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MULTIPLE SOLUTIONS FOR NONLINEAR PERIODIC PROBLEMS WITH DISCONTINUITIES

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ABSTRACT. In this paper we consider a periodic problem driven by the one dimensional p-Laplacian and with a discontinuous right hand side. We pass to a multivalued problem, by filling in the gaps at the discontinuity points. Then for the multivalued problem, using the nonsmooth critical point theory, we establish the existence of at least three distinct periodic solutions.

1. INTRODUCTION

In recent years there have been several works dealing with differential equations involving the one dimensional p-Laplacian. Most of them study the Dirichlet problem and prove mainly existence results and in some cases also multiplicity results. We refer to the papers of Boccardo-Drábek-Giachetti-Kučera [1], De Coster [5], Del Pino-Elgueta-Manasevich [6] and Zhang [18]. Periodic problems have been considered by Dang-Oppenheimer [4], Del Pino-Manasevich-Murua [7], Fabry-Fayyad [9], Guo [11], Manasevich-Mawhin [14] and Papageorgiou-Yannakakis [15]. We should point out that the last two works consider vector valued problems and in addition Manasevich-Mawhin [14] employ a more general differential operator, which is not necessarily homogeneous of some order and which includes as a special case the one-dimensional p-Laplacian. The problem of existence of multiple periodic solutions was investigated only by Del Pino-Manasevich-Murua [7], who using conditions on the interaction of the vector field f with the Fučík spectrum of the operator, proved the existence of multiple solutions when $(t,x) \to f(t,x)$ is continuous. For the semilinear case ($p = 2$) multiple periodic solutions were proved by Drábek-Invernizzi [8] and Fabry-Mawhin-Nkashama [10].

In this paper we consider a scalar nonlinear periodic problem with a discontinuous right hand side, driven by the one dimensional p-Laplacian. Using a variational approach based on the nonsmooth critical point theory of Chang [2], we prove the existence of at least three periodic solutions.

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2. Preliminaries

Let $T = [0, b]$ and consider the following periodic problem:

$$\left\{ \begin{array}{c} (|x'(t)|^{p-2}x'(t))' = f(t, x(t)) \quad \text{a.e. on} \quad T \\ x(0) = x(b)\,, \quad x'(0) = x'(b)\,, \quad 2 \le p < \infty\,. \end{array} \right\} \tag{1}$$

We do not assume that $f(t, \cdot)$ is continuous. So in order to develop an existence theory, we need to pass to a multivalued version of (1) by, roughly speaking, filling in the gaps at the discontinuity points of $f(t, \cdot)$. For this purpose we define

$$f_1(t, x) = \underline{\lim}_{y \to x} f(t, y), \quad f_2(t, x) = \overline{\lim}_{y \to x} f(t, y)$$

and

$$\hat{f}(t, x) = [f_1(t, x), f_2(t, x)]\,.$$

Then instead of (1) which need not have solutions, we consider the following periodic second order differential inclusion:

$$\left\{ \begin{array}{c} (|x'(t)|^{p-2}x'(t))' \in \hat{f}(t, x(t)) \quad \text{a.e. on} \quad T \\ x(0) = x(b)\,, \quad x'(0) = x'(b)\,, \quad 2 \le p < \infty\,. \end{array} \right\} \tag{2}$$

We study (2) using the nonsmooth critical point theory of Chang [2]. Chang's theory is based on the subdifferential of Clarke [3] for locally Lipschitz functions. Let X be a Banach space. A function

$$\varphi : X \to \mathbb{R}$$

is said to be "locally Lipschitz", if for every bounded set $B \subseteq X$, there exists $k_B > 0$ such that

$$|\varphi(x) - \varphi(y)| \le k_B \|x - y\|$$

for all $x, y \in B$. The function

$$\varphi^0 : X \times X \to \mathbb{R}$$

defined by

$$\varphi^0(x; h) = \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{\varphi(y + \lambda h) - \varphi(y)}{\lambda}$$

is called the generalized directional derivative of φ. It is easy to check that $\varphi^0(x; \cdot)$ is sublinear and continuous. So by the Hahn-Banach theorem $\varphi^0(x; \cdot)$ is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x) \subseteq X^*$ given by

$$\partial\varphi(x) = \left\{ x^* \in X^* : \; (x^*, h) \le \varphi^0(x; h) \quad \text{for all} \quad h \in X \right\}\,.$$

