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ON (σ,τ)-DERIVATIONS IN PRIME RINGS

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Abstract. Let R be a 2-torsion free prime ring and let σ,τ be automorphisms of R. For any $x,y\in R$, set $[x,y]_{\sigma,\tau}=x\sigma(y)-\tau(y)x$. Suppose that d is a (σ,τ)-derivation defined on R. In the present paper it is shown that (i) if R satisfies $[d(x),x]_{\sigma,\tau}=0$, then either $d=0$ or R is commutative (ii) if I is a nonzero ideal of R such that $[d(x),d(y)]=0$, for all $x,y\in I$, and d commutes with both σ and τ, then either $d=0$ or R is commutative. (iii) if I is a nonzero ideal of R such that $d(xy)=d(yx)$, for all $x,y\in I$, and d commutes with τ, then R is commutative. Finally a related result has been obtain for (σ,τ)-derivation.

1. Introduction

Throughout the present paper R will denote an associative ring with center $Z(R)$. For any $x,y\in R$ the symbol $[x,y]$ represents commutator $xy-yx$ and for a non-empty subset S of R, we put $C_R(S)=\{x\in R\mid [x,s]=0$, for all $s\in S\}$. The set of all commutators of elements of S will be written as $[S,S]$. Recall that R is prime if $aRb=(0)$ implies that $a=0$ or $b=0$. Let σ and τ be any two automorphisms of R. For any $a,b\in R$ we set $[a,b]_{\sigma,\tau}=a\sigma(b)-\tau(b)a$. An additive mapping $d:R\to R$ is called a derivation if $d(xy)=d(x)y+xd(y)$, for all $x,y\in R$. An additive mapping $d:R\to R$ is called a (σ,τ)-derivation if $d(xy)=d(x)\sigma(y)+\tau(x)d(y)$ holds for all $x,y\in R$. Of course a $(1,1)$-derivation where 1 is the identity map on R is a derivation. A mapping $F:R\to R$ is said to be centralizing if $[F(x),x]\in Z(R)$, for all $x\in R$, in the special case when $[F(x),x]=0$, the mapping F is said to be commuting on R. Mapping $F:R\to R$ is said to be (σ,τ)-centralizing (resp. (σ,τ)-commuting) if $[F(x),x]_{\sigma,\tau}\in Z(R)$ (resp. $[F(x),x]_{\sigma,\tau}=0$) holds for all $x\in R$. Of course a $(1,1)$-centralizing (resp. $(1,1)$-commuting) mapping is a centralizing (resp. commuting) on R. There are several results in the existing literature dealing with centralizing and commuting mappings in rings. The study of centralizing mappings was initiated by Posner [11] which states that the existence of a nonzero centralizing derivation on a prime

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ring forces the ring to be commutative (Posner's second theorem). In an attempt to generalize the above result Vukman [12] proved that if R is a 2-torsion free prime ring and $d : R \to R$ a nonzero derivation such that the map $x \mapsto [d(x), x]$ is commuting on R, then R is commutative. In the present paper it is shown that the conclusion of the above theorem holds if for a (σ, τ)-derivation d the mapping $x \mapsto d(x)$ is (σ, τ)-commuting. In fact we have proved the following.

Theorem 1. *Let R be a 2-torsion free prime ring. Suppose there exists a (σ, τ)-derivation $d : R \to R$ such that $[d(x), x]_{\sigma,\tau} = 0$, for all $x \in R$. Then either $d = 0$ or R is commutative.*

A famous result due to Herstein [9] states that if R is prime ring of characteristic not 2 which admits a nonzero derivation d such that $[d(x), d(y)] = 0$, for all $x, y \in R$, then R is commutative. Motivated by this result, recently Bell and Daif [5] studied derivation d satisfying $d(xy) = d(yx)$, for all $x, y \in R$. Now our object is to generalize these two results for (σ, τ)-derivations as follows:

