ARCHIVUM MATHEMATICUM (BRNO)
Tomus 38 (2002), 183 – 200

HETEROCLINIC ORBITS IN PLANE DYNAMICAL SYSTEMS

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Abstract. We consider general second order boundary value problems on the whole line of the type $u'' = h(t,u,u')$, $u(-\infty) = 0, u(+\infty) = 1$, for which we provide existence, non-existence, multiplicity results. The solutions we find can be reviewed as heteroclinic orbits in the (u,u') plane dynamical system.

1. Introduction

The main aim of this paper is to state existence and multiplicity results for the boundary value problem on the whole line

$$\text{(P)} \qquad \begin{cases} u'' = h(t,u,u') \\ u(-\infty) = 0\,, \quad u(+\infty) = 1 \end{cases}$$

where h is a continuous function on R^3 such that $h(t,0,0) = h(t,1,0) = 0$ for all $t \in R$.

Our initial motivation for treating (P) came from the study of travelling wavefronts for reaction-diffusion equations which arise from chemical and biological models (see [7] and references therein contained). In this context problem (P) is generally autonomous and one is interested in finding monotone solutions for it. Notice that the investigation of travelling wave solutions for different type of dynamics is still under intensive research; see e.g. [1] and [3] for recent results dealing with lattice differential equations.

On the other hand, the solvability of (P) also comes from the existence of nontrivial stationary solutions for semi-linear parabolic equations (see e.g. [11]).

Moreover, the study of (P) has interesting applications in the field of plane dynamical systems. In fact, since the solutions of (P) satisfy $\lim_{t\to\pm\infty} u'(t) = 0$ (see Lemma 4.1 for the case when h is autonomous), they can be reviewed as heteroclinic connections between the singular points $(0,0)$ and $(1,0)$. See also [2]

2000 *Mathematics Subject Classification*: Primary: 34C37; Secondary: 34B16, 34B40, 37C29.
Key words and phrases: nonlinear boundary value problems, heteroclinic solutions, lower and upper solutions, singular boundary value problems.
Received September 26, 2000.

and [4] for the existence of heteroclinic orbits of delay differential equations and their monotonicity properties.

Finally, we underline that, in the autonomous case, the existence of monotone solutions of (P) is equivalent (see Lemma 4.2) to the existence of positive solutions for a singular non-autonomous first order problem of the type

$$\text{(Q)} \qquad \begin{cases} x' = f(t,x) \\ x(0^+) = x(1^-) = 0 \end{cases}$$

where f is singular for $x = 0$.

Being (Q) of the first order, it is in a certain sense an over-determined problem and, according to our knowledge, it was never previously investigated.

On the contrary, a quite wide literature is available for second order singular boundary value problems both on finite intervals and on a half-line (see [5], [9] and references therein contained).

As regards problem (P), up to now a general theory was not completely developed. One can find a result of general type in [9, Thm. 14.3], where the existence of an ordered couple of upper and lower solutions for (P), together with a growth condition on h are assumed. Moreover, other results for special types of equations were established, such as the recent one by Ortega and Tineo [10], concerning the so-called Landesman-Lazer equations.

In this paper we propose a comparison-type approach for investigating (P), which proceeds by the following scheme.

First of all in Section 2 we recall a classical result on differential inequalities and we adapt a recent one due to Marcelli and Rubbioni [8] to the case of non-compact domains.

Then, in Section 3 we handle the associated first order singular problem (Q), with f essentially superlinear (possibly linear) in x for x large enough, and f singular in x when $x \to 0^+$, of the same type as $\frac{1}{x^\beta}$ with $0 < \beta \le 1$. Combining the upper and lower-solutions method discussed in Section 2 and phase-plane techniques, we obtain for (Q) both an existence and a non-existence result.

Then in Section 4, we are able to treat problem (P) in the autonomous case, and provide existence, non-existence and uniqueness results of heteroclinic orbits in the plane (u, u'). In particular, we obtain the following existence theorem.

Theorem A. *Given a continuous function $h : R^2 \to R$ satisfying $h(0,0) = h(1,0) = 0$, consider the boundary value problem*

$$\text{(1)} \qquad \begin{cases} u'' = h(u,u') \\ u(-\infty) = 0\,, \quad u(+\infty) = 1\,. \end{cases}$$

Assume the existence of a constant $L > 0$ such that one of the following pairs of conditions holds

$$\begin{array}{lll} h(u,0) < 0 & \textit{for all} & u \in \,]0,1[\\ h(u,v) \ge 2\sqrt{L}v - Lu & \textit{for all} & (u,v) \in [0,1] \times [0,+\infty[\,, \end{array}$$

or

$$\begin{array}{lll} h(u,0) > 0 & \textit{for all} & u \in \,]0,1[\\ h(u,v) \le -2\sqrt{L}v + L(1-u) & \textit{for all} & (u,v) \in [0,1] \times [0,+\infty[\,, \end{array}$$

then (P) *has a strictly monotone solution.*

We also state non-existence and uniqueness conditions.

Finally, Section 5 is devoted to the study of problem (P) in the non-autonomous case. We assume the existence of autonomous functions h_1 and h_2, satisfying the conditions of Theorem A and such that

$$h_1(u,v) \le h(t,u,v) \le h_2(u,v) \quad \text{for all} \quad (u,v) \in \,]0,1[\, \times \,]0,+\infty[\,.$$

By means of a suitable fixed-point technique, also used in [11], we are able to prove the existence of a one-parameter family of solutions of the non-autonomous problem (P).

2. About our comparison-type approach

This section deals with some existence results for first order equations, obtained by means of upper and lower solution techniques, which are a key tool for constructing our approach used to investigate (P).

Given a continuous function $g : \,]0,1[\, \times R \to R$ and the first order equation

$$x' = g(t,x) \tag{2}$$

we recall that a function $\gamma \in C^1(]0,1[)$ is said to be a lower solution (an upper solution) for (2) in $]0,1[$ if

$$\gamma'(t) \le g(t,\gamma(t)) \quad (\text{alternatively} \quad \gamma'(t) \ge g(t,\gamma(t))\,) \quad \text{for all} \quad t \in \,]0,1[\,.$$

The following theorem is a direct consequence of recent comparison-type results obtained by Marcelli and Rubbioni [8].

