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ON THE MINIMAL DISPLACEMENT OF POINTS UNDER MAPPINGS

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ABSTRACT. New contributions concerning the minimal displacement of points under mappings (defect of fixed point) are obtained.

1. Introduction

Given a set X in a Banach space and $T : X \to X$ that has no fixed points, it is natural to see how near T has fixed points, that is estimation of the quantity $\inf \{\|x - T(x)\| : x \in X\}$ is required. The study of this problem called minimal displacement of points under mappings, was started in the papers [6], [7], [8], [13], [14] and in the book [9], p. 210-218. In this paper new contributions to this topic are obtained. Many examples illustrating the concepts are presented. In Section 2 we give definitions and examples concerning the concepts of defect of fixed point and of best almost-fixed point for a mapping. In Section 3 we study the concepts introduced in Section 2 for various classes of mappings. Section 4 deals with some applications to various kinds of equations that have no solutions. Also, we use the concept of fixed point to introduce and study the concept of defect of property of fixed point for topological spaces.

2. Definitions and examples

Let us introduce the following.

Definition 2.1. ([9], p. 210). Let (X, d) be a metric space and $M \subseteq X$. The defect of fixed point of $f : M \to X$ is defined by $e_d(f; M) = \inf \{d(x, f(x)); x \in M\}$.

If there exists $x_0 \in M$ with $e_d(f; M) = d(x_0, f(x_0))$, then x_0 will be called the best almost-fixed point for f on M.

Remark 2.1. 1) If $f : M \to X$ has a fixed point, i.e. $\exists\, x_0 \in M$ with $f(x_0) = x_0$ then $e_d(f; M) = 0$. On the other hand, the defect can be 0 without to have

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f fixed point. For example, if $M = X = [1, +\infty)$ and $f(x) = x + \dfrac{1}{x}$ then $e_d(f; [1, +\infty)) = 0$ (where the metric d is generated by the absolute value $|\cdot|$) but f has no fixed point on M.

2) Let us suppose that $e_d(f; X) = 0$. Then obviously there exists a sequence $(x_n)_{n\in\mathbf{N}}$ in X, such that $d(x_n, f(x_n)) \xrightarrow{n\to\infty} 0$, sequence which is called asymptotically f-regular. Although this condition is not sufficient for the existence of fixed points for f, imposing some additional assumptions on f can be derived fixed point theorems (see e.g. [4], [10], [15]).

3) If $M \subseteq X$ is compact and $f : M \to X$ is continuous, then introducing $F : M \to \mathbf{R}_+$ defined by $F(x) = d(x, f(x))$, it easily follows that F is continuous on M and as a consequence, there exists $x_0 \in M$ with $e_d(f; M) = d(x_0, f(x_0))$. Also, in this case $e_d(f; M) = 0$ if and only if f has a fixed point.

4) If M is not compact or f is not continuous on M, then in general does not exist $x_0 \in M$ with $e_d(f; M) = d(x_0, f(x_0))$. The following two simple counterexamples show us the above statement: $M = X = (0,1), d(x,y) = |x - y|, f : X \to X, f(x) = x^2$ and $M = X = [0,1], d(x,y) = |x-y|, f : X \to X, f(x) = 1$ if $x = 0, f(x) = 0$ if $x \in (0,1]$, respectively.

5) Let $(X, \|\cdot\|)$ be a normed space and $M \subseteq X$ be nonempty compact set. In [5] it is proved that for any continuous map $f : M \to X$, there exists a point $x_0 \in M$ such that $\|x_0 - f(x_0)\| = \inf\{\|f(x_0) - y\| ; y \in M\}$. It follows that if moreover $f : M \to (X \setminus M)$, then $0 < e_d(f; M) \leq \|x_0 - f(x_0)\|$ (where the metric d is generated by norm $\|\cdot\|$). Indeed, let us suppose that $e_d(f; M) = 0$. It follows that there exist $x_n \in M, n \in \mathbf{N}$, such that $\|x_n - f(x_n)\| \xrightarrow{n\to\infty} 0$. Because M is compact too, there is $y_0 \in M$ such that $\|y_0 - f(y_0)\| = 0$, i.e. $y_0 = f(y_0)$, which is a contradiction because $y_0 \in M$ and $f(y_0) \in X \setminus M$.

