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FIXED AND COINCIDENCE POINTS OF HYBRID MAPPINGS

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Abstract. The purpose of this note is to provide a substantial improvement and appreciable generalizations of recent results of Beg and Azam; Pathak, Kang and Cho; Shiau, Tan and Wong; Singh and Mishra.

In [6], Pathak, Kang and Cho obtained some results improving several known results on coincidence and fixed point theorems. In this note, we wish to provide a substantial improvement of their main results ([6, Theorems 2.1, 2.8, 3.1, 3.2 and 3.3]) by totally disregarding the assumptions of continuity of mappings and replacing the completeness of the space by a set of weaker conditions. In our Theorem 2 we also dropped the weak compatibility requirement from their Theorem 2.1.

Let (X, d) be a metric space. Let $(CB(X), H)$ and $(CL(X), H)$ denote respectively the hyperspaces (cf. Nadler [8]) of nonempty closed bounded subsets of X, and nonempty closed subsets of X, where H is the Hausdorff metric induced by d, i.e.,

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$$

for all $A, B \in CB(X)$ (or $CL(X)$), where $d(x, A) = \inf_{y\in A} d(x, y)$.

It is well known that $(CB(X), H)$ and $(CL(X), H)$ are complete metric spaces, whenever (X, d) is complete. Of course, $(CB(X), H)$ and $(CL(X), H)$ are metric spaces.

The following is the main result of Pathak et. al. [6, Theorem 2.1].

Theorem 1. *Let (X, d) be a complete metric space, $f : X \to X$, and $T : X \to CB(X)$ be a f-weak compatible continuous mappings such that $T(X) \subset f(X)$ and*

$$H(Tx, Ty) \le h[a \cdot L(x, y) + (1 - a) \cdot N(x, y)] \tag{1}$$

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for all x, y *in* X, *where* $0 \le h < 1$, $0 \le a \le 1$,

$$L(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \tfrac{1}{2}[d(fx,Ty)+d(fy,Tx)]\}$$

and

$$N(x,y) = [\max\{d^2(fx,fy), d(fx,Tx)d(fy,Ty), d(fx,Ty)d(fy,Tx), \tfrac{1}{2}d(fx,Tx)d(fy,Tx), \tfrac{1}{2}d(fx,Ty)d(fy,Ty)\}]^{\frac{1}{2}}.$$

Then there exists a point $t \in X$ *such that* $ft \in Tf$.

We now improve the above theorem substantially in the following two results.

Theorem 2. *Let* Y *be an arbitrary non-empty set,* (X,d) *a metric space,* $f : Y \to X$ *and* $T : Y \to CL(X)$ *such that* $T(Y) \subset f(Y)$, *and* (1) *is satisfied for all* x, y *in* Y, $0 \le h < 1$, $0 \le a \le 1$. *If one of* $T(Y)$ *or* $f(Y)$ *is a complete subspace of* X, *then there exists a point* $t \in Y$ *such that* $ft \in Tt$.

Proof. Let $x_0 \in Y$ be arbitrary. Choose a point $x_1 \in Y$ such that $fx_1 \in Tx_0$. This choice is permissible since $Tx_0 \subset f(Y)$. If $h = 0$, we obtain $d(fx_1, Tx_1) \le k \cdot H(Tx_0, Tx_1) = 0$, i.e., $fx_1 \in Tx_1$ since fx_1 is closed. This is what we required to prove. Assume that $0 < h < 1$ and set $k = \frac{1}{\sqrt{h}}$. By Nadler's remark in [7], we may choose a point $x_2 \in Y$ such that $d(fx_2, fx_1) \le k \cdot H(Tx_1, Tx_0)$. In general, having chosen $x_n \in Y$, we may choose $x_{n+1} \in Y$ such that $fx_{n+1} \in Tx_n$ and $d(fx_{n+1}, fx_n) \le kH(Tx_n, Tx_{n-1})$, $n = 1, 2, \ldots$. Now setting $x = x_{n+1}$ and $y = x_n$ in (1), it can be easily verified that $\{fx_n\}$ is a Cauchy sequence in $f(Y)$.

