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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NEUTRAL NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study asymptotic behavior of solutions of second order neutral functional differential equation of the form

$$\Big(x(t) + px(t-\tau)\Big)'' + f(t, x(t)) = 0\,.$$

We present conditions under which all nonoscillatory solutions are asymptotic to $at + b$ as $t \to \infty$, with $a, b \in R$. The obtained results extend those that are known for equation

$$u'' + f(t, u) = 0\,.$$

Introduction

We shall study the asymptotic behavior for $t \to \infty$ of nonoscillatory solutions of the following nonlinear neutral differential equation

$$\Big(x(t) + px(t-\tau)\Big)'' + f(t, x(t)) = 0\,. \tag{1}$$

Recently, the problem of establishing conditions for all nonoscillatory solutions of nonlinear differential equation

$$u''(t) + f(t, u) = 0$$

to behave like linear function $at + b$ as $t \to \infty$ has been of great interest. This problem has been treated by Cohen [3] and Tong [8], who used Bihari's inequality to achieve their results. Efforts in this direction have been undertaken by Naito [4] and Philos and Purnaros [5], who showed that solutions of

$$u''(t) + a(t)f(u) = 0$$

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behave like solutions of equation $u'' = 0$. Those results have been generalized by Rogovchenko in [6]. Since neutral differential equations often preserve some properties of associated non-neutral equations, it is natural to expect that Rogovchenko's results are extendable to equation (1). This is the aim of this paper.

By a solution of (1) we mean a continuous function x on $[T_x, \infty)$ such that $x(t) + px(t-\tau)$ is twice continuously differentiable and $x(t)$ satisfies (1) for $t \geqslant T_x$. As is customary a nontrivial solution of (1) is called oscillatory if it has arbitrarily large zeros, otherwise, it is called nonoscillatory. In the sequel, it is assumed that (1) possesses such nontrivial solutions.

All inequalities in this paper are assumed to hold eventually, i.e. they are satisfied for all sufficiently large t.

Main Results

In what follows we shall use the following lemma, which gives useful information about properties of nonoscillatory solutions of (1).

Lemma 1. *Let $y(t) > 0$ (or $y(t) < 0$) eventually and define*

$$w(t) = y(t) + p\,\frac{t-\tau}{t}\,y(t-\tau)\,, \quad 0 \le p < 1\,,\ \tau > 0\,. \tag{2}$$

If $\lim_{t\to\infty} w(t) = c$, then $\lim_{t\to\infty} y(t) = \frac{c}{1+p}$.

Proof. Suppose that $y(t) > 0$. Then $c \ge 0$ and it is easy to verify that

$$\limsup_{t\to\infty} y(t) \ge \frac{c}{1+p} \quad \text{and}$$
$$\liminf_{t\to\infty} y(t) \le \frac{c}{1+p}\,.$$

Assume that

$$\limsup_{t\to\infty} y(t) = \lim_{n\to\infty} y(\bar{t}_n) = \frac{c+q_1}{1+p} \quad \text{and}$$
$$\liminf_{t\to\infty} y(t) = \lim_{n\to\infty} y(\underline{t}_n) = \frac{c-q_2}{1+p}\,.$$

where $q_1 \ge 0$, $q_2 \ge 0$. We shall prove that $q_1 = q_2 = 0$.

(a) Suppose that $q_1 \ge q_2 \ge 0$ and $q_1 > 0$. It follows from (2) that for any $\varepsilon > 0$

$$w(t) \ge y(t) + p\,\frac{t-\tau}{t}\,\frac{c-q_2-\varepsilon}{1+p}\,.$$

Taking $t = \bar{t}_n$ and letting $n \to \infty$, we get

$$c \ge \frac{c+q_1}{1+p} + p\,\frac{c-q_2-\varepsilon}{1+p}\,.$$

That is

$$q_1 \le q_2 p + p\varepsilon \,.$$

Setting $\varepsilon = [(1-p)q_2]/(2p)$ we are led to

$$q_1 \le p(2q_2 - q_1) \le pq_2 < q_2$$

a contradiction.