Example 2.1. Let B denote the unit ball in the space

$$C[-1,1] = \{x : [-1,1] \to \mathbf{R} \mid x \text{ is continuous on } [-1,1]\}$$

with the uniform metric $d(x,y) = \sup\{|x(t) - y(t)| : t \in [-1,1]\}$, and for fixed $k > 1$ set

$$\alpha(t) = \begin{cases} -1\,, & \text{if} \quad -1 \leq t \leq -\frac{1}{k} \\ kt\,, & \text{if} \quad -\frac{1}{k} \leq t \leq \frac{1}{k} \\ 1\,, & \text{if} \quad \frac{1}{k} \leq t \leq 1\,. \end{cases}$$

If the mapping $T : B \to B$ is defined by

$$(Tx)(t) = \alpha(\max\{-1, \min\{1, x(t) + 2t\}\})$$

then $d(x, Tx) \geq 1 - \dfrac{1}{k}$ for each $x \in C[-1,1]$ (see [9], p. 212). This implies $e_d(T; B) \geq 1 - \dfrac{1}{k}$.

Example 2.2. Let $X = c_0$ be the Banach space of the sequences that converge to 0 endowed with the norm $\|x\| = \sup\{|x_n| ; n \in \mathbf{N}\}, x = (x_n)_{n\in\mathbf{N}}$. If $f : X \to X$

is defined by

$$f(x_1, x_2, ..., x_n, ...) = \left(1, |x_2|^{\frac{1}{2}}, ..., |x_n|^{\frac{1}{2}}, ...\right) + \left(1, \frac{1}{2}, ..., \frac{1}{n}, ...\right)$$

then f is continuous and for every $k \in \mathbf{R}$, the equation $f(x) = kx$ has no solutions (see [12], p. 67).

Let $M_0 \subset X$,

$$M_0 = \{x = (x_1, x_2, ..., x_n, ...) \in c_0 : (|x_n|)_{n \geq 1} \text{ is decreasing}\}$$

and $f_0 : M_0 \to X, f_0(x) = f(x) + x$. Then $d(f_0(x), x) = \max\left\{2, |x_2|^{\frac{1}{2}} + \frac{1}{2}\right\}$, where $d(f_0(x), x) = \|f_0(x) - x\|$ and

$$e_d(f_0; M_0) = \inf\{d(f_0(x), x) : x \in M_0\} = 2\,.$$

The set of the best almost-fixed points for f_0 on M_0 is given by

$$\left\{x = (x_1, x_2, ..., x_n, ...) \in M_0 : |x_2| \leq \frac{9}{4}\right\}.$$

Let $M_1 \subset X$,

$$M_1 = \left\{x = (x_1, x_2, ..., x_n, ...) \in c_0 : (x_n)_{n \geq 1} \text{ is increasing}\right\}$$

and $f_1 : M_1 \to X, f_1(x) = f(x)$. Because $x = (x_1, x_2, ..., x_n, ...) \in M_1$ implies $x_n \leq 0, \forall n \geq 1$ we obtain that $\left(|x_n|^{\frac{1}{2}} - x_n\right)_{n \geq 2}$ is a decreasing and positive sequence, therefore

$$d(f_1(x), x) = \max\left\{2 - x_1, |x_2|^{\frac{1}{2}} + \frac{1}{2} - x_2\right\}.$$

The defect of fixed point of f_1 on M_1 is

$$e_d(f_1; M_1) = \inf\{d(f_1(x), x) : x \in M_1\} = 2$$

and the unique best almost-fixed point for f_1 on M_1 is $x_0 = (0, 0, ..., 0, ...)$.

Example 2.3. Let $F : C[0,1] \to C[0,1]$ be defined by

$$(Fx)(t) = (\max\{t, |x(t) - x(0)|\})^{\frac{1}{2}}\,.$$

The function F is continuous and for every $k \in \mathbf{R}$, the equation $Fx = kx$ has no solutions (see [12], p. 67).

Let $A : C[0,1] \to C[0,1]$ be defined by

$$(Ax)(t) = (Fx)(t) + (1 - k)\,x(t)$$

and let $x \in C[0,1]$. We denote $m = \inf\limits_{t \in [0,1]} x(t)\,, M = \sup\limits_{t \in [0,1]} x(t)\,.$ We get

$$\begin{aligned} |(Ax)(t) - x(t)| &= |(Fx)(t) - kx(t)| \leq |(Fx)(t)| + |kx(t)| \\ &= (\max(t, |x(t) - x(0)|))^{\frac{1}{2}} + |k|\,|x(t)| \end{aligned}$$

for every $t \in [0,1]$. Because the above inequalities become equalities if $x(t) \geq 0, \forall t \in [0,1], k \leq 0$ and 1 is the maximum point of x or if $x(t) \leq 0, \forall t \in [0,1], k \geq 0$ and 1 is the minimum point of x, denoting