If $f(Y)$ is a complete subspace of X, then $\{fx_n\}$ has a limit in $f(Y)$. Call it μ. Let $t \in f^{-1}\mu$. Then $ft = \mu$. By (1),

$$\begin{aligned}
&d(fx_{n+1}, Tt) \le H(Tx_n, Tt)\\
&\le h\big[a \cdot \max\{d(fx_n, ft), d(fx_n, Tx_n), d(ft, Tt), \tfrac{1}{2}[d(fx_n, Tt) + d(ft, Tx_n)]\}\\
&\quad + (1-a)\max\{d^2(fx_n, ft), d(fx_n, Tx_n)\cdot d(ft, Tt), d(fx_n, Tt)\cdot d(ft, Tx_n),\\
&\quad\ \tfrac{1}{2}d(fx_n, Tx_n)\cdot d(ft, Tx_n), \tfrac{1}{2}d(fx_n, Tt)\cdot d(ft, Tt)\}^{\frac{1}{2}}\big]\\
&\le h\big[a \cdot \max\{d(fx_n, ft), d(fx_n, fx_{n+1}), d(ft, Tt), \tfrac{1}{2}[d(fx_n, Tt) + (ft, fx_{n+1})]\}\\
&\quad + (1-a)\{\max\{d^2(fx_n, ft), d(fx_n, fx_{n+1})\cdot d(ft, Tt), d(fx_n, Tt)\cdot d(ft, fx_{n+1}),\\
&\quad\ \tfrac{1}{2}d(fx_n, fx_{n+1})\cdot d(ft, fx_{n+1}), \tfrac{1}{2}d(fx_n, Tt)d(ft, Tt)\}^{\frac{1}{2}}\big].
\end{aligned}$$

Passing the limits as $n \to \infty$ we have

$$d(ft, Tt) \le h \cdot \left[a \cdot d(ft, Tt) + \frac{(1-a)}{\sqrt{2}} \cdot d(ft, Tt)\right].$$

Since $h \cdot \left[a + \frac{1-a}{\sqrt{2}}\right] < 1$, it follows that $ft \in Tt$. When $T(Y)$ is a complete subspace of X, by noting the fact that $T(Y) \subset f(Y)$, this case essentially pertains to the previous case. This completes the proof. □

Recall that a point $z \in X$ is a hybrid fixed point of $f : X \to X$ and $T : X \to CL(X)$ if $fz \in Tfz$.

We are now in a position to state and prove a hybrid fixed point theorem from Theorem 2.

Theorem 3. *Let (X,d) be a metric space, $f: X \to X$ and $T: X \to CL(X)$ such that $T(X) \subset f(X)$, and (1) is satisfied for all $x, y \in X$, $0 \leq h < 1$, $0 \leq a \leq 1$. If $T(X)$ or $f(X)$ is a complete subspace of X, then there exists a point $t \in X$ such that $ft \in Tt$. Further, if T is a f-weakly compatible and either $f(ft) = ft$, or $t \in Tt$, then f and T have a common fixed point, indeed, $ft \in Tft$.*

Proof. By Theorem 2 (when $Y = X$), there exists a point $t \in X$ such that $ft \in Tt$. If T is f-weakly compatible, then by Lemma 2.6 [6], $fTt = Tft$.

Now, if $f(ft) = ft$, then $ft \in Tt$ implies $f(ft) \in f(Tt) = Tft$. Thus, if $f(ft) = ft$, then ft is a fixed point of T.

Again, if $t \in Tt$, then $ft \in fTt = Tft$; i.e., ft is a fixed point of T.

Hence, in either case, t is a common fixed point of f and T.

It is pertinent to say that Theorem 3 above has a big potential for applications to Pareto type of maximization problems. In fact, Corley [3, Theorem 1] has shown that a hybrid fixed point is a maximal in certain Pareto maximization problems (see, [3, p. 529]). The following is an extension of the main results of Das and Naik [4, Theorem 2.1] and Pathak [5, Theorem 4].

Theorem 4 ([6, Theorem 2.8]). *Let (X,d) be a complete metric space, and let $f, T: X \to X$ be f-weak compatible mappings such that $T(X) \subset f(X)$ and*

$$d(Tx, Ty) \leq h\big[a \cdot L(x,y) + (1-a)N(x,y)\big] \tag{2}$$

for all x, y in X, where $0 \leq h < 1$, $0 \leq a \leq 1$,

$$L(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\}$$

and

$$\begin{aligned} N(x,y) = \big[\max\{&d^2(fx,fy), d(fx,Tx)\cdot d(fy,Ty), d(fx,Ty)\cdot d(fy,Tx), \\ &d(fx,Tx)\cdot d(fy,Tx), d(fy,Ty)\cdot d(fx,Ty)\}\big]^{\frac{1}{2}}. \end{aligned}$$

If one of f or T is continuous, then there exists a unique common fixed point of f and T.

We improve the above theorem by removing the assumptions of continuity of either f or T and replacing the completeness of the space X by a set of weaker conditions as follows.