The multifunction

$$\partial\varphi : X \to 2^{X^*} \setminus \{\emptyset\}$$

is called the generalized or Clarke subdifferential of φ. If

$$\varphi, \psi : X \to \mathbb{R}$$

are locally Lipschitz functions, then

$$\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x)$$

and

$$\partial(\lambda\varphi)(x) = \lambda\partial\varphi(x) \quad \text{for all} \quad \lambda \in \mathbb{R}\,.$$

If φ is also convex, then the generalized subdifferential of φ coincides with the subdifferential in the sense of convex analysis. Also if $\varphi \in C^1(X, \mathbb{R})$, then

$$\partial\varphi(x) = \{\varphi'(x)\}\ .$$

A point $x \in X$ is a critical point of φ, if

$$0 \in \partial\varphi(x)\,.$$

If x is a local extremum of φ, then x is a critical point of φ.

The smooth critical point theory, uses a compactness condition, known as the "Palais-Smale condition"(PS-condition). In the present nonsmooth setting this condition takes the following form:

"every sequence $\{x_n\}_{n\geq 1}$ such that

- $|\varphi(x_n)| \leq M$ for all $n \geq 1$
  and
- $m(x_n) = \inf[\|x^*\| : \ x^* \in \partial\varphi(x_n)] \to 0$,

has a strongly convergent subsequence." We will call this the "nonsmooth PS-condition".

The following extension of the "Saddle Point Theorem" is essentially due to Chang [2] (see also Hu-Papageorgiou [13], Theorem III.6.19, p. 312).

Theorem 1. *If X is a reflexive Banach space, $X = Y \oplus V$ with* $\dim Y < \infty$

$$\varphi : X \to \mathbb{R}$$

is locally Lipschitz, satisfies the nonsmooth PS-condition and there exist constants $\beta_1 < \beta_2$ and a neighborhood U of 0 in Y such that

$$\varphi|_{\partial U} \leq \beta_1 \quad \textit{and} \quad \varphi|_V \geq \beta_2\,,$$

then φ has a critical point $x \in X$ and $\varphi(x) \geq \beta_2$.

Our hypotheses on the discontinuous nonlinearity $f(t, x)$ are the following:

$H(f)$: $f : T \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that

(i) f_1, f_2 are N-measurable functions (i.e. for all $x : T \to \mathbb{R}$ measurable, $t \to f_k(t, x(t))$, $k = 1, 2$, are measurable);

(ii) there exist $a, \gamma \in L^q(T)$ $(\frac{1}{p} + \frac{1}{q} = 1)$ and $c > 0$ such that for almost all $t \in T$ and all $x \in \mathbb{R}$ we have

$$\gamma(t) \leq f(t, x) \leq a(t) + c|x|^{p-1}\,;$$

(iii) there exists $0 < c_1 < \frac{1}{pb^p}$ such that for almost all $t \in T$ and all $x \in \mathbb{R}$,

$$F(t, x) \geq -c_1|x|^p$$

where $F(t, x) = \int_0^x f(t, r)dr$ (the potential function corresponding to f);

(iv) $\lim_{|r|\to\infty} \int_0^b F(t, r)\, dt = +\infty$;

(v) there exists $\rho > 0$ such that for almost all $t \in T$ and all $|x| = \rho$, we have

$$F(t,x) < 0\,;$$

(vi) there exist

$$g : \mathbb{R} \to \mathbb{R}$$

bounded below, continuous at 0 and γ-subadditive (i.e. $g(x+y) \le \gamma(g(x) + g(y))$, $\gamma > 0$, $x, y \in \mathbb{R}$), $h \in L^1(T)$ with $\int_0^b (\frac{1}{\gamma} h^+(t) - \gamma h^-(t))\, dt \ge 0$ and $M_1 > 0$ such that for almost all $t \in T$ and all $|x| \ge M_1$,

$$F(t,x) \ge g(x)h(t)\,.$$

Remark 1. If f is time-invariant, then because f_1 is lower semicontinuous and f_2 is upper semicontinuous, hypothesis $H(f)$ (i) is satisfied. This is is also the case if $f(t,\cdot)$ is monotone nondecreasing, because then