Theorem 2.1. *Given a continuous function* $g : \,]0,1[\, \times \,]0,+\infty[\, \to R$*, let us consider the first order equation* (2)*. Assume there exist two positive functions* $\gamma_1(t)$ *and* $\gamma_2(t)$ *which are respectively a lower and an upper solution for* (2) *on* $]0,1[$*. Then* (2) *admits a solution* $\gamma(t)$ *satisfying*

$$\min\{\gamma_1(t),\gamma_2(t)\} \le \gamma(t) \le \max\{\gamma_1(t),\gamma_2(t)\}, \quad \textit{for all} \quad t \in \,]0,1[\,.$$

Proof. For all $t \in \,]0,1[$, let us define

$$m(t) = \min\{\gamma_1(t),\gamma_2(t)\}\,, \quad M(t) = \max\{\gamma_1(t),\gamma_2(t)\}\,.$$

Let $(I_n)_n$ be an increasing sequence of compact intervals whose union is $]0,1[$.

For every $n \in N$ we can apply [8], Corollary 6 (see also [8], Remark 1) in the interval I_n and we obtain a solution ψ_n of (2) in I_n satisfying $m(t) \le \psi_n(t) \le M(t)$ for all $t \in I_n$. We denote by ϕ_n the maximal solution of (2) in I_n which lies between the two functions m and M and extend it to the whole interval $]0,1[$ in a constant way.

For each $n \in N$ put

$$J_n = \big[\min_{t\in I_n} m(t),\ \max_{t\in I_n} M(t)\big]\,.$$

Given $\bar{n} \in N$, observe that for all $t \in I_{\bar{n}}$ and every $n \geq \bar{n}$ we have

$$|\phi_n'(t)| = |g(t,\phi_n(t)| \leq \max_{(t,x)\in I_{\bar{n}}\times J_{\bar{n}}} |g(t,x)| < +\infty \quad \text{ and } \quad \phi_n(t) \geq \phi_{n+1}(t)\,.$$

Thus the sequence $(\phi_n)_n$ is equi-lipschitzian and definitively monotone non-decreasing in any compact subset of $]0,1[$. Hence it uniformly converges on compact sets of $]0,1[$ to a continuous function γ on $]0,1[$. Therefore, γ satisfies equation (2) and

$$m(t) \leq \gamma(t) \leq M(t) \quad \text{for all} \quad t \in\,]0,1[\,. \qquad \square$$

We recall now a classical comparison-type result (see e.g. [12])

Lemma 2.2. *Let $g : [0,1]\times R \to R$ be a continuous function and let $\gamma \in C^1([0,1])$ be such that*

$$\gamma'(t) \leq g(t,\gamma(t))\,, \qquad t \in [0,1]\,.$$

Then, if $\gamma(a) \leq x_0$, we have $\gamma(t) \leq \varphi(t)$, $t \in [0,1]$, where φ is the maximal solution of the initial value problem

$$x'(t) = g(t,x), \quad x(a) = x_0\,.$$

Moreover, if $\gamma(b) \geq x_1$, we have $\gamma(t) \geq \theta(t)$, $t \in [0,1]$, where θ is the minimal solution of the terminal value problem

$$x' = g(t,x) \qquad x(b) = x_1\,.$$

3. On a singular first order problem

Given the continuous function $f :]0,1[\times]0,+\infty[\to R$ we are now interested to discuss the solvability of the following boundary value problem

$$\text{(3)} \qquad \begin{cases} x' = f(t,x) \\ x(0^+) = x(1^-) = 0\,. \end{cases}$$

We deal with the case when f is infinite when $x = 0$, that is (3) is a singular problem. More precisely, throughout this section we shall assume the existence of a constant $\beta > 0$ such that

$$\text{(4)} \qquad \limsup_{(t,x)\to(t_0,0)} x^\beta f(t,x) < 0 \quad \text{for all} \quad t_0 \in\,]0,1[$$

or

$$\text{(5)} \qquad \liminf_{(t,x)\to(t_0,0)} x^\beta f(t,x) > 0 \quad \text{for all} \quad t_0 \in\,]0,1[\,.$$

Hence, in both cases,

$$\lim_{x\to 0} |f(t,x)| = +\infty \quad \text{for all } t \in\,]0,1[\,.$$

As a typical example of our dynamics f near the points $(t, 0)$ with $t \in]0, 1[$, we can take

$$f(t, x) = \frac{F(t, x)}{x^\beta}$$

where $F :]0, 1[\times [0, +\infty[\to R$ is an arbitrary continuous function, such that $F(t, 0) \neq 0$ for every $0 < t < 1$. Indeed, when $F(t, 0) < 0$ in $]0, 1[$ then condition (4) holds, whereas $F(t, 0) > 0$ in $]0, 1[$ implies condition (5).

Our approach in dealing with the existence of solutions of problem (3), is based on a suitable combination between methods of upper and lower-solutions and phase-plane techniques.

First we consider the case when f is negative near the t-axis, i.e. condition (4) holds, and by means of a comparison-type technique, we get in Thm. 3.2 both an existence and a non-existence result for problem (3). Subsequently, the case when f is positive, i.e. condition (5) holds, will be treated by means of a suitable change of variables, reducing the problem to the previous one and obtaining analogous existence, non-existence results (see Thm. 3.4).

The following lemma shows that, for any given $t_0 \in]0, 1[$, the maximal right existence interval of every solution of the Cauchy problem

$$\left\{ \begin{array}{l} x' = f(t, x) \\ x(t_0) = \alpha \end{array} \right. \tag{6}$$

can be as small as one desires, provided α is sufficiently small.

Lemma 3.1. *Assume condition* (4). *Then, for all* $t_0 \in]0, 1[$ *and* $\epsilon > 0$ *it is possible to find* $\bar{\alpha} = \bar{\alpha}(t_0, \epsilon)$ *such that for every solution* $x(t)$ *of the Cauchy problem* (6), *with* $\alpha \in]0, \bar{\alpha}[$, *there exists a value* $\tau_x \in]t_0, t_0 + \epsilon[$ *satisfying*

$$\lim_{t \to \tau_x^-} x(t) = 0$$

i.e. $[t_0, \tau_x[\subset [t_0, t_0 + \epsilon[$ *is the maximal right existence interval of the solution* x.