Theorem 5. *Let (X,d) be a metric space and let $f, T: X \to X$ be such that $T(X) \subset f(X)$ and (2) is satisfied for all $x, y \in X$ and $0 \leq h < 1$, $0 \leq a \leq 1$. If one of $T(X)$ or $f(X)$ is a complete subspace of X, then there exists a point $t \in X$ such that $ft = Tt$. Further,*

(i) *if there exist $v, w \in X$ such that $fv = Tv$ and $fw = Tw$, then $fv = fw$,*

(ii) *if T is f-weak compatible, then f and T have a unique common fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Following the proof technique of [4], [5] and [6], we find a Cauchy sequence $\{fx_n\}$ in $f(X)$ where $Tx_n = fx_{n+1}$, $n = 0, 1, 2, \ldots$. If $f(X)$ is complete, then $\{fx_n\}$ has a limit in $f(X)$ say u. Let $t \in f^{-1}u$ so that $ft = u$. The construction of $\{Tx_n\}$ shows that it also converges to ft. Setting $x = x_n$ and $y = t$ in (2) and passing the limit as $n \to \infty$, we obtain $d(ft, Tt) \leq h[a \cdot d(ft, Tt) + (1-a)\, d(ft, Tt)]$. So $ft = Tt$.

Now assume that $v, w \in X$ are such that $fv = Tv$ and $fw = Tw$, then by (2),

$$d(fv, fw) = d(Tv, Tw) \leq h \cdot [ad(fv, fw) + (1-a)\, d[fv, fw)]\,,$$

and so $fv = fw$. This proves (i).

Finally, if T is f-weak compatible, then appealing to the Lemma 2.6 of [6], $fTt = Tft$. Since $ft = Tt$, it follows that $fft = fTt = Tft$, and so ft is a coincidence point of f and T. Therefore by (i), we have $ft = fft$ proving that ft is a common fixed point of f and T. By using (2), the uniqueness of the common fixed point can be easily verified.

Let Y be an arbitrary non-empty set, (X, d) a metric space, $f : Y \to X$ and $T : Y \to CL(X)$ such that $T(Y) \subset f(Y)$ satisfying the following condition:

$$H^r(Tx, Ty) \leq \alpha_1(d(fx, Tx))d^r(fx, Tx) + \alpha_2(d(fy, Ty))d^r(fy, Ty) \tag{3}$$

for all $x, y \in Y$, where $\alpha_i : R \to [0, 1)$ $(i = 1, 2)$ and r is some fixed positive real number.

If there exists a sequence $\{x_n\}$ in Y such that $\lim\limits_{n\to\infty} d(fx_n, Tx_n) = 0$, then $\{x_n\}$ is said to be asymptotically T-regular with respect to f. If $Y = X$ and $f = \mathrm{id}_X$ (the identity map on X), then the sequence $\{x_n\} \subset X$ is said to be asymptotically T-regular. □

We now state and prove the following theorem which improves the corresponding result of Pathak, Kang and Cho [6, Theorem 3.1].

Theorem 6. *Let Y be an arbitrary non-empty set, (X, d) a metric space, $f : Y \to X$ and $T : Y \to CL(X)$ such that $T(Y) \subset f(Y)$ and (3) is satisfied for all x, y in Y, where $\alpha_i : R \to [0, 1)$ and r is some fixed positive real number. If one of $T(Y)$ or $f(Y)$ is a complete subspace of X and if there exists an asymptotically T-regular sequence $\{x_n\}$ with respect to f in Y, then there exists a point $x^* \in Y$ such that $fx^* \in Tx^*$. Moreover, $Tx_n \to Tx^*$ as $n \to \infty$.*

Proof. Suppose $f(Y)$ is a complete subspace of X. By (3),

$$\begin{aligned} H^r(Tx_n, Tx_m) \leq{} & \alpha_1(d(fx_n, Tx_n))d^r(fx_n, Tx_n) \\ & + \alpha_2(d(fx_m, Tx_m))d^r(fx_m, Tx_m)\,. \end{aligned}$$

Here right hand side tends to $0 \to$ as $n, m, \to \infty$.

This show that $\{Tx_n\}$ is a Cauchy sequence in $T(Y)$, but $T(Y) \subset f(Y)$ and $f(Y)$ is complete. It follows that $\{Tx_n\}$ is a Cauchy sequence in a complete metric space $(T(Y), H)$. Hence there exists $K^* \in CL(X)$ such that $H(Tx_n, K^*) \to 0$ as $n \to \infty$. Suppose $k^* \in K^*$, and let $x^* \in F^{-1}(k^*)$, then $fx^* = k^* \in K^*$. Again, by (3)