$$\begin{aligned} C_+[0,1] &= \{x \in C[0,1] : x(t) \geq 0, \forall t \in [0,1]\}, \\ C_-[0,1] &= \{x \in C[0,1] : x(t) \leq 0, \forall t \in [0,1]\}, \\ C_+^*[0,1] &= \{x \in C_+[0,1] : x(1) \geq x(t), \forall t \in [0,1]\}, \\ C_-^*[0,1] &= \{x \in C_-[0,1] : x(1) \leq x(t), \forall t \in [0,1]\}, \end{aligned}$$

and considering the uniform metric d, we obtain

$$e_d\left(A; C_+^*[0,1]\right) = \inf\left\{(\max(1, M - x(0)))^{\frac{1}{2}} - kM : x \in C_+^*[0,1]\right\} = 1, \ \forall k \leq 0$$

and

$$e_d\left(A; C_-^*[0,1]\right) = \inf\left\{(\max(1, x(0) - m))^{\frac{1}{2}} - km : x \in C_-^*[0,1]\right\} = 1, \ \forall k \geq 0.$$

A best almost-fixed point for F on $C_+^*[0,1]$ and on $C_-^*[0,1]$ is the same, namely the constant zero function (if $k \neq 0$ then it is unique).

Now, let us consider the metric $d : C[0,1] \to \mathbf{R}$ defined by

$$d(x,y) = \left(\int_0^1 (x(t) - y(t))^2 \, dt\right)^{\frac{1}{2}}.$$

We observe that $x(t) \geq 0, \forall t \in [0,1]$ and $k \leq 0$ or $x(t) \leq 0, \forall t \in [0,1]$ and $k \geq 0$ imply

$$(Fx)(t) - kx(t) = (\max(t, |x(t) - x(0)|))^{\frac{1}{2}} - kx(t) \geq t^{\frac{1}{2}} - kx(t) \geq t^{\frac{1}{2}}, \forall t \in [0,1]$$

with equalities when $x(t) = 0, \forall t \in [0,1]$. By using the above notations we have

$$\begin{aligned} e_d(A; C_+[0,1]) &= \inf\left\{\left(\int_0^1 ((Fx)(t) - kx(t))^2 \, dt\right)^{\frac{1}{2}} : x \in C_+[0,1]\right\} \\ &= \left(\int_0^1 t \, dt\right)^{\frac{1}{2}} = 1, \ \forall k \leq 0 \end{aligned}$$

and

$$\begin{aligned} e_d(A; C_-[0,1]) &= \inf\left\{\left(\int_0^1 ((Fx)(t) - kx(t))^2 \, dt\right)^{\frac{1}{2}} : x \in C_-[0,1]\right\} \\ &= \left(\int_0^1 t \, dt\right)^{\frac{1}{2}} = 1, \ \forall k \geq 0. \end{aligned}$$

A best almost-fixed point for A is the constant function zero.

3. Main results

Taking into account the Remark 2.1, 1), 2), 3), in this section we consider some results for mappings f that satisfy $e_d(f;M) > 0$, i.e. for mappings that neither have no fixed points nor asymptotically regular sequences. The first result is the following.

Theorem 3.1. *Let (X,d) be a metric space and $f : X \to X$ be a nonexpansive mapping, i.e. $d(f(x), f(y)) \leq d(x,y)$, for all $x, y \in X$. Then $e_d(f^n;X) \leq ne_d(f;X)$ and $\inf\{d(f^n(x), f^{n+1}(x));\ x \in X\} = e_d(f;X)$, $\forall n \in \{1,2,...\}$, where f^n denotes the n-th iterate of f.*

Proof. Firstly, by applying iteratively the triangle inequality, we easily obtain $d(x, f^n(x)) \leq nd(x, f(x))$. Secondly, by $e_d(f;X) \leq d(f^n(x), f^{n+1}(x)) \leq d(f^{n-1}(x), f^n(x)) \leq \cdots \leq d(x, f(x))$, passing to infimum, we obtain the desired equality. □

Remark 3.1. 1) If (X,d) is compact and $f : X \to X$ is contractive, i.e. $d(f(x), f(y)) < d(x,y)$, for all $x, y \in X, x \neq y$, then it is known that f has a fixed point $x_0 = f(x_0)$ and in this case obviously we have $e_d(f;X) = e_d(f^n;X) = 0$, $\forall n \in \mathbb{N}$. But a nonexpansive mapping even on a compact metric space (X,d) has no in general a fixed point, so the statement of Theorem 3.1 is not a trivial one.