Theorem 2. *Let R be a 2-torsion free prime ring, and I a nonzero ideal of R. If R admits a (σ, τ)-derivation d such that $[d(x), d(y)] = 0$, for all $x, y \in I$ and d commutes with both σ, τ, then either $d = 0$ or R is commutative.*

Theorem 3. *Let R be a 2-torsion free prime ring, and I a nonzero ideal of R. If R admits a nonzero (σ, τ)-derivation d such that $d(xy) = d(yx)$, for all $x, y \in I$ and d commutes with τ, then R is commutative.*

2. Proof of the Main Results

Throughout the present paper, we shall make extensive use of the following basic commutator identities:

$$[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y$$

and

$$[x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z)\,.$$

To facilitate our discussion, we begin with the following lemmas.

Lemma 2.1 ([1, Lemma 3]). *Let R be a prime ring, I a nonzero ideal of R and $a \in R$. If R admits a (σ, τ)-derivation d such that $ad(I) = (0)$ (or $d(I)a = (0)$), then either $d = 0$ or $a = 0$.*

Lemma 2.2. *Let R be a 2-torsion free prime ring, I be a nonzero ideal of R. If R admits a (σ, τ)-derivation d such that $d^2(I) = (0)$ and d commutes with both σ, τ, then $d = 0$.*

Proof. For any $x \in I$, we have $d^2(x) = 0$. Replacing x by xy, we get $d^2(x)\sigma^2(y) + \tau(d(x))d(\sigma(y)) + d(\tau(x))\sigma(d(y)) + \tau^2(x)d^2(y) = 0$, for all $x, y \in I$ and hence using the fact that $d^2(I) = (0)$ and d commutes with both σ, τ, the above relation yields that $\tau(d(x))\sigma(d(y)) = 0$, for all $x, y \in I$ i.e. $\sigma^{-1}(\tau(d(x)))d(y) = 0$, for all $x, y \in I$.

Thus application of Lemma 2.1 gives that either $d = 0$ or $\sigma^{-1}(\tau(d(x))) = 0$. If $\sigma^{-1}(\tau(d(x))) = 0$, for all $x \in I$, then $d(x) = 0$, for all $x \in I$. For any $r \in R$, replace x by xr, to get $d(x)\sigma(r) + \tau(x)d(r) = 0$, for all $x \in I$ and hence $x\tau^{-1}(d(r)) = 0$, for all $x \in I$, $r \in R$ i.e. $IR\tau^{-1}(d(r)) = (0)$. Since I is a nonzero ideal of R and R is prime the above relation yields that $\tau^{-1}(d(r)) = 0$, for all $r \in R$ and hence $d = 0$. $\square$

Proof of Theorem 1. Let us introduce a mapping $B(\cdot,\cdot) : R \times R \to R$ by the relation $B(x,y) = [d(x), y]_{\sigma,\tau} + [y, d(x)]_{\sigma,\tau}$, for all $x, y \in R$. Obviously $B(\cdot,\cdot)$ is symmetric (that is $B(x,y) = B(y,x)$, for all $x, y \in R$) and additive in both the arguments. Notice that

$$\begin{aligned} B(xy, z) &= [d(xy), z]_{\sigma,\tau} + [d(z), xy]_{\sigma,\tau} \\ &= B(x,z)\sigma(y) + \tau(x)B(y,z) + d(x)\sigma([y,z]) + \tau([x,z])d(y)\,, \end{aligned} \tag{1}$$

for all $x, y, z \in R$.

Now, introduce a mapping f from R into itself by $f(x) = B(x,x)$, for all $x \in R$. We have $f(x) = 2[d(x), x]_{\sigma,\tau}$ for all $x \in R$. The mapping f satisfies the relation

$$\begin{aligned} f(x+y) &= 2[d(x+y), x+y]_{\sigma,\tau} \\ &= 2[d(x), x]_{\sigma,\tau} + 2[d(y), x]_{\sigma,\tau} + 2[d(x), y]_{\sigma,\tau} + 2[d(y), y]_{\sigma,\tau} \\ &= f(x) + f(y) + 2B(x,y)\,, \end{aligned} \tag{2}$$

for all $x, y \in R$.

