{R}^n$. Then $T^\mu Y = T^A\mathbb{R}^m \times T^B\mathbb{R}^n = A^m \times B^n$. Let $1, u_1, \dots, u_a$ or $1, v_1, \dots, v_b$ be a basis in A or B with nilpotent u's or v's and $x^i, z_1^i, \dots, z_a^i$ or $y^p, w_1^p, \dots, w_b^p$ be the induced coordinates on $T^A\mathbb{R}^m$ or $T^B\mathbb{R}^n$, respectively. Consider a constant vector field $W = \eta^p \frac{\partial}{\partial y^p}$. Since its flow is formed by translations, we have $\mathcal{T}^\mu W = \eta^p \frac{\partial}{\partial y^p} + 0$. Then $\widetilde{L}(v_d)\mathcal{T}^\mu W = \eta^p \frac{\partial}{\partial w_d^p}$, $d = 1, \dots, b$. Similarly, if we consider a constant vector field $X = \xi^i \frac{\partial}{\partial x^i}$, we have $\mathcal{T}^\mu X = \xi^i \frac{\partial}{\partial x^i} + 0$. Then $\lambda(u_c)\mathcal{T}^\mu X = \xi^i \frac{\partial}{\partial z_c^i} + 0$, $c = 1, \dots, a$. Since ξ^i and η^p are arbitrary, this implies the coordinate form of our claim. □

The manifold case, [1], [4], implies directly

Lemma 7. *Let D be a projectable $(1,k)$-tensor field on Y. Then*

$$\mathcal{T}^\mu D(\lambda(a_1)\mathcal{T}^\mu X_1, \dots, \lambda(a_k)\mathcal{T}^\mu X_k) = \lambda(a_1 \dots a_k)\mathcal{T}^\mu(D(X_1, \dots, X_k))$$

for every projectable vector fields $X_1, \dots, X_k$ on Y and every $a_1, \dots, a_k \in A$. □

If at least one of the vector fields $X_1, \dots X_k$ is vertical, then $D(X_1, \dots, X_k)$ is also a vertical vector field on Y.

Lemma 8. *Let D be a projectable $(1,k)$-tensor field on Y, $X_1,\dots,X_s$ be projectable vector fields on Y and $W_{s+1},\dots,W_k$ be vertical vector fields on Y, $s<k$. Then*

$$\begin{aligned}(26)\quad &\mathcal{T}^\mu D(\lambda(a_1)\mathcal{T}^\mu X_1,\dots,\lambda(a_s)\mathcal{T}^\mu X_s,\widetilde{L}(b_{s+1})\mathcal{T}^\mu W_{s+1},\dots,\widetilde{L}(b_k)\mathcal{T}^\mu W_k)\\ &\quad=\widetilde{L}(\mu(a_1)\dots\mu(a_s)b_{s+1}\dots b_k)\mathcal{T}^\mu(D(X_1,\dots,X_s,W_{s+1},\dots,W_k)).\end{aligned}$$

Proof. We have $D(c_1X_1,\dots,c_sX_s,c_{s+1}W_{s+1},\dots,c_kW_k) = (c_1\dots c_k)D(X_1,\dots,X_s,W_{s+1},\dots,W_k)$, where $c_1,\dots,c_s\in i_1\mathbb{R}$ and $c_{s+1},\dots,c_k,(c_1\dots c_k)\in i_2\mathbb{R}$. Applying $\mathcal{T}^\mu$, we obtain (26) analogously to the proof of Lemma 4. □

The Frölicher-Nijenhuis bracket of a projectable tangent valued k-form P and a projectable tangent valued l-form Q is a projectable tangent valued $(k+l)$-form $[P,Q]$, [10].

Proposition 6. *For the Frölicher-Nijenhuis bracket of two projectable tangent valued forms P and Q on Y, we have*

$$(27)\qquad [\lambda(a)\mathcal{T}^\mu P,\lambda(a')\mathcal{T}^\mu Q]=\lambda(aa')\mathcal{T}^\mu([P,Q]),\qquad a,a'\in A.$$

If Q is vertical valued, then

$$(28)\qquad [\lambda(a)\mathcal{T}^\mu P,\widetilde{L}(b)\mathcal{T}^\mu Q]=\widetilde{L}(\mu(a)b)\mathcal{T}^\mu([P,Q]),\qquad a\in A, b\in B.$$

If both P and Q are vertical valued, then

$$(29)\qquad [\widetilde{L}(b)\mathcal{T}^\mu P,\widetilde{L}(b')\mathcal{T}^\mu Q]=\widetilde{L}(bb')\mathcal{T}^\mu([P,Q]),\qquad b,b'\in B.$$