2) If $f : X \to X$ is α-Lipschitz with $\alpha > 1$, i.e. $d(f(x), f(y)) \leq \alpha d(x,y)$, $\forall x, y \in X$, then reasoning as in the proof of Theorem 3.1 we obtain the inequality

$$e_d(f;X) \leq \inf\{d(f^n(x), f^{n+1}(x));\ x \in X\} \leq \alpha^n e_d(f;X),\ \forall n \geq 1\,.$$

Let us denote

$$\mathcal{F} = \{g : X \to X; g \text{ has fixed point in } X\},$$

where (X,d) is supposed to be compact. A natural question is to find the best approximation of a function $f : X \to X, f \notin \mathcal{F}$, by elements in $\mathcal{F}$. In this sense, we can define

$$E_{\mathcal{F}}(f) = \inf\{D(f,g); g \in \mathcal{F}\},$$

where $D(f,g) = \sup\{d(f(x), g(x)); x \in X\}$.

The following lower estimate for $E_{\mathcal{F}}(f)$ holds.

Theorem 3.2. *We have $e_d(f;X) \leq E_{\mathcal{F}}(f)$, for any $f : X \to X$.*

Proof. Let $g \in \mathcal{F}$ be with the fixed point y. We get

$$d(y, f(y)) \leq d(y, g(y)) + d(g(y), f(y)) \leq D(f,g),$$

i.e.

$$e_d(f;X) \leq D(f,g)$$

for all $g \in \mathcal{F}$. As a consequence, $e_d(f;X) \leq E_{\mathcal{F}}(f)$, which proves the theorem. □

Given a metric space (X,d) not necessarily compact and $f : X \to X$, an important problem is to establish the existence of points $x_0 \in X$ with $e_d(f;X) = d(x_0, f(x_0))$. Notice that [6], [7], [8], [9], [13], [14], do not treat this problem.

In this sense might be useful the following results.

Theorem 3.3. *Let (X, d) be a complete metric space and $f : X \to X$ be continuous on X with $e_d(f; X) > 0$. Let $\varepsilon > 0$ be arbitrary and $x \in X$ with*

$$e_d(f; X) \leq d(x, f(x)) \leq e_d(f; X) + \varepsilon\,.$$

Then there exists $x_\varepsilon \in X$ such that $d(x_\varepsilon, x) \leq 1, 0 < d(x_\varepsilon, f(x_\varepsilon)) \leq d(x, f(x))$ and $0 < d(x_\varepsilon, f(x_\varepsilon)) < d(y, f(y)) + \varepsilon d(x_\varepsilon, y)$, for all $y \in X, y \neq x_\varepsilon$.

Proof. Let us define $F : X \to \mathbf{R}_+, F(x) = d(x, f(x))$. By hypothesis, F is continuous on X and bounded from below. Applying the well-known Ekeland's variational principle to F (see e.g. [2], [3] or [1], p. 33, Th. 3.1) we obtain the statement in the theorem. □

By replacing d with $\varepsilon^{-\frac{1}{2}} d$, in Theorem 3.3 we immediately obtain:

Theorem 3.4. *Let (X, d) be a complete metric space and $f : X \to X$ be continuous on X, with $e_d(f; X) > 0$. Let $\varepsilon > 0$ be arbitrary and $x \in X$ be with*

$$0 < e_d(f; X) \leq d(x, f(x)) \leq e_d(f; X) + \varepsilon\,.$$

Then there exists $x_\varepsilon \in X$ such that $d(x_\varepsilon, x) \leq \sqrt{\varepsilon}, 0 < d(x_\varepsilon, f(x_\varepsilon)) \leq d(x, f(x))$ and $0 < d(x_\varepsilon, f(x_\varepsilon)) < d(y, f(y)) + \sqrt{\varepsilon} d(x_\varepsilon, y)$, for all $y \in X, y \neq x_\varepsilon$.

Deeper results can be obtained if we consider $(X, \|\cdot\|)$ as a real Banach space, because in this case we can use the differential calculus too in normed spaces.

Firstly we need some well-known concepts.

Definition 3.1. Let $(X, \|\cdot\|_1), (Y, \|\cdot\|_2)$ be normed spaces and $f : X \to Y$. We say that f is Gâteaux differentiable at a point $x \in X$, if there exists the limit $\lim\limits_{t\to 0, t\neq 0} \dfrac{f(x+th) - f(x)}{t}$, for all $h \in X$.