Throughout the proof we shall use the mappings B and f, as well as the relation (1) and (2) without specific references. The assumption of the theorem can be rewritten as

$$f(x) = 0\,, \quad \text{for all} \quad x \in R\,. \tag{3}$$

Linearization of (3) gives that $f(x) + f(y) + 2B(x,y) = 0$, for all $x, y \in R$ and hence $2B(x,y) = 0$, for all $x, y \in R$. Since char $R \neq 2$, we get $B(x,y) = 0$, for all $x, y \in R$. Replacing y by xy in the above relation, we obtain

$$B(x, xy) = f(x)\sigma(x) + \tau(x)B(x,y) + d(x)\sigma([x,y]) = 0\,,$$

for all $x, y \in R$ and hence using (3) and the fact that $B(x,y) = 0$, we get

$$d(x)\sigma([x,y]) = 0\,, \quad \text{for all} \quad x, y \in R\,,$$

i.e. $\sigma^{-1}(d(x))[x,y] = 0$, for all $x, y \in R$. Again replace y by yz in the above expression, to get $\sigma^{-1}(d(x))y[x,z] = 0$, for all $x, y, z \in R$ and hence $\sigma^{-1}(d(x))R[x,z] = 0$, for all $x, z \in R$. Thus for each $x \in R$, either $\sigma^{-1}(d(x)) = 0$ or $[x,z] = 0$, for all $z \in R$. This shows that additive group R is the union of two of its additive subgroups $A = \{x \in R \mid \sigma^{-1}(d(x)) = 0\}$ and $B = \{x \in R \mid [x,z] = 0, \ \text{for all } z \in R\}$. This implies that either $R = A$ or $R = B$. If $R = A$, then $\sigma^{-1}(d(x)) = 0$, for all $x \in R$, i.e. $d = 0$. On the other hand if $R = B$, then $[x,z] = 0$, for all $x, z \in R$, i.e. R is commutative. This completes the proof of the theorem. $\square$

Proof of Theorem 2. We have

$$[d(x), d(y)] = 0\,, \quad \text{for all} \quad x, y \in I\,. \tag{4}$$

Replacing y by xy in (4) and using (4), we get

$$d(x)[d(x), \sigma(y)] + [d(x), \tau(x)]d(y) = 0\,, \quad \text{for all} \quad x, y \in I\,.$$

Now for any $r \in R$, replace y by yr in the above expression to get

$$d(x)\sigma(y)[d(x), \sigma(r)] + [d(x), \tau(x)]\tau(y)d(r) = 0\,, \tag{5}$$

for all $x, y \in I$, $r \in R$. In view of (4) for $r = \sigma^{-1}(d(z))$, for any $z \in I$ (5) reduces to

$$[d(x), \tau(x)]\tau(y)\sigma^{-1}(d^2(z)) = 0\,, \quad \text{for all} \quad x, y, z \in I\,.$$

For any $s \in R$, replacing y by $y\tau^{-1}(s)$ in the above relation we get

$$[d(x), \tau(x)]\tau(y)R\sigma^{-1}(d^2(z)) = (0)\,,$$

for all $x, y, z \in I$, $s \in R$. This implies that either $\sigma^{-1}(d^2(z)) = 0$ or $[d(x), \tau(x)] \cdot\tau(y) = 0$, for all $x, y \in I$. If $\sigma^{-1}(d^2(z)) = 0$, for all $z \in I$, then $d^2(z) = 0$ for all $z \in I$ and hence by Lemma 2.2 we get the required result. On the other hand if $[d(x), \tau(x)]\tau(y) = 0$, for all $x, y \in I$, then $\tau^{-1}([d(x), \tau(x)])y = 0$, for all $x, y \in I$ and hence $\tau^{-1}([d(x), \tau(x)])RI = (0)$, for all $x \in I$. Since I is a nonzero ideal of R and R is prime the above relation yields that $\tau^{-1}([d(x), \tau(x)]) = 0$, for all $x \in I$ and hence

$$[d(x), \tau(x)] = 0\,, \quad \text{for all} \quad x \in I\,. \tag{6}$$

Linearizing (6), we get

$$[d(x), \tau(y)] + [d(y), \tau(x)] = 0 \quad \text{for all} \quad x, y \in I\,. \tag{7}$$