$$f_1(t,x) = \lim_{n\to\infty} f(t, x - \frac{1}{n}) \quad \text{and} \quad f_2(t,x) = \lim_{n\to\infty} f(t, x + \frac{1}{n})$$

and so f_1, f_2 are both measurable, in particular then N-measurable. It is well known that the eigenvalue $\lambda_1 > 0$ is simple and isolated. Hypothesis $H(f)$ (iv) is a coercivity condition on the averaged potential and finally hypothesis $H(f)$ (vi) was first used by Tang [17] in the context of smooth semilinear second order periodic systems. The functions

$$g(x) = \frac{1}{1+|x|} \quad \text{and} \quad g(x) = |x|^r + \beta, \quad r \ge 1,\ \beta \ge 0$$

are γ-subadditive.

3. Auxiliary results

Let

$$W^{1,p}_{\text{per}}(T) = \{x \in W^{1,p}(T) :\ x(0) = x(b)\}$$

and consider the energy functional

$$\varphi : W^{1,p}_{\text{per}}(T) \to \mathbb{R}$$

defined by

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p + \int_0^b F(t, x(t))\, dt\,.$$

We know that φ is locally Lipschitz (see Chang [2]).

Proposition 1. *If hypotheses $H(f)$ hold, then φ is bounded below.*

Proof. From hypotheses $H(f)$ (ii) and (vi), we have that for almost all $t \in T$ and all $x \in \mathbb{R}$

$$F(t,x) \ge g(x)h(t) - \gamma_1(t)$$

with $\gamma_1 \in L^1(T)$ (in fact $\gamma_1(t) = M_1|\gamma(t)| + \max_{0\le x\le M_1}(g(x)h(t))$). Note that

$$W^{1,p}_{\text{per}}(T) = \mathbb{R} \oplus V\,,$$

where

$$V = \left\{ x \in W^{1,p}_{\text{per}}(T) : \int_0^b x(t)\,dt = 0 \right\}$$

(i.e. if $x \in W^{1,p}_{\text{per}}(T)$, then $x = \overline{x} + v$ with $\overline{x} \in \mathbb{R}$ and $v \in V$; see Hu-Papageorgiou [13], Proposition IV.7.8, p. 502). Exploiting the γ-subadditivity of g we have

$$\begin{aligned}
&\int_0^b F(t, x(t))\,dt \\
&\quad \geq \int_0^b g(x(t))h(t)\,dt - \int_0^b \gamma_1(t)\,dt \\
&\quad \geq \int_{h \geq 0} \left(\frac{1}{\gamma} g(\overline{x}) - \gamma g(-v(t)) \right) h(t)\,dt + \int_{h<0} \gamma(g(\overline{x}) + g(v(t)))h(t)\,dt - \|\gamma_1\|_1 \\
&\quad \geq g(\overline{x}) \int_0^b \left(\frac{1}{\gamma} h^+(t) - \gamma h^-(t) \right) dt - \int_0^b \gamma g(-v(t)) h^+(t)\,dt \\
&\qquad - \int_0^b \gamma g(v(t)) h^-(t)\,dt - \|\gamma_1\|_1 .
\end{aligned}$$

From Tang [17] (see inequality (7), p. 302), we know that for all $x \in \mathbb{R}$,

$$g(x) \leq \beta_3(1 + |x|)$$

for some $\beta_3 > 0$. Hence we obtain

$$\begin{aligned}
\varphi(x) &\geq \frac{1}{p}\|x'\|_p^p + g(\overline{x}) \int_0^b (\frac{1}{\gamma} h^+(t) - \gamma h^-(t))\,dt - 2\beta_3(1 + \|v\|_\infty)\|h\|_1 - \|\gamma_1\|_1 \\
&\geq \frac{1}{p}\|x'\|_p^p - \beta_4(1 + \|x'\|_p) - \beta_5 , \quad \text{for some} \quad \beta_4,\ \beta_5 > 0 . \qquad (3)
\end{aligned}$$

Here in obtaining the last inequality, we have used the fact that $g(\cdot)$ is bounded below and the Poincarè-Wirtinger inequality which says that $\|v\|_\infty \leq \beta_0 \|x'\|_p$ for some $\beta_0 > 0$ and all $x \in W^{1,p}_{\text{per}}(T)$. From (3) it follows that φ is bounded below on $W^{1,p}_{\text{per}}(T)$. □