Proof. Given $t_0 \in]0, 1[$ and $\epsilon > 0$, by (4) it is possible to find $\bar{\sigma} \in]0, \epsilon[$ and $M > 0$ such that

$$f(t, x) \leq -\frac{M}{x^\beta} \quad \text{for all } (t, x) \in [t_0, t_0 + \bar{\sigma}] \times]0, \bar{\sigma}] \,. \tag{7}$$

Denoting by $\bar{\alpha} = \min\{(\bar{\sigma} M(\beta + 1))^{\frac{1}{\beta+1}}, \bar{\sigma}\}$ and given $0 < \alpha < \bar{\alpha}$, let us consider the Cauchy problem

$$\left\{ \begin{array}{l} x' = -\frac{M}{x^\beta} \\ x(t_0) = \alpha \,. \end{array} \right. \tag{8}$$

It is easy to check that the solution $\gamma(t) = (\alpha^{\beta+1} - (\beta + 1)M(t - t_0))^{\frac{1}{\beta+1}}$ of (8) is defined for all $t_0 \leq t \leq t_0 + \frac{\alpha^{\beta+1}}{M(\beta+1)}$ and since $\alpha < \bar{\alpha}$, we have $\frac{\alpha^{\beta+1}}{M(\beta+1)} < \bar{\sigma}$.

For α as before, consider now a solution $\xi(t)$ of (6), defined on its maximal right interval $[t_0, \tau_x[$. Since $\alpha \leq \bar{\sigma}$, we have

$$\xi'(t_0) = f(t_0, \alpha) \leq -\frac{M}{\alpha^\beta} < 0$$

and taking account of (7), we deduce that $\xi(t)$ is decreasing on $[t_0, \tau_x[\cap[t_0, \bar{\sigma}]$. Therefore, we have $0 < \xi(t) \le \alpha$ on the same interval.

Consequently, for all $t \in [t_0, t_0 + \frac{\alpha^{\beta+1}}{M(\beta+1)}[$, it holds

$$\xi'(t) = f(t, \xi(t)) \le -\frac{M}{\xi(t)^\beta}$$

hence ξ is a lower solution for the equation in (8) and by Lemma 2.2 we have $\xi(t) \le \gamma(t)$ for all t, implying $\tau_x < t_0 + \bar{\sigma} \le t_0 + \epsilon$. □

Now we can state and prove the main result of this section.

Theorem 3.2. *Consider the boundary value problem* (3) *with* f *satisfying* (4).

i) *If there exists* $L > 0$ *such that*

$$f(t,x) \ge 2\sqrt{L} - \frac{Lt}{x} \quad \text{for all} \quad (t,x) \in]0,1[\times]0,+\infty[\,, \tag{9}$$

then (3) *has a strictly positive solution.*

ii) *If there exist* $0 < M < L$ *and* $\epsilon > 0$ *such that*

$$f(t,x) \le 2\sqrt{M} - \frac{Lt}{x} \quad \text{for all} \quad (t,x) \in]0,\epsilon] \times]0,\epsilon]\,, \tag{10}$$

then (3) *does not admit solutions.*

Proof. i) Assume condition (9) and consider the function $\xi = \sqrt{L}t$, defined for $t \in [0,1]$. According to (9), for any $t \in]0,1[$ it holds

$$\xi'(t) = 2\sqrt{L} - \frac{Lt}{\xi(t)} \le f(t, \xi(t))\,; \tag{11}$$

hence $\xi(t)$ is a strictly positive lower-solution on $]0,1[$ for the equation in (3). Denoting by

$$\Omega = \{(t,x) : 0 < t \le 1,\ 0 \le x \le \xi(t)\}\,,$$

the set of points (t,x) between the graph of ξ and t-axis, we shall prove the existence of a solution of (3) which lies in Ω.

For any $n \in N$, consider the Cauchy problem

$$\begin{cases} x' = f(t,x) \\ x\left(1 - \frac{1}{2n}\right) = \alpha\,. \end{cases} \tag{12}$$

According to Lemma 3.1 it is possible to find $\bar{\alpha}_n > 0$ such that the maximal existence interval of each solution of (12) with $0 < \alpha < \bar{\alpha}_n$ is strictly contained in $]0, 1 - \frac{1}{2n+2}[$. Given

$$\alpha_n = \min\left\{\bar{\alpha}_n, \frac{\sqrt{L}}{2n}\right\}$$

denote by ξ_n the minimal solution of

$$\begin{cases} x' = f(t,x) \\ x\left(1 - \frac{1}{2n}\right) = \alpha_n\,. \end{cases} \tag{13}$$

First notice that $\xi_n(t) > 0$ is defined and positive in $]0, 1 - \frac{1}{2n}]$. Assume, in fact, the existence of $t_0 \in]0, 1 - \frac{1}{2n}[$ such that $\lim_{t \to t_0^+} \xi(t) = 0$ and $\xi(t) > 0$ for all $t \in]t_0, 1 - 1/2n]$. According to condition (4), it is possible to find two positive constants λ and σ such that

$$f(t, x) < -\frac{\lambda}{x^\beta} \quad \text{for all} \quad (t, x) \in [t_0, t_0 + \sigma] \times]0, \sigma],$$

therefore, for t in a right neighborhood of t_0, it holds

$$\xi_n'(t) = f\big(t, \xi_n(t)\big) < -\frac{\lambda}{(\xi_n(t))^\beta} < 0$$

in contradiction with $\xi(t_0^+) = 0$. Consequently each ξ_n turns out to be defined on all $]0, 1 - 1/2n]$; moreover, since $\xi_n(1 - \frac{1}{2n}) = \alpha_n \le \frac{\sqrt{L}}{2n} \le \sqrt{L}(1 - \frac{1}{2n}) = \xi(1 - \frac{1}{2n})$ for all n, by (11) we can apply Lemma 2.2 to show that

$$\xi_n(t) \le \xi(t) \quad \text{for all} \quad t \in]0, 1 - \frac{1}{2n}], \quad n \in N.$$

Since, in addition, $0 < \alpha_n < \bar{\alpha}_n$, the maximal existence interval of ξ_n is $]0, \tau_n[$ with $\tau_n \in]1 - \frac{1}{2n}, 1 - \frac{1}{2n+2}[$ and it holds $\lim_{t \to \tau_n^-} \xi_n(t) = 0$. Put $\xi_n(t) = 0$ for all $t \in [\tau_n, 1]$, each ξ_n is continuous on $]0, 1]$ and satisfies

$$\big(t, \xi_n(t)\big) \in \Omega \quad \text{for all} \quad t \in]0, 1].$$

By means of a recursive process we obtain now a solution p of problem (3). Define the following monotone non-decreasing sequence of continuous functions $p_n :]0, 1[\to R$

$$p_1(t) = \xi_1(t), \quad p_{n+1}(t) = \max\{p_n(t), \xi_{n+1}(t)\}, \quad n \in N$$

and denote by $p(t)$ its limit. Notice that $\big(t, p(t)\big) \in \Omega$ for all $t \in]0, 1[$.