Now replacing y by yx in (7) and using (7), we get $d(x)[\sigma(y), \tau(x)] = 0$, for all $x, y \in I$. For any $r_1 \in R$, again replace y by $y\sigma^{-1}(r_1)$, to get $d(x)\sigma(y)[r_1, \tau(x)] = 0$, for all $x, y \in I$, $r_1 \in R$ and hence $\sigma^{-1}(d(x))y\sigma^{-1}([r_1, \tau(x)]) = 0$ i.e. $\sigma^{-1}(d(x)) \cdot IR\sigma^{-1}([r_1, \tau(x)]) = (0)$. The primeness of R implies that for each $x \in I$ either $\sigma^{-1}(d(x))I = (0)$ or $\sigma^{-1}([r_1, \tau(x)]) = 0$. If $\sigma^{-1}(d(x))I = (0)$, then $\sigma^{-1}(d(x))RI = (0)$. Since I is a nonzero ideal of R and R is prime the above relation yields that $\sigma^{-1}(d(x)) = 0$ and hence $d(x) = 0$. Thus for each $x \in I$, either $d(x) = 0$ or $[r_1, \tau(x)] = 0$, for all $r_1 \in R$. Now let $A = \{x \in I \mid d(x) = 0\}$, $B = \{x \in I \mid [r_1, \tau(x)] = 0, \text{ for all } r_1 \in R\}$. Then A and B are additive subgroups of I and $I = A \cup B$. But a group can not be a union of two its proper subgroups and hence $I = A$ or $I = B$. If $I = A$, then $d(x) = 0$, for all $x \in I$. For any $s_1 \in R$, replace x by xs_1, to get $\tau(x)d(s_1) = 0$, for all $x \in I$ and hence $IR\tau^{-1}(d(s_1)) = (0)$. Again primeness of R implies that $\tau^{-1}(d(s_1)) = 0$, for all $s_1 \in R$, and hence $d = 0$. On the other hand if $I = B$, then that $\tau(x) \in Z(R)$, for all $x \in I$ and hence $x \in Z(R)$, for all $x \in I$ i.e. $I \subseteq Z(R)$. But if R is prime which has a nonzero central ideal, then R is commutative. □

Proof of Theorem 3. Let $c \in I$ be a constant i.e. an element such that $d(c) = 0$ and let z be an arbitrary element of I. The condition that $d(cz) = d(zc)$ yields that $\tau(c)d(z) = d(z)\sigma(c)$. Now for each $x, y \in I$, $[x, y]$ is a constant and hence

$$\tau([x, y])d(z) = d(z)\sigma([x, y])\,, \quad \text{for all} \quad x, y, z \in I\,. \tag{8}$$

We have $d(xy)=d(yx)$, for all $x,y\in I$. This can be rewritten as

$$(9)\qquad [d(x),y]_{\sigma,\tau}=[d(y),x]_{\sigma,\tau}\,,\quad \text{for all}\quad x,y\in I\,.$$

Replacing x by x^2 in (9) and using (9), we get

$$(10)\qquad d(x)\sigma([x,y])+\tau([x,y])d(x)=0\,,\quad \text{for all}\quad x,y\in I\,.$$