Proposition 2. *If hypotheses $H(f)$ hold, then φ satisfies the nonsmooth PS-condition.*

Proof. Let

$$\{x_n\}_{n \geq 1} \subseteq W^{1,p}_{\text{per}}(T)$$

be such that

$$|\varphi(x_n)| \leq M_2$$

for all $n \geq 1$ and

$$m(x_n) \to 0 \quad \text{as} \quad n \to \infty .$$

Let

$$x_n^* \in \partial\varphi(x_n)$$

be such that

$$\|x_n^*\| = m(x_n) .$$

Its existence is guaranteed by the fact that $\partial\varphi(x_n) \subseteq W^{1,p}_{\mathrm{per}}(T)^*$ is w-compact and the norm functional is weakly lower semicontinuous. Let

$$A : W^{1,p}_{\mathrm{per}}(T) \to W^{1,p}_{\mathrm{per}}(T)^*$$

be the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_0^b |x'(t)|^{p-2}x'(t)y'(t)\,dt$$

for all $x, y \in W^{1,p}_{\mathrm{per}}(T)$ (here by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair

$$(W^{1,p}_{\mathrm{per}}(T), W^{1,p}_{\mathrm{per}}(T)^*))\,.$$

Using the elementary inequality which says that if $a_1,\ a_2 \in \mathbb{R}$ and $p \geq 2$, we have that

$$(|a_1|^{p-2}a_1 - |a_2|^{p-2}a_2)(a_1 - a_2) \geq 2^{2-p}|a_1 - a_2|^p$$

we see that A is monotone. Clearly it is also demicontinuous (i.e. strong to weak sequentially continuous), hence A is maximal monotone (see Hu-Papageorgiou [12], Corollary III. 1.35, p. 309). We have

$$x_n^* = A(x_n) + u_n$$

with $u_n \in L^q(T),\ f_1(t, x_n(t)) \leq u_n(t) \leq f_2(t, x_n(t))$ a.e. on T $n \geq 1$ (see Chang [2] or Hu-Papageorgiou [13], pp. 316-317). From the choice of the sequence $\{x_n\}_{n\geq 1}$ we have that

$$\left|\langle A(x_n), y\rangle - \int_0^b u_n(t)y(t)\,dt\right| \leq \varepsilon_n\|y\| \tag{4}$$

for all $y \in W^{1,p}_{\mathrm{per}}(T)$ with $\varepsilon_n \downarrow 0$. Let $y \equiv 1$. We obtain

$$\begin{aligned}
&\left|\int_0^b u_n(t)\,dt\right| \leq \beta_6 \quad \text{for all} \quad n \geq 1 \quad \text{and some} \quad \beta_6 > 0\\
&\Rightarrow \left|\int_0^b u_n^+(t) - u_n^-(t)\,dt\right| \leq \beta_6\\
&\Rightarrow \int_0^b u_n^+(t)\,dt \leq \beta_6 + \int_0^b u_n^-(t)\,dt \leq \beta_6 + \int_0^b \gamma^-(t)\,dt \leq \beta_6 + \|\gamma\|_1 = \beta_7\,.
\end{aligned}$$

So we deduce that $\{u_n\}_{n\geq 1} \subseteq L^1(T)$ is bounded.

Next let

$$x_n = \overline{x}_n + v_n\,,$$

with $\overline{x}_n \in \mathbb{R}$ ($\overline{x}_n = \frac{1}{b}\int_0^b x_n(t)\,dt$) and $v_n \in V$. In (4) let $y = v_n$. We obtain

$$\begin{aligned}
&\|v_n'\|_p^p - \int_0^b u_n(t)v_n(t)\,dt \leq \varepsilon_n\|v_n\|\,,\\
&\Rightarrow \|v_n'\|_p^p - \|u_n\|_1\|v_n\|_\infty \leq \varepsilon_n\|v_n\|\,.
\end{aligned}$$