(b) Suppose that $q_2 \ge q_1 \ge 0$ and $q_2 > 0$. Then (2) implies

$$w(t) \le y(t) + p\,\frac{c + q_1 + \varepsilon}{1+p}\,, \quad \varepsilon > 0\,.$$

Putting $t = \underline{t}_n$ and letting $n \to \infty$ we see that

$$c \le \frac{c - q_2}{1+p} + p\,\frac{c + q_1 + \varepsilon}{1+p}\,.$$

Then $q_2 \le pq_1 + p\varepsilon$. Setting $\varepsilon = [(1-p)q_2]/2p$ and proceeding similarly as above we get desired contradiction. The proof is complete now. □

Theorem 1. *Suppose that $0 \le p < 1$, $\tau > 0$ and $f(t,u)$ satisfies*

(i) *$f(t,u)$ is continuous in $D = \{(t,u);\ t \in [t_0,\infty),\ u \in R\}$, where $t_0 \ge 1$*

(ii) *there exist continuous functions $h, g\colon R_+ \to R_+$ such that*

$$|f(t,u)| \le h(t) g\left(\frac{|u|}{t}\right) \quad on \quad D\,,$$

where for $s > 0$, the function $g(s)$ is positive and nondecreasing and

$$\int_{t_0}^{\infty} h(s)ds < \infty$$

and

$$G(x) = \int_{t_0}^{x} \frac{ds}{g(s)} \to \infty \quad as \quad x \to \infty\,.$$

Then every nonoscillatory solution $x(t)$ of (1) is asymptotic to $at + b$, where a, b are real constants.

Proof. Assume that $x(t)$ is a nonoscillatory solution of (1). Set

$$z(t) = x(t) + px(t-\tau)\,, \tag{3}$$

then $|z(t)| > |x(t)|$ and it follows from (1) that

$$z''(t) = -f\bigl(t, x(t)\bigr)\,.$$

If we denote $z(t_0) = c_1$, $z'(t_0) = c_2$, then integrating the previous equality two times from t_0 to t we get

$$z'(t) = c_2 - \int_{t_0}^{t} f\big(s, x(s)\big)\, ds\ , \tag{4}$$
$$z(t) = c_2(t - t_0) + c_1 - \int_{t_0}^{t} (t-s) f\big(s, x(s)\big)\, ds\,.$$

It follows that

$$|z(t)| \le \big(|c_1| + |c_2|\big)t + t\int_{t_0}^{t} \big|f\big(s, x(s)\big)\big|\, ds \tag{5}$$

and in view of (ii) it is obvious that

$$\big|f\big(t, x(t)\big)\big| \le h(t) g\left(\frac{|x(t)|}{t}\right) \le h(t) g\left(\frac{|z(t)|}{t}\right)\,.$$

Thus from (5)

$$\frac{|z(t)|}{t} \le |c_1| + |c_2| + \int_{t_0}^{t} h(s) g\left(\frac{|z(s)|}{s}\right)\, ds\,. \tag{6}$$

Applying Bihari's inequality to (6), we get

$$\frac{|z(t)|}{t} \le G^{-1}\left(G\Big(|c_1| + |c_2|\Big) + \int_{t_0}^{t} h(s)\, ds\right)\,,$$

where $G^{-1}(x)$ is the inverse function of $G(x)$. We put

$$k_1 = G\Big(|c_1| + |c_2|\Big) + \int_{t_0}^{\infty} h(s)\, ds < \infty\,.$$

Since $G^{-1}(x)$ is increasing, we conclude that

$$\frac{|z(t)|}{t} \le k_2 = G^{-1}(k_1) < \infty\,.$$

On the other hand, by (ii) we have

$$\int_{t_0}^{t} \big|f\big(s, x(s)\big)\big|\, ds \le \int_{t_0}^{t} h(s) g\left(\frac{|x(s)|}{s}\right)\, ds \le \int_{t_0}^{t} h(s) g\left(\frac{|z(s)|}{s}\right)\, ds$$
$$\le g(k_2)\int_{t_0}^{\infty} h(s)\, ds < k_3\,.$$

Therefore $\int_{t_0}^{\infty} \big|f\big(s, x(s)\big)\big| ds$ exists and from (4) we see that there exists an $a_1 \in R$ such that

$$\lim_{t\to\infty} z'(t) = a_1 .$$

Then by the l'Hospital's rule we verify that

$$\lim_{t\to\infty} \frac{z(t)}{t} = \lim_{t\to\infty} z'(t) = a_1 .$$

Now we put $w(t) = z(t)/t$, then (3) implies

$$w(t) = y(t) + p\,\frac{t-\tau}{t}\,y(t-\tau)\,,$$

where $y(t) = x(t)/t$. Lemma 1 insures that

$$\lim_{t\to\infty} \frac{x(t)}{t} = a = \frac{a_1}{1+p}\,.$$

The proof is complete.

Remark 1. If in the proof of Theorem 1 we choose c_2 sufficiently large then $\lim_{t\to\infty} z'(t) \neq 0$ and the corresponding solution $x(t)$ of Eq. (1) is asymptotic to $at + b$, where $a \neq 0$.