Proof. L. Mangiarotti and M. Modugno, [10], deduced the following expression of $[P,Q]$ in terms of the bracket of projectable vector fields

$$\begin{aligned}&[P,Q](X_1,\dots,X_{k+l})\\ &\quad=\frac{1}{k!l!}\sum_\sigma \operatorname{sign}\sigma\,[P(X_{\sigma1},\dots,X_{\sigma k}),Q(X_{\sigma(k+1)},\dots,X_{\sigma(k+l)})]\\ &\quad+\frac{-1}{k!(l-1)!}\sum_\sigma \operatorname{sign}\sigma\,Q([P(X_{\sigma1},\dots,X_{\sigma k}),X_{\sigma(k+1)}],X_{\sigma(k+2)},\dots)\\ (30)\quad&\quad+\frac{(-1)^{kl}}{(k-1)!l!}\sum_\sigma \operatorname{sign}\sigma P([Q(X_{\sigma1},\dots X_{\sigma l}),X_{\sigma(l+1)}],X_{\sigma(l+2)},\dots)\\ &\quad+\frac{(-1)^{k-1}}{(k-1)!(l-1)!2}\sum_\sigma \operatorname{sign}\sigma\,Q(P([X_{\sigma1},X_{\sigma2}],X_{\sigma3},\dots],X_{\sigma(k+2)},\dots)\\ &\quad+\frac{(-1)^{(k-1)l}}{(k-1)!(l-1)!2}\sum_\sigma \operatorname{sign}\sigma P(Q([X_{\sigma1},X_{\sigma2}],X_{\sigma3},\dots],X_{\sigma(l+2)},\dots)\end{aligned}$$

In all three cases, we express the value of the right hand side on $\lambda(a_1)\mathcal{T}^\mu X_1$, $\dots,\lambda(a_s)\mathcal{T}^\mu X_s, \widetilde{L}(b_1)\mathcal{T}^\mu W_1,\dots,\widetilde{L}(b_h)\mathcal{T}^\mu W_h$, $s+h=k+l$. In the first case, if $h=0$, we use Proposition 3 and Lemma 7 and deduce that each term is equal to the value of $\mathcal{T}^\mu$ on the corresponding term of (30) multiplied by $\lambda(aa'a_1\dots a_{k+l})$. For $h>0$, Propositions 3–5 and Lemmas 7 and 8 imply that the multiplication factor is $\widetilde{L}(\mu(aa'a_1\dots a_s)b_1\dots b_h)$. Using Lemma 6, we obtain (27). In the second and third cases we proceed in the same way. □

APPENDIX: Jet composition in coordinates

We realized in Section 2 that the coordinate formula for $T^{r,s,q}_{k,l}f$ is reduced to the coordinate formula for the composition of jets. We deduce the coordinate composition formula for the semiholonomic jets and we discuss its special form in the holonomic case.

Let M and N be two manifolds. The space of non-holonomic r-jets $\widetilde{J}^r(M,N)$ is defined by the induction $\widetilde{J}^1(M,N)=J^1(M,N)$ and $\widetilde{J}^r(M,N)$ is the first jet prolongation of the fibered manifold $\alpha:\widetilde{J}^{r-1}(M,N)\to M$, where α is the source jet projection. In other words, the elements of $\widetilde{J}^r(M,N)$ are of the form $j^1_x\sigma$, where σ is a local map $M\to\widetilde{J}^{r-1}(M,N)$ satisfying $\alpha\circ\sigma=\mathrm{id}_M$. Let P be another manifold. The composition $B\circ A\in\widetilde{J}^r_x(M,P)_z$ of $A\in\widetilde{J}^r_x(M,N)_y$, $A=j^1_x\sigma(u)$ and $B\in\widetilde{J}^r_y(N,P)_z$, $B=j^1_y\varrho(v)$ is defined by the following induction. Let β denote the target jet projection. Then $\beta\circ\sigma$ is a local map of M into N and $\sigma(u)$ and $\varrho(\beta(\sigma(u)))$ are composable non-holonomic $(r-1)$-jets. Then one defines

$$B\circ A=j^1_x(\varrho(\beta(\sigma(u)))\circ\sigma(u)) \tag{31}$$

with the composition of non-holonomic $(r-1)$-jets on the right hand side, [3].