We prove now that the convergence is uniform on any compact interval $[a, b] \subset]0, 1[$. Indeed, let $\bar{n}$ be such that $b < 1 - \frac{1}{2\bar{n}}$ and put

$$r = \min_{a \le t \le b} p_{\overline{n}}(t) \quad \text{and} \quad K = \max_{(t,x) \in [a,b] \times [r, \sqrt{L}]} |f(t, x)|.$$

For every $t \in [a, b]$ and $n > \bar{n}$, it holds $0 < r \le p_n(t) \le \sqrt{L}$ and then $|p_n'(t)| = |f\big(t, p_n(t)\big)| \le K$. Therefore $\big(p_n\big)_n$ is equi-lipschitzian and then it uniformly converges to p in $[a, b]$; hence $p(t)$ is continuous and is a solution of the equation in (3) for $t \in [a, b]$. By the arbitrariness of $[a, b]$, p solves the equation in (3) on all $]0, 1[$.

Now it remains to prove that p satisfies the limit conditions. Since $0 < p(t) \le \sqrt{L}t$ for all $t \in]0, 1[$, we immediately obtain $\lim_{t \to 0^+} p(t) = 0$.

Let us assume, by contradiction, that

$$\limsup_{t \to 1^-} p(t) = l > 0$$

and take $c, d \in R$ with $0 < c < d < l$. According to (9), it is then possible to find $k > 0$ such that

$$f(t,x) \geq -k \quad \text{for every} \quad t \in]0,1[\,, x \in [c, \sqrt{L}]\,. \tag{14}$$

Put $\delta = (d-c)/k$, let $t_0 \in]1-\delta, 1[$ be such that $p(t_0) > d$. Let $m \in N$ satisfying $p_m(t_0) > d$, $1 - 1/2m > t_0$ and $\sqrt{L}/2m < c$. By virtue of the continuity of p_m, since $p_m(1-1/2m) \leq \sqrt{L}/2m$, there exists a value $\bar{t} \in]t_0, 1[$ such that $p_m(\bar{t}) = c$ and $p_m(t) \geq c$ for every $t \in [t_0, \bar{t}]$. Therefore, by (14), we deduce that

$$c = p_m(\bar{t}) \geq p_m(t_0) - k(\bar{t} - t_0) > d + c - d = c\,,$$

a contradiction. Hence $\limsup\limits_{t\to 1^-} p(t) = 0$ and since p is strictly positive on $]0,1[$, this implies $\lim\limits_{t\to 1^-} p(t) = 0$, so that p solves problem (3).

ii) Assume now condition (10) and consider the second order linear equation

$$t'' - 2\sqrt{M}t' + Lt = 0\,. \tag{15}$$

Denoting by $\gamma = \sqrt{L-M}$, it is easy to show that

$$t(s) = \eta e^{\sqrt{M}s}(\cos(\gamma s) - \frac{\sqrt{M}}{\gamma}\sin(\gamma s))$$

is a solution of (15) for every $\eta > 0$. In addition it holds $t(s) > 0$ for all $s \in]\bar{s}, 0]$ where $\bar{s} = \frac{1}{\gamma}[\arctan\frac{\gamma}{\sqrt{M}} - \pi]$, and $t(\bar{s}) = 0$; moreover we have $t'(0) = 0$ and $t'(s) > 0$ for all $s \in [\bar{s}, 0[$.

In particular, the function $t = t(s)$ is invertible for $\bar{s} \leq s \leq 0$; let $s = s(t)$ be its inverse which is defined for $t \in [0, \eta]$ and put $\xi(t) = t'(s(t))$. Since $t'(s) = -\eta e^{\sqrt{M}s}(\frac{M}{\gamma} + \gamma)\sin(\gamma s)$, $s \in [\bar{s}, 0]$, we can choose the positive constant $\eta < \epsilon$ in such a way that $t'(s) \leq \epsilon$ in $[\bar{s}, 0]$.

Consequently, according to (15) and (10), we have

$$\xi'(t) = \frac{t''(s(t))}{t'(s(t))} = 2\sqrt{M} - \frac{Lt}{\xi(t)} \geq f(t, \xi(t))\,.$$

Therefore, the function ξ is an upper solution for the equation in (3) on the interval $]0, \eta[$.

Let us now assume, by contradiction, that (3) admits the solution $\psi(t)$. Since $\xi(0) - \psi(0^+) > 0$ and $\xi(\eta) - \psi(\eta) < 0$, according to the continuity of ξ and ψ on $]0, \eta]$, it is possible to find a value $t^* \in]0, \eta[$ satisfying

$$\xi(t) - \psi(t) > 0 \quad \text{for all} \quad 0 \leq t < t^* \quad \text{and} \quad \xi(t^*) = \psi(t^*)\,.$$

Consequently, by (10) we have

$$\xi'(t) - \psi'(t) = 2\sqrt{M} - \frac{Lt}{\xi(t)} - f(t, \psi(t)) \geq Lt\{\frac{1}{\psi(t)} - \frac{1}{\xi(t)}\} > 0, \quad \text{for all} \quad t \in]0, t^*[$$

in contradiction with $\xi(t^*) = \psi(t^*)$. □

Remark 3.3. Notice that condition (9) essentially states that $\xi(t) = \sqrt{L}t$ is a linear lower solution on all $]0,1[$ for the equation in (3) which passes through the origin.

Condition (9) could be replaced by the following more general one

$$f(t,x) \geq g(t,x) \quad \text{for all} \quad (t,x) \in\,]0,1[\,\times\,]0,+\infty[$$

with $g : [0,1] \times [0,+\infty[\, \to R$ continuous and such that the Cauchy problem

$$\begin{cases} x' = g(t,x) \\ x(0) = 0 \end{cases}$$

has a positive solution γ on all $]0,1[$.

In particular, a linear lower solution passing through the origin exists, for the equation in (3), whenever

$$f(t,x) \geq M - \frac{Lt}{x} \quad \text{for all} \quad (t,x) \in\,]0,1[\,\times\,]0,+\infty[$$

and

$$M \geq 2\sqrt{L}\,;$$

in this sense, $2\sqrt{L}$ is the best constant in condition (9).

On the other hand, condition (10) essentially ensures the existence of an upper solution for the equation in (3) having its maximal existence interval which is contained in $]0,\epsilon[$.