From the Poincarè-Wirtinger inequality, we know that $\|v\|_\infty \leq \beta_0\|v'\|_p$ for some $\beta_0 > 0$ and all $v \in V$. So

$$\|v_n'\|_p^p - \beta_8\|v_n'\|_p \leq \varepsilon_n'\|v_n'\|_p$$

for some $\beta_8 > 0$, $\varepsilon_n' > 0$ with $\varepsilon_n' \downarrow 0$. From this inequality it follows that

$$\{v_n' = x_n'\}_{n\geq 1} \subseteq L^p(T)$$

is bounded. We claim that

$$\{x_n\}_{n\geq 1} \subseteq C(T)$$

is bounded. Suppose not. Then we must have

$$L_n = \max_T x_n \to +\infty \quad \text{or} \quad l_n = \min_T x_n \to -\infty$$

as $n \to \infty$. Suppose the first holds (the analysis is similar if the second possibility is in effect). We have

$$\frac{1}{p}\|x_n'\|_p^p + \int_0^b \int_0^{L_n} f(t,r)\,dr\,dt + \int_0^b \int_{L_n}^{x_n(t)} f(t,r)\,dr\,dt \leq M_2 \,. \tag{5}$$

By hypothesis $H(f)$ (ii) we have

$$\begin{aligned}
\int_0^b \int_{L_n}^{x_n(t)} f(t,r)\,dr\,dt \quad &\geq \quad \int_0^b \int_{L_n}^{x_n(t)} \gamma(t)\,dr\,dt = \int_0^b \gamma(t)(x_n(t) - L_n)\,dt \\
&\geq \quad -\|\gamma\|_1 \|x_n - L_n\|_\infty \\
\Rightarrow -\int_0^b \int_{L_n}^{x_n(t)} f(t,r)\,dr\,dt \quad &\leq \quad \|\gamma\|_1 \|x_n - L_n\|_\infty \,.
\end{aligned}$$

Let $t_n \in T$ be such that $x_n(t_n) = L_n$, $n \geq 1$. We have

$$x_n(t_n) - x_n(t) = \int_t^{t_n} x_n'(s)\,ds\,, \quad t \in T$$

$$\Rightarrow |L_n - x_n(t)| \leq \|x_n'\|_1 \leq \beta_9\,,$$

for all $n \geq 1$, all $t \in T$ and some $\beta_9 > 0$,

$$\Rightarrow -\int_0^b \int_{L_n}^{x_n(t)} f(t,r)\,dr\,dt \leq \beta_{10} \quad \text{for some} \quad \beta_{10} > 0\,.$$

Therefore returning to (5), we see that

$$\int_0^b F(t, L_n)\,dt \leq \beta_{11}$$

for all $n \geq 1$ and some $\beta_{11} > 0$. But because $L_n \to +\infty$, from hypothesis $H(f)$ (iv), we have that

$$\int_0^b F(t, L_n)\,dt \to +\infty\,,$$

a contradiction. This proves that

$$\{x_n\}_{n\geq 1} \subseteq C(T)$$

is bounded and so we infer that

$$\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(T)$$

is bounded. By passing to a subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \quad \text{in} \quad W^{1,p}_{\text{per}}(T)$$

and

$$x_n \to x \quad \text{in} \quad L^p(T)\,.$$

Denoting by $(\cdot,\cdot)_{pq}$ the duality brackets for the pair

$$(L^p(T), L^q(T))\,,$$

we have

$$\lim[\langle A(x_n), x_n - x\rangle - (u_n, x_n - x)_{pq}] \le \lim \varepsilon_n \|x_n - x\| = 0$$

(here $\|x_n - x\|$ is the Sobolev norm of $x_n - x \in W^{1,p}_{\text{per}}(T)$)

$$\Rightarrow \overline{\lim}\langle A(x_n), x_n - x\rangle \le 0\,.$$

This inequality follows from the fact that $(u_n, x_n - x)_{pq} \to 0$ (since $\{u_n\}_{n\ge1} \subseteq L^q(T)$ is bounded and $x_n \to x$ in L^p. But A being maximal monotone, is generalized pseudomonotone (see Hu-Papageorgiou [12], p. 365) and so

$$\|x'_n\|_p^p = \langle A(x_n), x_n\rangle \to \langle A(x), x\rangle = \|x'\|_p^p\,.$$