The inclusion $J^r(M,N)\subset\widetilde{J}^r(M,N)$ is defined by $j^r_xf\mapsto j^1_x(j^{r-1}f)$ and (31) coincides with the composition of holonomic jets. The subspace of semiholonomic r-jets $\overline{J}^r(M,N)\subset\widetilde{J}^r(M,N)$ is defined by the following induction. An element $j^1_x\sigma\in\widetilde{J}^r(M,N)$ is said to be semiholonomic, if

(i) σ is a local section $M\to\overline{J}^{r-1}(M,N)$,
(ii) σ satisfies $\sigma(x)=j^1_x(\pi^{r-1}_{r-2}\circ\sigma)$,

where $\pi^{r-1}_{r-2}:\overline{J}^{r-1}(M,N)\to\overline{J}^{r-2}(M,N)$ is the canonical projection. The composition of two semiholonomic jets is semiholonomic as well. The coordinates of an element $A\in\overline{J}^r_0(\mathbb{R}^m,\mathbb{R}^n)_0$ are

$$a^p_i,\ a^p_{ij},\dots,a^p_{i_1\dots i_r} \tag{32}$$

that are arbitrary in all subscripts. We have $J^r(M,N)\subset\overline{J}^r(M,N)$ and this inclusion is characterized by symmetry in all subscripts.

Consider $B\in\overline{J}^r_0(\mathbb{R}^n,\mathbb{R}^p)_0$ with coordinates

$$b^a_p,\ b^a_{pq},\dots,b^a_{p_1\dots p_r} \tag{33}$$

Write $C=B\circ A$, $C=(c^a_i,c^a_{ij},\dots,c^a_{i_1\dots i_r})$.

Proposition 7. *For every $s \le r$, consider the set $Q(r,s)$ of all ordered decompositions of r into s summands $r = r_1 + \dots + r_s$. For every $\pi = (r_1, \dots, r_s)$, consider all associated orderings $\varrho \in \pi$, $\varrho = ((j_1, \dots j_{r_1}), \dots, (j_{r_1} + \dots + j_{r_h+1}, \dots, j_{r_1} + \dots + j_{r_h} + j_{r_{h+1}}), \dots, (j_{r-r_s+1}, \dots, j_r))$ such that the first terms satisfy $j_1 < \dots < j_{r_1} + \dots + j_{r_h+1} < \dots < j_{r-r_s+1}$ and each subsequence is increasing, i.e. $j_{r_1} + \dots + j_{r_h+1} < \dots < j_{r_1} + \dots + j_{r_h} + j_{r_{h+1}}$ for all $h = 1, \dots, s$. Then we have*

$$c^a_{i_1 \dots i_r} = \sum_{s=1}^{r} \sum_{\pi \in Q(r,s)} \sum_{\varrho \in \pi} b^a_{p_1 \dots p_s} a^{p_1}_{i_{j_1} \dots i_{j_{r_1}}} \dots a^{p_s}_{i_{j_{r-r_s+1}} \dots i_{j_r}} . \tag{34}$$

Proof. We proceed by induction. The case $r = 1$ is trivial. If we analyze (31) with (i) and (ii), we find that the formula for $c_{i_1 \dots i_r i_{r+1}}$ is obtained by the following procedure. Each product of $s+1$ elements is replaced by $s+1$ terms, where we gradually replace $b^a_{p_1 \dots p_s}$ by $b^a_{p_1 \dots p_s p_{s+1}} a^{p_{s+1}}_{i_{r+1}}$, $a^{p_1}_{i_{j_1} \dots i_{j_{r_1}}}$ by $a^{p_1}_{i_{j_1} \dots i_{j_{r_1}} i_{r+1}}, \dots, a^{p_s}_{i_{j_{r-r_s+1}} \dots i_{j_r}}$ by $a^{p_s}_{i_{j_{r-r_s+1}} \dots i_{j_r} i_{r+1}}$ and in each case all other terms remain unchanged. This procedure is compatible with passing from r to $r+1$ in (34). □

In the holomonic case, all a's and b's in (34) are symmetric in all subscripts. Then (34) can be rewritten as

$$c^a_{i_1 \dots i_r} = \sum_{s=1}^{r} \sum_{(I_1, \dots, I_s)} b^a_{p_1 \dots p_s} a^{p_1}_{I_1} \dots a^{p_s}_{I_s} , \tag{35}$$

where the inner sum is extended to all partitions $(I_1, \dots, I_s)$ of the set $\{i_1, \dots, i_r\}$ into s subsets. This formula was deduced by D. R. Grigore and D. Krupka, [5], see also [12].

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