We examine now the case when f is positive near the t-axis, i.e. condition (5) holds.

Theorem 3.4. *Consider the boundary value problem* (3) *with* f *satisfying* (5).

i) *If there exists* $L > 0$ *such that*

$$f(t,x) \leq -2\sqrt{L} + \frac{L(1-t)}{x} \quad \textit{for all} \quad (t,x) \in]0,1[\times]0,+\infty[\,, \tag{16}$$

then (3) *has a strictly positive solution.*

ii) *If there exist* $0 < M < L$ *and* $\epsilon > 0$ *such that*

$$f(t,x) \geq -2\sqrt{M} + \frac{L(1-t)}{x} \quad \textit{for all} \quad (t,x) \in [1-\epsilon,1[\,\times\,]0,\epsilon]\,, \tag{17}$$

then (3) *does not admit solutions.*

Proof. For all $(t,x) \in\,]0,1[\,\times\,]0,+\infty[$, define $f_1(t,x) := -f(1-t,x)$. It is immediate to verify that f_1 satisfies condition (4). Moreover, if $\xi_1(t)$ is a solution of the problem

$$\begin{cases} x' = f_1(t,x) \\ x(0^+) = x(1^-) = 0\,, \end{cases}$$

then the function $\xi(t) = \xi_1(1-t)$ is a solution of (3) and viceversa. Hence the result immediately follows from Thm. 3.2. □

Remark 3.5. Condition (16) essentially states that the linear function $\psi(t) = \sqrt{L}(1-t)$ is an upper solution on all $]0,1[$ of the equation in (3) which passes through the point $(1, 0)$. As in the case when f satisfies condition (4) (see Remark 3.3), it could be replaced by a more general one involving an arbitrary upper solution of the equation in (3). Moreover, similar arguments of those at the end of Remark 3.3 could be made in this case.

We conclude this section with a result about the uniqueness of the solution of the boundary value problem (3).

Theorem 3.6. *Let $f :]0,1[\times]0,+\infty[\to R$ be a continuous function such that $f(t,\cdot)$ is monotone non-decreasing for each $t \in]0,1[$ (alternatively $f(t,\cdot)$ is monotone non-increasing for each $t \in]0,1[$). Then (3) has, at most, one solution.*

Proof. Assume $f(t,\cdot)$ non-decreasing for each $t \in]0,1[$ and let ξ_1, ξ_2 be two distinct solutions of (3). Suppose without restriction that $\xi_1(\bar{t}) < \xi_2(\bar{t})$ for some $\bar{t} \in]0,1[$ and put $t^* = \sup\{t \in [\bar{t},1[\, : \xi_1(s) < \xi_2(s)$ for all $s \in [\bar{t},t[\}$. Then $\xi_1(t) < \xi_2(t)$ for each $t \in [\bar{t},t^*[$, $\xi_1(t^*) = \xi_2(t^*)$ and according to the monotonicity of f one has

$$\xi_1'(t) = f(t,\xi_1(t)) \le f(t,\xi_2(t)) = \xi_2'(t) \quad \text{for all} \quad t \in [\bar{t},t^*[$$

in contradiction with $\xi_1(t^*) = \xi_2(t^*)$.

In a similar way, when $f(t,\cdot)$ is non-increasing, reasoning in a left neighborhood of $\bar{t}$ one gets a contradiction. □

4. On the second order boundary problem: the autonomous case

We are now ready to discuss the solvability of problem (1). The key ingredient to do this is Lemma 4.2 which essentially reduces (1) to a singular first order problem of the type studied in Section 3.

First of all we prove that any monotone solution u of (1) has indeed both its first and second derivative bounded.

Lemma 4.1. *Every monotone solution $u(t)$ of (1) satisfies*

$$\lim_{t\to-\infty} u'(t) = \lim_{t\to+\infty} u'(t) = 0\,.$$

Proof. Let u be a monotone solution of problem (1). Since u is also bounded, we get $\liminf\limits_{t\to-\infty} u'(t) = 0$; denote by $M = \limsup\limits_{t\to-\infty} u'(t)$. If we assume by contradiction that $M \ne 0$, then given $a \in]0,M[$ it is possible to find two monotone diverging sequences of negative numbers $(t_n)_n$ and $(\tau_n)_n$ with $t_{n+1} < \tau_n < t_n$ for all $n \in N$ satisfying

$$u'(t_n) = a,\; u'(\tau_n) = \frac{a}{2} \quad \text{and} \quad \frac{a}{2} \le u'(t) \le a \quad \text{for all} \quad t \in [\tau_n, t_n]\,.$$

According to the continuity of h it is possible to find $k \in R$ such that

$$k = \max_{(u,v)\in[0,1]\times[0,a]} h(u,v)\,.$$

Consequently we obtain

$$t_n - \tau_n \geq \frac{a}{2k} \quad \text{for all} \quad n \in N$$

implying

$$\int_{-\infty}^{0} u'(t)\, dt \geq \sum_{n=1}^{\infty} (t_n - \tau_n) \frac{a}{2} = +\infty$$

in contradiction with $u(-\infty) = 0$; hence $M = 0$ and $\lim_{t \to -\infty} u'(t) = 0$.

In a similar way one also gets the result for $t \to +\infty$. □

Lemma 4.2. *Assume $h(u, 0) \neq 0$ for all $u \in]0, 1[$. Then, problem (1) admits a non-decreasing solution if and only if the singular boundary value problem*

$$\text{(18)} \qquad \begin{cases} \dot{z} = \frac{h(u,z)}{z} & \left(\dot{z} = \frac{dz}{du}\right) \\ z(0^+) = z(1^-) = 0\,. \end{cases}$$

has a strictly positive solution.

Proof. (*Necessary condition*) Let u be a non-decreasing solution of (1). First notice that $u'(t) > 0$ for all $t \in R$ satisfying $u(t) \in]0, 1[$; assuming in fact the existence of $t_0 \in R$ with $u(t_0) \in]0, 1[$ and $u'(t_0) = 0$, we obtain $u''(t_0) = h(u(t_0), 0) \neq 0$, hence u' should change its sign in a neighborhood of t_0, in contradiction with the monotonicity of u. It is then possible to define the inverse function $t = t(u)$ of u which is of class C^1 on all $]0, 1[$; moreover, putting $\zeta(u) = u'\big(t(u)\big)$ we obtain

$$\dot{\zeta}(u) = \frac{u''(t)}{u'(t)} = \frac{h(u, \zeta(u))}{\zeta(u)}\,.$$

Consequently, by means of Lemma 4.1 we get $\lim_{u \to 0^+} \zeta(u) = 0$ and $\lim_{u \to 1^-} \zeta(u) = 0$ hence ζ is a solution of (18).