But $L^p(T)$ is uniformly convex. So

$$x'_n \to x' \quad \text{in} \quad L^p(T)$$

(Kadec-Klee property, see Hu-Papageorgiou [12], Lemma I.1.74, p. 28) and from this we infer that

$$x_n \to x \quad \text{in} \quad W^{1,p}_{\text{per}}(T)\,.$$

Therefore φ satisfies the nonsmooth PS-condition. □

Proposition 3. *If hypotheses $H(f)$ hold, then*

$$\varphi|_V \ge 0\,.$$

Proof. Let $v \in V$. Using hypothesis $H(f)$ (iii) and since by the Wirtinger inequality (see Mawhin-Willem [19], p. 8)

$$\|v\|_p^p \le b^p\|v'\|_p^p \quad \text{for all} \quad v \in V\,,$$

we obtain

$$\varphi(v) \ge \frac{1}{p}\|v'\|_p^p - c_1\|v\|_p^p \ge \frac{1}{p}\|v'\|_p^p - \frac{c_1}{b^p}\|v'\|_p^p$$

(recall $c_1 < \frac{1}{pb^p}$). □

4. The multiplicity theorem

In this section we state and prove our theorem on the existence of multiple solutions for (2).

Theorem 2. *If hypotheses $H(f)$ hold, then problem* (2) *has at least three distinct solutions.*

Proof. Let

$$U^{\pm} = \left\{ x \in W^{1,p}_{\mathrm{per}}(T) : \; x = c + v \,, \quad \text{with} \quad c > 0 \quad (\text{resp.} \quad c < 0) \,, \quad v \in V \right\} \,.$$

We claim that φ attains its infimum on both open sets U^+, U^-. To this end let

$$m_+ = \inf[\varphi(x) : \; x \in U^+] = \inf[\varphi(x) : \; x \in \overline{U^+}\,] \,.$$

Let

$$\overline{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in \overline{U^+} \\ +\infty & \text{otherwise} \end{cases}$$

It is clear that $\overline{\varphi}$ is lower semicontinuous and bounded below on $W^{1,p}_{\mathrm{per}}(T)$ (see Proposition 1). So we can apply the Ekeland variational principle (see for example Hu-Papageorgiou [12], p. 519), to obtain

$$\{x_n\}_{n \geq 1} \subseteq U^+$$

such that $\overline{\varphi}(x_n) = \varphi(x_n) \downarrow m_+$ and

$$\begin{aligned} \overline{\varphi}(x_n) &\leq \overline{\varphi}(y) + \varepsilon_n \|x_n - y\| \quad \text{for all} \quad y \in W^{1,p}_{\mathrm{per}}(T) \,, \quad \text{with} \quad \varepsilon_n \downarrow 0 \\ \Rightarrow \varphi(x_n) &\leq \varphi(y) + \varepsilon_n \|x_n - y\| \quad \text{for all} \quad y \in U^+ \,. \end{aligned}$$

Let $w \in W^{1,p}_{\mathrm{per}}(T)$ and $\lambda > 0$. Since U^+ is open in $W^{1,p}_{\mathrm{per}}(T)$ and $x_n \in U^+$, we can find $\delta > 0$ (depending on $n \geq 1$) such that for all $\lambda \in [0, \delta]$, $x_n + \lambda w \in U^+$. Hence we have for all $\lambda \in (0, \delta]$

$$\begin{aligned} -\varepsilon_n \|w\| &\leq \frac{\varphi(x_n + \lambda w) - \varphi(x_n)}{\lambda} \\ \Rightarrow -\varepsilon_n \|w\| &\leq \varphi^0(x_n; w) \,. \end{aligned}$$

Invoking lemma 1.3 of Szulkin [16], we can find

$$y_n^* \in W^{1,p}_{\mathrm{per}}(T)^* \,,$$

with $\|y_n^*\| \leq 1$ such that

$$\langle \varepsilon_n y_n^*, w \rangle \leq \varphi^0(x_n; w) \quad \text{for all} \quad w \in W^{1,p}_{\mathrm{per}}(T)$$

$$\Rightarrow x_n^* = \varepsilon_n y_n^* \in \partial\varphi(x_n) \,, \; n \geq 1 \,, \quad \text{and} \quad \|x_n^*\| \to 0 \,.$$

By virtue of Proposition 2, we may assume that

$$x_n \to y_1 \quad \text{in} \quad W^{1,p}_{\mathrm{per}}(T) \,.$$

Then

$$\varphi(x_n) \to \varphi(y_1) = m_+$$

and $y_1 \in \overline{U^+}$. If $y_1 \in \partial U^+ = V$, then from Proposition 3 we have

$$0 \leq \varphi(y_1) \,,$$

while from hypothesis $H(f)$ (v) we have that

$$\varphi(y_1) = m_+ < 0 \,,$$

a contradiction. So

$$y_1 \in \operatorname{int} \overline{U^+} = U^+$$

and this proves that y_1 is a local minimum of φ on $W^{1,p}_{\text{per}}(T)$, hence

$$0 \in \partial\varphi(y_1)\,.$$

Similarly we obtain $y_2 \in U^-$ with

$$0 \in \partial\varphi(y_2)\,.$$

Evidently $y_1 \neq y_2$.