We say that f is Gâteaux derivable on $M \subseteq X$ if for each $x \in X$ there exists a mappings denoted $\nabla f(x) \in L(X, Y) = \{G : X \to Y; G\text{-linear}\}$, such that

$$\lim_{t\to 0, t\neq 0} \frac{f(x+th) - f(x)}{t} = (\nabla f(x))(h)\,,$$

for all $h \in X$. The mapping $\nabla f(x)$ is called the gradient of f at the point x.

We present

Theorem 3.5. *Let $(X, \langle , \rangle)$ be a real Hilbert space and $f : X \to X$ be Gâteaux derivable on X with $e_d(f; X) > 0$. Then for each $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that*

$$0 < e_d(f; X) \leq \|x_\varepsilon - f(x_\varepsilon)\| \leq e_d(f; X) + \varepsilon$$

and

$$\left\langle \frac{x_\varepsilon - f(x_\varepsilon)}{\|x_\varepsilon - f(x_\varepsilon)\|}, (1_X - \nabla f(x_\varepsilon))(h) \right\rangle \leq \sqrt{\varepsilon}\,,$$

where $1_X(h) = h$, for all $h \in X, \|x\| = \sqrt{\langle x, x\rangle}, x \in X$ and the metric d is generated by norm $\|\cdot\|$.

Proof. Let us denote $F(x) = \|x - f(x)\| > 0, x \in X$. Obviously $F : X \to \mathbf{R}_+$ is bounded from below. Then

$$\lim_{t\to 0, t\neq 0} \frac{F(x+th) - F(x)}{t} = \lim_{t\to 0, t\neq 0} \left(\frac{F^2(x+th) - F^2(x)}{t} \cdot \frac{1}{F(x+th) + F(x)} \right)$$
$$= \left(\nabla F^2(x)\right)(h) \cdot \frac{1}{2F(x)} .$$

But because $F^2(x) = \langle x - f(x), x - f(x) \rangle$, simple calculations show us that $\left(\nabla F^2(x)\right)(h) = 2\langle x - f(x), (1_X - \nabla f(x))(h) \rangle, \forall h \in X$.

As a consequence, $\exists\ (\nabla F(x))(h) = \left\langle \frac{x - f(x)}{\|x - f(x)\|}, (1_X - \nabla f(x))(h) \right\rangle$, for all $x \in X$ and all $h \in X$. Then taking in the last inequality of Theorem 3.4, $y = x_\varepsilon \pm th, h \in X, t \neq 0$, and passing with $t \to 0$ (see also e.g. [1], Theorem 3.4, p. 35), we obtain

$$0 < e_d(f; X) \leq \|x_\varepsilon - f(x_\varepsilon)\| \leq e_d(f; X) + \varepsilon$$

and

$$|(\nabla F(x))(h)| \leq \sqrt{\varepsilon} ,$$

which proves the theorem. □

Because the main problem is to prove the existence of minimum points for the nonlinear functional $F(x) = \|x - f(x)\|$, obviously we can use well-known variational methods.

In this sense, it is immediate the following.

Theorem 3.6. *Let $(X, \langle , \rangle)$ be a real Hilbert space, $\|x\| = \sqrt{\langle x, x \rangle}, x \in X$, the metric generated by norm $\|\cdot\|$ denoted by d and $M \subseteq X$.*

(i) *Let $f : M \to X$ be Gâteaux derivable on $x_0 \in M, x_0$ interior point of M, with $e_d(f; M) > 0$. If $\|x_0 - f(x_0)\| = e_d(f; M), x_0 \in M$, then*

$$\langle x_0 - f(x_0), (1_X - \nabla f(x_0))(h) \rangle = 0 ,$$

for all $h \in X$.

(ii) *If $F(x) = \|x - f(x)\|, x \in X$ is moreover Gâteaux derivable on $M \subset X$ open convex, then $\|x_0 - f(x_0)\| = e_d(f; M)$ if and only if*

$$\langle x_0 - f(x_0), (1_X - \nabla f(x_0))(h) \rangle = 0 ,\ \forall h \in X .$$

Remark 3.2. As a consequence, the best almost-fixed points of f on M must be among the solutions of the equation $\langle x - f(x), (1_X - \nabla f(x))(h) \rangle = 0, \forall h \in X$.

4. Applications

Firstly, the concepts and results in the previous sections allow us to approach the study of equations that have no solutions.