(*Sufficient condition*) Let ζ be a strictly positive solution of (18); let u be a solution of the Cauchy problem

$$\begin{cases} u' = \zeta(u) \\ u(0) = 1/2\,; \end{cases}$$

it is easy to see that, without loss of generality, we can continue u on all the real line. Since ζ it is strictly positive, then u is monotone. Hence by virtue of Lemma 4.1 we deduce that both $u'(\pm\infty)$ exist and they are necessarily equal to zero. Moreover, whenever $u(t) \in]0, 1[$, we obtain

$$u''(t) = \dot{\zeta}(u) u'(t) = h(u(t), u'(t))\,.$$

Finally, if $t_1 = \inf\{t : u(t) > 0\}$, since $h(0, 0) = 0$, we have

$$\lim_{t \to t_1^+} u''(t) = \lim_{t \to t_1^+} h(u(t), u'(t)) = 0\,;$$

similarly, if $t_2 = \sup\{t : u(t) < 1\}$, we obtain $\lim_{t \to t_2^-} u''(t) = 0$; therefore u is a solution of (1). □

We are now ready to prove an existence and non-existence result for problem (1). In particular, we are able to treat both the case when all points $(u,0)$, with $0<u<1$, are exit points and when they are all entrance points for the plane vector field $(u,v)\to(v,h(u,v))$.

Theorem 4.3. *Given the continuous function $h: R^2\to R$ satisfying $h(0,0)=h(1,0)=0$, let us consider the boundary value problem (1).*

(Existence) Assume there exists a constant $L>0$ such that one of the following pairs of conditions holds

$$h(u,0)<0 \qquad \text{for all} \quad u\in\,]0,1[\tag{19}$$

$$h(u,v)\geq 2\sqrt{L}v-Lu \qquad \text{for all} \quad (u,v)\in[0,1]\times[0,+\infty[\,, \tag{20}$$

or

$$h(u,0)>0 \qquad \text{for all} \quad u\in\,]0,1[\tag{21}$$

$$h(u,v)\leq -2\sqrt{L}v+L(1-u) \qquad \text{for all} \quad (u,v)\in[0,1]\times[0,+\infty[\,, \tag{22}$$

then problem (1) has a strictly monotone solution.

(Uniqueness) Assume one of the following conditions

$$\frac{h(u,v)}{v} \text{ monotone non-decreasing in } v \text{ for each } u\in\,]0,1[\tag{23}$$

or

$$\frac{h(u,v)}{v} \text{ monotone non-increasing in } v \text{ for each } u\in\,]0,1[\,, \tag{24}$$

then (1) has at most one solution, up to a time-shift.

(Non-existence) Assume the existence of constants $L>M>0$ and $\epsilon>0$ such that one of the following pairs of conditions holds

$$\begin{array}{lll} h(u,0)<0 & \text{for all} & u\in\,]0,1[\\ h(u,v)\leq 2\sqrt{M}v-Lu & \text{for all} & (u,v)\in[0,\epsilon]\times[0,\epsilon]\,, \end{array} \tag{25}$$

or

$$\begin{array}{lll} h(u,0)>0 & \text{for all} & u\in\,]0,1[\\ h(u,v)\geq -2\sqrt{M}v+L(1-u) & \text{for all} & (u,v)\in[1-\epsilon,1[\,\times\,]0,\epsilon]\,, \end{array} \tag{26}$$

then (1) does not admit monotone solutions.

Proof. Put

$$f(t,x)=\frac{h(t,x)}{x} \quad \text{for all} \quad (t,x)\in\,]0,1[\,\times\,]0,+\infty[\,. \tag{27}$$

Observe that f respectively satisfies condition (4) or (5), with $\beta=1$, according to the sign of function h.

Applying Lemma 4.2, the proof follows from the results in Section 3.
(Existence) Assume conditions (19) and (20); then f satisfies (9) and the result follows from Thm. 3.2 i). Assume now, (21) and (22) and define f as in (27); then f satisfies (16) and the result follows from Thm. 3.4 i).

(*Uniqueness*) First notice that, since h does not depend explicitly on t, whenever $u(t)$ is a solution of (1) and $b \in R$, then also $u(t+b)$ is a solution of (1). On the other hand, when condition (23) or condition (24) are satisfied, then the function $f(t,\cdot)$ defined as in (27) is monotone, of the same type of h; therefore, according to Thm. 3.6, the corresponding first order problem is at most uniquely solvable, hence two solutions of (1) may only differ for a time-shift.

(*Non-existence*) Under condition (25) the result follows from Thm. 3.2 ii) while under condition (26) it follows from Thm. 3.4 ii). □

Remark 4.4. Notice that the existence result in Thm. 4.3 can be obtained also under one-side growth conditions of non-linear type. More precisely, it continues to hold even if one weakens conditions (20) and (22) by assuming the existence of a constant $\beta > 0$ such that

$$(20)' \qquad h(u,v) \geq 2\sqrt{L}v^{\beta} - Luv^{\beta-1} \quad \text{for all} \quad (u,v) \in [0,1]\times]0,+\infty[$$

and, respectively,

$$(22)' \qquad h(u,v) \leq -2\sqrt{L}v^{\beta} + L(1-u)v^{\beta-1} \quad \text{for all} \quad (u,v) \in [0,1]\times]0,+\infty[.$$

In fact, it suffices to apply Thms. 3.2, 3.4 to the function $f(t,x) = \dfrac{h(t,x)}{x^{\beta}}$.

Non-existence conditions (25), (26) can be weakened in a similar way.

5. The non-autonomous case

Given a continuous function $h : R^3 \to R$, we are now interested in discussing the non-autonomous boundary value problem

$$(28) \qquad \begin{cases} u'' = h(t,u,u') \\ u(-\infty) = 0\,, \quad u(+\infty) = 1\,. \end{cases}$$

By using the existence results provided in the previous section for the autonomous case, we are now able to obtain the following existence and multiplicity result for problem (28).