In view of (8) the above yields that $2\tau([x,y])d(x)=0$, for all $x,y\in I$. This implies that

$$(11)\qquad \tau([x,y])d(x)=0\,,\quad \text{for all}\quad x,y\in I\,.$$

Now, replacing y by yz in(11) and using (11), we find that $[x,y]z\tau^{-1}(d(x))=0$, for all $x,y,z\in I$ and hence $[x,y]IR\tau^{-1}(d(x))=(0)$, for all $x,y\in I$. Thus, primeness of R implies that for each $x\in I$, either $[x,y]I=(0)$ or $\tau^{-1}(d(x))=0$. Now, let $A=\{x\in I\mid [x,y]I=(0)$, for all $y\in I\}$, $B=\{x\in I\mid \tau^{-1}(d(x))=0\}$. Clearly, both A and B are additive subgroups of I whose union is I. By Brauer's trick we have either $I=A$ or $I=B$. If $I=B$, then $\tau^{-1}(d(x))=0$, for all $x\in I$ and hence $d(x)=0$, for all $x\in I$. For any $r\in R$, replace x by xr, to get $\tau(x)d(r)=0$, for all $x\in I$. This implies that $IR\tau^{-1}(d(r))=(0)$, for all $r\in R$. Since $I\neq(0)$, and R is prime the above relation yields that $\tau^{-1}(d(r))=0$, for all $r\in R$ and hence $d=0$, a contradiction. On the other hand if $I=A$, then $[x,y]I=(0)$, for all $x,y\in I$ i.e. $[x,y]RI=(0)$. Again since $I\neq(0)$, we get $[x,y]=0$, for all $x,y\in I$ and hence by the corollary of Lemma 1.1.5 of [10], R is commutative. □

The following example shows that the conclusion of the above theorem need not be true if I is a one sided ideal of R even in the case if d is assumed to be a derivation on R.

Example. Let R be a ring of 2×2 matrices over a field F; let $I=\begin{pmatrix}1&0\\0&0\end{pmatrix}R=\left\{\begin{pmatrix}a&b\\0&0\end{pmatrix}\,\middle|\, a,b\in F\right\}$. Let d be the inner derivation of R given by $d(x)=x\begin{pmatrix}0&1\\0&0\end{pmatrix}-\begin{pmatrix}0&1\\0&0\end{pmatrix}x$, for all $x\in R$. It is readily verified that d satisfies the property $d(xy)=d(yx)$, for all $x,y\in I$. However, R is not commutative.

Theorem 4. *Let R be a 2-torsion free prime ring and σ,τ be automorphisms of R. Suppose that d_1 and d_2 are two (σ,τ)-derivations of R such that $d_1\sigma=\sigma d_1, d_1\tau=\tau d_1, d_2\sigma=\sigma d_2$ and $d_2\tau=\tau d_2$. If $d_1d_2(R)=0$, then either $d_1=0$ or $d_2=0$.*

Proof. We have

$$(12)\qquad d_1d_2(x)=0\,,\quad \text{for all}\quad x\in R\,.$$

Replacing x by xy in (12) and using (12), we get

$$\tau(d_2(x))\sigma(d_1(y))+\tau(d_1(x))\sigma(d_2(y))=0\,,\quad \text{for all}\quad x,y\in R\,.$$

Again replace x by $\tau^{-1}(d_2(x))$ in the above expression and use (12), to get $d_2^2(x)\sigma(d_1(y)) = 0$, for all $x, y \in R$ and hence $\sigma^{-1}(d_2^2(x))d_1(y) = 0$, for all $x, y \in R$. Thus by Lemma 2.1 either $\sigma^{-1}(d_2^2(x)) = 0$, for all $x \in R$ or $d_1 = 0$. If $\sigma^{-1}(d_2^2(x)) = 0$, for all $x \in R$, then $d_2^2(x) = 0$, for all $x \in R$. Replacing x by xy and using the fact that $d_2^2(R) = 0$, we get $2\tau(d_2(x))\sigma(d_2(y)) = 0$, for all $x, y \in R$ and hence $\tau(d_2(x))\sigma(d_2(y)) = 0$. Again replace y by $\sigma^{-1}(y)$, to get $\tau(d_2(x))d_2(y) = 0$, for all $x, y \in R$ and hence again application of Lemma 2.1 gives that $d_2 = 0$ or $\tau(d_2(x)) = 0$, for all $x \in R$. If $\tau(d_2(x)) = 0$, for all $x \in R$, then $d_2 = 0$. This completes the proof of our theorem. $\square$

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