Indeed, because each equation $E(x) = 0_X$ in a normed space $(X, \|\cdot\|)$ can be written as $f(x) = x$, where $f(x) = E(x) + x$, for the case when $E(x) = 0_X$ has no solutions in X, a best almost-fixed point y of f will also be called best almost-solution of the equation $E(x) = 0_X$, satisfying $\|E(y)\| = \inf\{\|E(x)\| ; x \in X\}$.

We consider the case of algebraic equations.

Theorem 4.1. *Let $P_m(x)$ be an algebraic polynomial of degree $m \in \mathbf{N}$, with real coefficients such that the equation $P_m(x) = 0$ has no real solutions. Then m is even, there exists at least one and at most $\frac{m}{2}$ best almost-real solutions of the equation $P_m(x) = 0$.*

Proof. The fact that m must be even is obvious. Denote $f(x) = P_m(x) + x$. Let $x_0 \in \mathbf{R}$ be such that $e_d(f) = |x_0 - f(x_0)| = |P_m(x_0)| > 0$ (the metric d is generated by absolute value $|\cdot|$). Because the scalar product on $\mathbf{R}$ is the usual one, $(1_X - \nabla f(x_0))(h) = (1 - f'(x_0)) \cdot h$, by Theorem $3.6, (i)$, it easily follows that $f'(x_0) = 1$, i.e. $P_m'(x_0) = 0$. But the degree of $P_m'(x)$ is odd, so the equation $P_m'(x) = 0$ has at least one solution. We are interested in the maximum number of points $x_k \in \mathbf{R}$ that satisfy

$$e_d(f; \mathbf{R}) = \inf\{|P_m(x)|; x \in \mathbf{R}\} = |P_m(x_k)|, k \in \{1, ..., p\}.$$

Obviously that $P_m'(x_k) = 0, k \in \{1, ..., p\}$, where $x_k, k \in \{1, ...p\}$ are considered in increasing order.

We show that if $e_d(f; \mathbf{R}) = |P_m(x_k)|$, then $e_d(f; \mathbf{R}) \neq |P_m(x_{k+1})|$, which will prove the theorem. Indeed, because $P_m(x)$ has no real solutions, it follows that $P_m(x) > 0$, for all $x \in \mathbf{R}$ or $P_m(x) < 0$, for all $x \in \mathbf{R}$. Let us suppose, for example, that $P_m(x) > 0, \forall x \in \mathbf{R}$ (the case $P_m(x) < 0, \forall x \in \mathbf{R}$ is similar). If $e_d(f; \mathbf{R}) = P_m(x_k) = P_m(x_{k+1})$, then we get that there is $\xi \in (x_k, x_{k+1})$ with $P_m'(\xi) = 0$, i.e. x_k and x_{k+1} are not consecutive, a contradiction. The theorem is proved. □

Now, we will consider the following integral equation which appears in statistical mechanics

$$u(x) = 1 + \lambda \int_x^1 u(s - x)\, u(s)\, ds \qquad x \in [0, 1],$$

where λ is a real parameter, $\lambda > -1$. In [16], p. $236 - 237$ it is proved that for $\lambda > \frac{1}{2}$ the above equation has no solutions $u \in C[0, 1]$.

Let us consider $C[0, 1]$ endowed with the uniform metric

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}.$$

Denoting $A : C[0, 1] \to C[0, 1]$ by

$$(Au)(x) = 1 + \lambda \int_x^1 u(s - x)\, u(s)\, ds,$$

where $\lambda > \frac{1}{2}$, it follows that A has no fixed points in $C[0, 1]$.

Theorem 4.2. *Let $A : M \to C[0, 1]$, where $\lambda > \frac{1}{2}, A$ is defined as above and*

$$M = \{u \in C[0, 1] : u \textit{ is derivable, } u'(x) \geq 0, \forall x \in [0, 1], 0 \leq u(0) \leq u(1) \leq 1\}.$$

Then $e_d(A;M) = \dfrac{4\lambda - 1}{4\lambda}$ *and a best almost-fixed point for* A *on* M *is* $u : [0,1] \to \mathbf{R}$, $u(x) = \dfrac{1}{2\lambda}, \forall x \in [0,1]$.