Example 1. Consider the nonlinear differential equation

$$(7)\quad \left(x(t) + \frac{1}{2}\,x(t-1)\right)'' - \left(\frac{2}{t^3} + \frac{1}{(t-1)^3}\right)\cdot\left(1 + \frac{t^4}{(t^2+1)^2}\right)\cdot\frac{x^2(t)}{x^2(t)+t^2} = 0\,,$$

where $t \geq 2$. Set $h(t) = 2\left(\frac{2}{t^3} + \frac{1}{(t-1)^3}\right)$ and $g(u) = \frac{u^2}{u^2+1}$. Then applying Theorem 1 we deduce that for any nonoscillatory solution $x(t)$ of (7) there exist real a, b such that $x(t) - (at+b) \to 0$ as $t \to \infty$. Observe that

$$x(t) = t + \frac{1}{t}$$

is the solution of (7) which is asymptotic to t as $t \to \infty$.

Corollary 1. *Consider the equation*

$$(8)\qquad \big(x(t) + px(t-\tau)\big)'' + a(t)x(t) = 0$$

where $0 \leq p < 1$, $\tau > 0$ *and*

$$\int^{\infty} t|a(t)|\,dt < \infty\,.$$

Then every nonoscillatory solution of (8) *is asymptotic to* $at + b$ *as* $t \to \infty$.

Proof. The conclusion of Corollary 1 follows from Theorem 1 with

$$h(t) = t|a(t)| \quad\text{and}\quad g(u) = u\,.$$

Remark 2. For $p = 0$ Corollary 1 corresponds to the well known result (see e.g. [1, Theorem 5, page 114] or [6, Corollary1]) for equation $x'' + a(t)x = 0$.

Corollary 2. *Consider the equation*

$$(9) \qquad \big(x(t) + px(t-\tau)\big)'' + a(t)x^{\alpha}(t) = 0\,, \quad 0 < \alpha < 1$$

where $0 \le p < 1$, $\tau > 0$, α *is a quotient of two odd integers and*

$$\int^{\infty} t^{\alpha}|a(t)|dt < \infty\,.$$

Then every nonoscillatory solution of (8) *is asymptotic to* $at + b$ *as* $t \to \infty$.

Proof. Apply Theorem 1 with

$$h(t) = t^{\alpha}|a(t)| \quad \text{and} \quad g(u) = u^{\alpha}\,. \qquad \square$$

Example 2. Consider the following kind of Emden-Fowler equation

$$(10) \qquad \big(x(t) + px(t-\tau)\big)'' + t^{\beta}x^{\alpha}(t) = 0\,, \quad 0 < \alpha < 1$$

where $0 \le p < 1$, $\tau > 0$ and α is a quotient of two odd integers. Then by Corollary 2 every nonoscillatory solution of (10) is asymptotic to $at + b$ as $t \to \infty$ provided that $\alpha + \beta + 1 < 0$.

If we let $h(t) = t\, f_u(t,0)$ and $g(u) = u$ in Theorem 1 we get the following asymptotic criterion, which extends Cohen's result known for $u'' + f(t,u) = 0$ to Eq. (1):

Corollary 3. *Let* $0 \le p < 1$, $\tau > 0$. *Assume that* $f(t,u)$ *satisfies* (i) *and*

(iii) *the derivative* $f_u(t,u)$ *exists on* D *and* $f_u(t,u) > 0$ *on* D.
(iv) $|f(t,u)| \le f_u(t,0)|u|$ *on* D.

In addition suppose that

$$\int^{\infty} t f_u(t,0)\,dt < \infty\,.$$

Then every nonoscillatory solution of (1) *is asymptotic to* $at + b$ *as* $t \to \infty$.

As a matter of fact we are able to extend conclusions of Theorem 1 to more general equation.

Theorem 2. *Suppose that* $0 \le p < 1$, $\tau > 0$ *and* $f(t,u,v)$ *satisfies*

(i) $f(t,u,v)$ *is continuous in* $D = \{(t,u,v);\ t \in [t_0,\infty),\ u,v \in R\}$, *where* $t_0 \ge 1$
(ii) *there exist continuous functions* $h, g\colon R_+ \to R_+$ *such that*

$$|f(t,u,v)| \le h(t) g\left(\frac{|u|}{t}\right) |v| \quad on \quad D\,,$$

where function h *is the same as in case* (ii) *of Theorem 1 and* g *is positive, nondecreasing and*

$$\int_{t_0}^{x} \frac{ds}{sg(s)} \to \infty \quad as \quad x \to \infty\,.$$

Then every nonoscillatory solution $x(t)$ of

$$\Big(x(t)+px(t-\tau)\Big)''+f(t,x(t),x'(t))=0$$

is asymptotic to $at+b$, where a, b are real constants.

The proof of the theorem is analogous to that of Theorem 1 and thus is omitted.

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