Theorem 5.1. *Assume that there exist a constant $L > 0$ and two continuous functions $h_1, h_2 : R^2 \to R$ satisfying $h_i(0,0) = h_i(1,0) = 0$ for $i = 1,2$ and such that*

$$(29) \qquad 2\sqrt{L}v - Lu \leq h_1(u,v) \leq h(t,u,v) \leq h_2(u,v) \quad \textit{for all} \quad (t,u,v) \in R\times[0,1]\times[0,+\infty[$$

$$(30) \qquad h_2(u,0) < 0 \qquad \textit{for all} \quad u \in]0,1[\,.$$

Moreover, suppose that

$$(31) \qquad \frac{h(t,u,v)}{v} \quad \textit{monotone non-decreasing in } v \textit{ for each } (t,u) \in R\times]0,1[\,.$$

Then, problem (28) has a one-parameter family of distinct monotone solutions.

Proof. As a consequence of Thm. 3.2, it is easy to see that both the first order singular problems

$$(32) \qquad \begin{cases} \dot z = \frac{h_1(u,z)}{z} \\ z(0^+) = z(1^-) = 0 \end{cases} \qquad\qquad \begin{cases} \dot z = \frac{h_2(u,z)}{z} \\ z(0^+) = z(1^-) = 0 \end{cases}$$

are solvable. Let $\eta_1(u)$, $\eta_2(u)$ be two respective solutions of such problems.

For each $u \in]0,1[$, let us put

$$m(u) = \min\{\eta_1(u), \eta_2(u)\}, \qquad M(u) = \max\{\eta_1(u), \eta_2(u)\}$$

and denote by $\mathbf{D}$ the set of functions $g \in C([0,1])$ satisfying

$$m(u) \le g(u) \le M(u) \quad \text{for all} \quad u \in]0,1[\,.$$

Of course, each $g \in \mathbf{D}$ satisfies $g(0) = g(1) = 0$.

Let us fix $t_0 \in R$ and $u_0 \in]0,1[$. For every $g \in \mathbf{D}$ let $F_g :]0,1[\to R$ be the integral function

$$F_g(u) = \int_{u_0}^{u} \frac{1}{g(s)}\, ds + t_0\,.$$

Since $\dot F_g(u) = \frac{1}{g(u)} > 0$, F_g is invertible. Denoted by $]t_1, t_2[= F_g(]0,1[)$, let $w_g :]t_1,t_2[\to]0,1[$ be the inverse function of F_g, where $t_1, t_2 \in R \cup \{\pm\infty\}$. Of course, w_g is increasing with $\lim_{t \to t_1^+} w_g(t) = 0$ and $\lim_{t\to t_2^-} w_g(t) = 1$. Hence, if $t_1 > -\infty$ and/or $t_2 < +\infty$, we can continue the function w_g by putting $w_g(t) = 0$ for $t \le t_1$ and $w_g(t) = 1$ for $t \ge t_2$. In this way, we have $w_g \in C^1(R)$, in fact,

$$(33) \qquad \lim_{t\to t_1^+} w_g'(t) = \lim_{u\to 0^+} g(u) = 0, \qquad \text{and} \qquad \lim_{t\to t_2^-} w_g'(t) = \lim_{u\to 1^-} g(u) = 0\,.$$

Moreover, $w_g(t_0) = u_0$ and $w_g'(t) > 0$ if and only if $w_g(t) \in]0,1[$.
Define now

$$(34) \qquad h_g(u,v) = h(F_g(u), u, v) \quad \text{for all} \quad (u,v) \in]0,1[\times R\,.$$

For each $g \in \mathbf{D}$, the corresponding first order boundary value problem

$$(P_g) \qquad \begin{cases} \dot z = \frac{h_g(u,z)}{z} \\ z(0^+) = z(1^-) = 0 \end{cases}$$

has a unique solution $s_g \in \mathbf{D}$. In fact, according to (29) and (34) we obtain

$$\frac{h_g(u,v)}{v} \ge 2\sqrt{L} - \frac{Lu}{v} \quad \text{for all} \quad u \in]0,1[\quad \text{and} \quad v > 0\,.$$

In addition (30) implies $h_g(u,0) \le h_2(u,0) < 0$ for all $u \in]0,1[$. Hence all the assumptions of Thm. 3.2-i) are satisfied and (P_g) has a solution s_g. As a consequence of (31), the function $\frac{h_g(u,v)}{v}$ is monotone non-decreasing, with respect to v, for all $u \in]0,1[$; hence also Thm. 3.6 may be applied and s_g is the unique trajectory satisfying (P_g).

Observe now that according to condition (29) we obtain

$$\dot\eta_1(u) \le \frac{h_g(u, \eta_1(u))}{\eta_1(u)} \quad \text{for all} \quad u \in]0,1[$$

and

$$\dot{\eta}_2(u) \geq \frac{h_g(u, \eta_2(u))}{\eta_2(u)} \quad \text{for} \quad u \in]0,1[,$$

that is η_1 is a lower solution on all $]0,1[$ and η_2 is an upper solution on $]0,1[$ for the equation in (P_g). Thus, by applying Thm. 2.1 we deduce that $s_g \in \mathbf{D}$ for every $g \in \mathbf{D}$.

Hence, we can define the map $T : g \mapsto s_g$ from $\mathbf{D}$ into itself. Notice that if g^* is a fixed point for T, then the inverse function w_* of F_{g^*} is a solution of problem (28). In fact, put $t = F_{g^*}(u)$, we have $u = w_*(t)$, hence $u'(t) = w_*'(t) = g^*(u(t))$. Therefore,

$$w_*''(t) = \dot{g}*(u)u'(t) = \frac{h(F_{g^*}(u), u, g^*(u))}{g^*(u)}u'(t) = h(t, w_*(t), w_*'(t)) \tag{35}$$

whenever $w_*'(t) \in]0,1[$. Moreover, put $]t_1, t_2[= F_{g^*}(]0,1[)$, from (35), (33) and (29) it follows that

$$\lim_{t \to t_1^+} w_*''(t) = h(t_1, 0, 0) = 0, \quad \text{and} \quad \lim_{t \to t_2^-} w_*''(t) = h(t_2, 0, 0) = 0\,,$$

hence $w_* \in C^2(R)$ and solves problem (28).

In addition, $w_*(t_0) = u_0$ and since h is not autonomous, if we change the choice of $u_0 \in]0,1[$, keeping t_0 fixed, we obtain a different solution. In other words, we succeed in getting a one-parameter family of distinct solutions of (28).

In view of what we just observed, the theorem is proved if we show that the map T from $\mathbf{D}$ into itself admits a fixed point. Since the set $\mathbf{D}$ is convex and closed in the $C([0,1])$-norm, the assertion then follows from the next two propositions which show that the map T satisfies all the assumptions of Shauder-Tychonoff fixed point theorem.