Proof. We have

$$\begin{aligned} e_d(A;M) &= \inf\{\sup\{|(Au)(x) - u(x)| : x \in [0,1]\} : u \in M\} \\ &= \inf\left\{\sup\left\{\left|\lambda \int_x^1 u(s-x)\,u(s)\,ds - u(x) + 1\right| : x \in [0,1]\right\} : u \in M\right\}. \end{aligned}$$

The function $g : [0,1] \to \mathbf{R}$ defined by

$$g(x) = \lambda \int_x^1 u(s-x)\,u(s)\,ds - u(x) + 1$$

is monotone decreasing. Indeed,

$$g'(x) = -\lambda \int_x^1 u'(s-x)\,u(s)\,ds - u(0)\,u(x) - u'(x) \le 0$$

for every $u \in M$. Then

$$\begin{aligned} \sup\{|g(x)| : x \in [0,1]\} &= \max\left\{\Big|\sup_{x\in[0,1]} g(x)\Big|, \Big|\inf_{x\in[0,1]} g(x)\Big|\right\} \\ &= \max\left\{\left|\lambda \int_0^1 u^2(s)\,ds - u(0) + 1\right|, \big|1 - u(1)\big|\right\} \end{aligned}$$

and the conditions $0 \le u(0) \le u(1) \le 1$ imply

$$\sup\{|g(x)| : x \in [0,1]\} = \lambda \int_0^1 u^2(s)\,ds - u(0) + 1\,.$$

This means that the defect of fixed point of A is

$$\begin{aligned} e_d(A;M) &= \inf\left\{\lambda \int_0^1 u^2(s)\,ds - u(0) + 1 : u \in M\right\} \\ &= \inf\left\{\lambda u^2(0) - u(0) + 1 : u \in M, u(x) = u(0), \forall x \in [0,1]\right\} \\ &= \frac{4\lambda - 1}{4\lambda} \end{aligned}$$

and a best almost-fixed point for A on M is the function $u : [0,1] \to \mathbf{R}$, defined by $u(x) = \dfrac{1}{2\lambda}$. □

If we replace the uniform metric on $C[0,1]$ with the metric $d : C[0,1] \times C[0,1] \to \mathbf{R}$ defined by $d(u,v) = \left(\int_0^1 (u(x) - v(x))^2\,dx\right)^{\frac{1}{2}}$ we obtain

Corollary 4.3. *For the operator* A *defined above,* $\lambda > \dfrac{1}{2}$, *we have* $e_d(A;M) = 1$ *and a best almost-fixed point for* A *on* M *is* $u(x) = 0$, $\forall x \in [0,1]$.

Proof. We get

$$d(u,Au)=\Big(\int_0^1\Big(\lambda\int_x^1 u(s-x)\,u(s)\,ds-u(x)+1\Big)^2dx\Big)^{\frac{1}{2}}\geq -u(1)+1$$

by using the monotony of g (as above). The inequality becomes equality if $\lambda\int_x^1 u(s-x)\,u(s)\mathrm{d}s-u(x)=-u(1)$, almost everywhere $x\in[0,1]$. It follows $\lambda\int_x^1 u(s-x)\,u(s)\mathrm{d}s=u(x)-u(1)$, almost everywhere $x\in[0,1]$, $u\in M$. But $u(x)-u(1)\leq 0$ and $\lambda\int_x^1 u(s-x)\,u(s)\mathrm{d}s\geq 0$, which implies $u(x)=u(1)$, almost everywhere $x\in[0,1]$. Replacing in the above equation, it easily follows $u(x)=0$, almost everywhere $x\in[0,1]$, which by $u\in C[0,1]$ implies $u(x)=0,\forall x\in[0,1]$. Therefore

$$e_d(A;M)=\inf\{d(u,Au):u\in M\}=1\,,$$

and a best almost-fixed point for A on M is $u(x)=0$, $\forall x\in[0,1]$, which proves the corollary. □

Finally, we will use the defect of fixed point to introduce a similar concept for topological spaces. Firstly, we recall the following definition, well-known in the theory of fixed point (see for example [9], [16]).

Definition 4.1. The topological space $(X,\mathcal{T})$ has the property of fixed point if any continuous function $f:X\to X$ has fixed point.

The concept of defect of fixed point introduced in Section 2 suggests the following

Definition 4.2. Let (X,d) be a metric space and $\mathcal{T}_d$ be the topology on X generated by d. The defect of fixed point of topological space $(X,\mathcal{T}_d)$ is defined by

$$t(X,\mathcal{T}_d)=\sup\{e_d(f;X)\,|f:X\to X\text{ is continuous}\}.$$

Remark 4.1. We can reformulate the above definition in a more general frame, considering $(X,\mathcal{T})$ a metrizable space and d the corresponding metric.

Remark 4.2. If $(X,\mathcal{T}_d)$ has the property of fixed point then $e_d(f;X)=0$ for every continuous function $f:X\to X$, therefore $t(X,\mathcal{T}_d)=0$.