Moreover, hypothesis $H(f)$ (v), together with Propositions 2 and 3, allow the application of Theorem 1, which gives $y_3 \in W^{1,p}_{\text{per}}(T)$ such that

$$0 \in \partial\varphi(y_3)$$

and

$$\varphi(y_3) \geq 0\,.$$

Since $\varphi(y_1) = m_+ < 0$, $\varphi(y_2) = m_- < 0$, we see that $y_3 \neq y_1$, $y_3 \neq y_2$.

Now let $y = y_k$, $k = 1, 2, 3$. Since $0 \in \partial\varphi(y)$, we have

$$\begin{aligned}
&A(y) + u = 0 \quad \text{with} \quad u \in L^q(T),\ f_1(t, y(t)) \leq u(t) \leq f_2(t, y(t)) \\
&\quad \Rightarrow \langle A(y), \theta\rangle + (u, \theta)_{pq} = 0 \quad \text{for all} \quad \theta \in C_0^\infty(T) \\
&\quad \Rightarrow \left\{ \begin{array}{rcl} -(|y'(t)|^{p-2}y'(t))' + u(t) & = & 0 \ \text{ a.e. on } \ T \\ y(0) & = & y(b) \end{array} \right\}
\end{aligned}$$

(by the definition of distributional derivative).

Also we have

$$\langle A(y), w\rangle + (u, w)_{pq} = 0$$

for all $w \in W^{1,p}_{\text{per}}(T)$. From Green's identity (integration by parts), we have

$$\begin{aligned}
\langle A(y), w\rangle &= \int_0^b |y'(t)|^{p-2}y'(t)w'(t)\,dt \\
&= |y'(b)|^{p-2}y'(b)w(b) - |y'(0)|^{p-2}y'(0)w(0) \\
&\quad - \int_0^b (|y'(t)|^{p-2}y'(t))'w(t)\,dt \\
&= |y'(b)|^{p-2}y'(b)w(b) - |y'(0)|^{p-2}y'(0)w(0) - (u, w)_{pq} \\
\Rightarrow |y'(0)|^{p-2}&y'(0)w(0) = |y'(b)|^{p-2}y'(b)w(b)\,.
\end{aligned}$$

Let $w \in W^{1,p}_{\text{per}}(T)$ be such that $w(0) = w(b) = 1$. We have

$$|y'(0)|^{p-2}y'(0) = |y'(b)|^{p-2}y'(b)$$

and since the map $r \to |r|^{p-2}r$ is a homeomorphism on $\mathbb{R}^N$, we conclude that

$$y'(0) = y'(b)\,.$$

Therefore y_1, y_2, y_3 are three distinct solutions of (2). □

Remark 2. Clearly y_1, y_2, are nonzero, while y_3 is nonzero, provided that $f(t, 0) \neq 0$ for all $t \in T_0$, $|T_0| > 0$ ($|\cdot|$ denotes the Lebesgue measure on T).

As a simple example consider the following locally Lipschitz but nonsmooth potential function $F(x)$ (for simplicity we drop the t-dependence)

$$F(x)=\begin{cases} -c_1|x|^p, & |x|\le 1\\ |x|^p-(c_1+1), & |x|>1\end{cases}$$

where $0<c_1<\frac{1}{pb^p}$. Clearly hypotheses $H(f)$ (i) $\to$ (iv) are satisfied. Also if $p=1$, then hypothesis $H(f)$ (v) also holds. Finally let $0<\varepsilon\le c_1+1$ and $M_1=(\frac{c_1+1}{\varepsilon})^{\frac{1}{p}}$. Then for $|x|\ge M_1$ we have $|x|^p-(c_1+1)\ge(1-\varepsilon)|x|^p=g(x)$ and clearly g is 2^{p-1}-subadditive.

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