Proposition 5.2. *The set $T(\mathbf{D})$ endowed with the $C([0,1])$-norm, is relatively compact.*

Proof. According to Ascoli's theorem, $T(\mathbf{D})$ is relatively compact if and only if it is bounded and equicontinuous in each $u \in [0,1]$. Of course, since $T(\mathbf{D}) \subset \mathbf{D}$, then it is bounded.

Let us prove now that $T(\mathbf{D})$ is also equicontinuous in each point u. If $u = 0$ or $u = 1$, then the equicontinuity directly follows from the continuity of the function M. Otherwise, for all $u \in]0,1[$ and any $s_g \in T(\mathbf{D})$ it holds

$$\dot{s}_g(u) = \frac{h_g(u, s_g(u))}{s_g(u)}$$

with h_g defined as in (34). Hence, by (29) we have

$$2\sqrt{L} - \frac{L}{s_g(u)} \leq \dot{s}_g(u) \leq \frac{h_2(u, s_g(u))}{s_g(u)}, \quad \text{for} \quad u \in]0,1[.$$

Fixed a compact interval $[a,b] \subset]0,1[$, since $\min_{u \in [a,b]} s_g(u) > 0$, it is then easy to show that also the set $\{\dot{s}_g : g \in \mathbf{D}\}$ is bounded in the space $C([a,b])$ and this implies the equicontinuity of $T(\mathbf{D})$ in each $u \in [a,b]$. The assertion then follows from the arbitrariness of $[a,b] \subset]0,1[$. □

Proposition 5.3. *The operator T from $\mathbf{D}$ into itself is continuous in the $C([0,1])$-norm.*

Proof. Let $(g_n)_n$ be a sequence in $\mathbf{D}$ converging to $g \in \mathbf{D}$ in the $C([0,1])$-norm, i.e. uniformly on $[0,1]$. Introduce as previously the functions F_{g_n} and F_g defined for all $u \in]0,1[$.
For any $(u,v) \in]0,1[\times]0,+\infty[$, denote by

$$h_n(u,v) = h(F_{g_n}(u),u,v) \quad \text{and} \quad h_0(u,v) = h(F_g(u),u,v)\,.$$

First we prove that $(F_{g_n})_n$ uniformly converges to F on any compact $[a,b] \subset]0,1[$. Indeed, let $a_1 = \min\{a,u_0\}$, $b_1 = \max\{b,u_0\}$ and put $\mu = \min\limits_{a_1 \le u \le b_1} m(u)$; of course $\mu > 0$. For all $u \in [a_1,b_1]$ it holds $g_n(u) \ge \mu$ for each n and then also $g(u) \ge \mu$. Moreover, one has

$$|F_{g_n}(u) - F(u)| = \left| \int_{u_0}^{u} \left\{ \frac{1}{g_n(s)} - \frac{1}{g(s)} \right\} ds \right| \le (b_1 - a_1) \frac{\sup\limits_{a_1 \le \xi \le b_1} |g(\xi) - g_n(\xi)|}{\mu^2}$$

and the conclusion comes from the convergence assumption of the sequence $(g_n)_n$. Taking account of the uniform convergence of $(F_{g_n})_n$ on compact subsets of $]0,1[$ and the continuity of $h(t,u,v)$ on all R^3, it is easy to show that $h_n(u,v)$ uniformly converges to $h_0(u,v)$ on $[a,b] \times R$ for every compact $[a,b] \subset]0,1[$.

Put $s_{g_n} = T(g_n)$ and $s_g = T(g)$. As it is well-known, the continuity of T is equivalent to the convergence of $(s_{g_n})_n$ to s_g in the $C([0,1])$-norm.

Assume by contradiction that this is false. Then it is possible to find a subsequence, again simply denoted by $(s_{g_n})_n$, which does not converge to s_g in the $C([0,1])$-norm. This is the same as assuming the existence of a constant $\epsilon > 0$ and of a sequence $(t_n)_n$ of points in $]0,1[$ satisfying

$$|s_{g_n}(t_n) - s_g(t_n)| > \epsilon \quad \text{for all} \quad n \in N\,. \tag{36}$$

On the other hand, by Proposition 5.2 the set $(s_{g_n})_n$ is relatively compact, then it is possible to extract a subsequence $(s_{g_{n_k}})_k$ uniformly converging to a function $s \in \mathbf{D}$ on $[0,1]$. Since

$$\dot{s}_{g_n}(u) = \frac{h_n(u, s_{g_n}(u))}{s_{g_n}(u)} \quad \text{for each} \quad u \in]0,1[\,,$$

by virtue of the uniform convergence of $(h_n(u,v))_n$ on $[a,b] \times R$ for each $[a,b] \subset]0,1[$, we get

$$\dot{s}(u) = \frac{h_0(u, s(u))}{s(u)} \quad \text{for every} \quad u \in]0,1[\,.$$

Moreover, since $m(u) \le s_{g_{n_k}}(u) \le M(u)$ for all $u \in]0,1[$ and $k \in N$, by passing to the limit when $k \to +\infty$, we obtain

$$m(u) \le s(u) \le M(u) \quad \text{for any} \quad u \in]0,1[\,.$$

Hence s is a solution of (P_g) and since such a problem is uniquely solvable, it holds $s = s_g$, in contradiction with (36). $\square$

Remark 5.4. According to Thm. 4.3, we get the solvability of problem (28) also when conditions (29) and (30) are respectively replaced by the following ones

$$h_1(u,v) \le h(t,u,v) \le h_2(u,v) \le -\sqrt{L}v + L(1-u)$$
$$\text{for all} \quad (t,u,v) \in R \times]0,1[\times]0,+\infty[$$
$$h_1(u,0) > 0 \qquad \text{for all} \quad u \in]0,1[.$$

and instead of monotonicity condition (31) we require the reversed one. Moreover, analogous considerations to those in Remark 4.4 hold also in this case.

Remark 5.5. Under the additional assumption that

$$\frac{h_i(u,v)}{v} \quad \text{is monotone non-decreasing in } v \text{ for all } u \in]0,1[, \quad i = 1,2$$

and by means of Thm. 3.6 it is easy to see that both problems in (32) are uniquely solvable, that is the functions η_1 and η_2 defined in the proof of Thm. 5.1 are unique. Moreover, it is easy to check that the following relation holds between them

$$\eta_2(u) \le \eta_1(u) \quad \text{for all} \quad u \in [0,1] .$$

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