Example 4.1. We consider the Euclidean metric δ on $\mathbf{R}^2$ and we denote by $\mathcal{T}_\delta$ the Euclidean topology on $\mathbf{R}^2$. Because the defect of fixed point of the continuous function $f:\mathbf{R}^2\to\mathbf{R}^2$ defined by $f(x,y)=(x+a,y+b)$ is $e_\delta\left(f;\mathbf{R}^2\right)=\sqrt{a^2+b^2}$, we obtain $t\left(\mathbf{R}^2,\mathcal{T}_\delta\right)=+\infty$.

Nevertheless, there is a subset X dense in the topological space $\left(\mathbf{R}^2,\mathcal{T}_\delta\right)$ such that $t(X,\mathcal{T}_\delta|_X)=0$. Indeed, if we denote $C=\left\{(x,y)\in\mathbf{R}^2:x^2+y^2=1\right\}$ and $\{z_1,...,z_n,...\}$ a dense subset of C then the set $X=\bigcup_{n\in\mathbf{N}^*}X_n$, where $X_n=\{tnz_n:0\leq t\leq 1\}$, is dense in $\left(\mathbf{R}^2,\mathcal{T}_\delta\right)$ and any continuous function $f:X\to X$ has at least a fixed point (see [12], p. 145).

Example 4.2. There exist topological spaces that have as defect of fixed point real positive numbers. Indeed, let us consider $a, b, c, d \in \mathbf{R}$, $a < b < c < d$, $a+d = b+c$, $ad-bc \neq 0$ and the continuous function $f_0 : [a, b] \cup [c, d] \to [a, b] \cup [c, d]$ defined by $f_0(x) = x + \dfrac{ad-bc}{a-b}$ if $x \in [a, b]$ and $f_0(x) = x + \dfrac{bc-ad}{c-d}$ if $x \in [c, d]$. We have $e_d(f_0; [a, b] \cup [c, d]) = \dfrac{|ad-bc|}{b-a}$ (relative to the metric d generated by absolute value $|\cdot|$), therefore $t([a, b] \cup [c, d], \mathcal{T}_d) \geq \dfrac{|ad-bc|}{b-a}$. On the other hand, $e_d(f; [a, b] \cup [c, d]) \leq d - a$ for every function f defined on $[a, b] \cup [c, d]$ with values in $[a, b] \cup [c, d]$, which implies $t([a, b] \cup [c, d], \mathcal{T}_d) \leq d - a$.

By using Remark 4.2 and Remark 2.1, 3) we obtain the following characterization of topological spaces with the property of fixed point.

Theorem 4.4. *Let (X, d) be a compact metric space and $\mathcal{T}_d$ the topology on X generated by d. Then $(X, \mathcal{T}_d)$ has the property of fixed point if and only if $t(X, \mathcal{T}_d) = 0$.*

The property of fixed point is invariant by homeomorphisms, with other words it is a topological property. An analogous result can be proved for the defect of this property.

Theorem 4.5. (i) *If (X, d) and (X', d') are isometric then $t(X, \mathcal{T}_d) = t\left(X', \mathcal{T}_{d'}\right)$;*

(ii) $t(X, \mathcal{T}_d) \leq \operatorname{diam} X$;

Proof. (i) Let $f : X' \to X'$ be a continuous function and $i : X \to X'$ the isometry between (X, d) and (X', d'), that is the function i is bijective and $d(x, y) = d'(i(x), i(y))$, for every $x, y \in X$. Because i and i^{-1} are continuous (see [11], p. 123) the function $i^{-1} \circ f \circ i : X \to X$ is continuous and

$$d\left(\left(i^{-1} \circ f \circ i\right)(x), x\right) = d'(f(i(x)), i(x)), \ \forall x \in X.$$

We have

$$\begin{aligned} e_{d'}(f; X') &= \inf\{d'(f(x'), x') : x' \in X'\} = \inf\{d'(f(i(x)), i(x)) : x \in X\} \\ &= \inf\left\{d\left(\left(i^{-1} \circ f \circ i\right)(x), x\right) : x \in X\right\} = e_d\left(i^{-1} \circ f \circ i; X\right), \end{aligned}$$

therefore $t(X', \mathcal{T}_{d'}) \leq t(X, \mathcal{T}_d)$. We analogously obtain the converse inequality and the property is proved.

(ii) By

$$e_d(f; X) = \inf\{d(f(x), x) : x \in X\} \leq \operatorname{diam} X$$

for any continuous function $f : X \to X$, we obtain the inequality. $\square$

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