$$\begin{aligned} d(fx^*, Tx^*) &\leq H^r(K^*, Tx^*) \leq \lim_{n\to\infty} H^r(Tx_n, Tx^*) \\ &\leq \lim_{n\to\infty}[\alpha_1(d(fx_n, Tx_n))d^r(fx_n, Tx_n) + \alpha_2(d(fx^*, Tx^*))d^r(fx^*, Tx^*)] \\ &= \alpha_2(d(fx^*, Tx^*))d^r(fx^*, Tx^*), \end{aligned}$$

which yields $(1 - \alpha_2(d(fx^*, Tx^*)))d^r(fx^*, Tx^*) \leq 0$, i.e., $d(fx^*, Tx^*) = 0$, and so $fx^* \in Tx^*$. Now

$$\begin{aligned} H^r(K^*, Tx^*) &= \lim_{n\to\infty} H^r(x_n, Tx^*) \leq \alpha_2\big(d(fx^*, Tx^*)\big)d^r(fx^*, Tx^*) \\ &\leq d^r(fx^*, Tx^*) = 0\,. \end{aligned}$$

Thus, we obtain $Tx^* = K^* = \lim_{n\to\infty} Tx_n$. The other case, when $T(Y)$ is a complete subspace of X, essentially pertain to the previous case as $T(Y) \subset f(Y)$. This completes the proof. □

Remark. If $Y = X$ and $f = \mathrm{id}_x$ (the identity map of X), then we conclude from the above theorem that T has a fixed point x^* in X. Moreover, $Tx_n \to Tx^*$ as $n \to \infty$. Hence our result is a substantial generalization of the corresponding Theorem 3.1 of Pathak, Kang and Cho [6].

Our next result improves Theorem 3.2 of Pathak, Kang and Cho [6].

Theorem 7. *Let Y be an arbitrary non-empty set, (X, d) a metric space, $f : Y \to X$, $T : Y \to CL(X)$ such that $T(Y) \subset f(Y)$ and (3) is satisfied for all x, y in Y, where $\alpha_i : R \to [0, 1)$ and r is some fixed positive real number. If one of $T(Y)$ or $f(Y)$ is a complete subspace of X and there exists an asymptotically T-regular sequence $\{x_n\}$ in Y with respect to f and Tx_n is compact, for all $n \in N$, then there exists a point $t \in Y$ such that $ft \in Tt$.*

Proof. Let $y_n \in Tx_n$ be such that $d(fx_n, y_n) = d(fx_n, Tx_n)$. Since this sequence $\{x_n\}$ in Y is asymptotically T-regular with respect to f, it follows that a cluster point of $\{fx_n\}$ is cluster point of $\{fx_n\}$ and $\{y_n\}$. Suppose that y^* is such a cluster point of $\{fx_n\}$ and $\{y_n\}$. Then as in Theorem 6, there exists a point $x^* \in Y$ such that $fx^* \in Tx^*$. Now by (3)

$$\begin{aligned} d^r(y^*, Tx^*) &\leq \lim_{n\to\infty} H^r(Tx_n, Tx^*) \\ &\leq \lim_{n\to\infty} \big[\alpha_1(d(fx_n, Tx_n))d^r(fx_n, Tx_n) + \alpha_2(d(fx^*, Tx^*))d^r(fx^*, Tx^*)\big] \\ &= \lim_{n\to\infty} \big[\alpha_1(d(fx_n, Tx_n))d^r(fx_n, Tx_n)\big]\,, \end{aligned}$$

which implies that $y^* \in Tx^*$. Let $t \in f^{-1}y^*$. Then $ft = y^*$. By (3) again, we have

$$\begin{aligned} d^r(y^*, Tt^*) &\leq H^r(Tx^*, Tt) \\ &\leq \alpha_1(d(fx^*, Tx^*))d^r(fx^*, Tx^*) + \alpha_2(d(ft, Tt))d^r(ft, Tt)\,, \end{aligned}$$

which implies that

$$(1-\alpha_2(d(ft,Tt)))d^r(ft,Tt) \le 0\,.$$

We, therefore, have $ft \in Tt$. This completes the proof. □

Remark 2. If $Y = X$ and $f = \mathrm{id}_x$ (the identity map of X), then we conclude from the above theorem that each cluster point of $\{x_n\}$ is a fixed point of T, indeed, $y^* \in Ty^*$. Thus, our Theorem 7 is a legitimate and appreciable generalization of corresponding Theorem 3.2 of Pathak, Kang and Cho [6].

Theorem 8. *Let Y be an arbitrary non-empty set, (X,d) a metric space, $f : Y \to X$and $T : Y \to CL(X)$ such that $T(Y) \subset f(Y)$ and (3) is satisfied for all x, y in Y with $\alpha_1(d(fx,Tx)) + \alpha_2(d(fy,Ty)) \le 1$. If $\inf\{d(fx,Tx) : x \in Y\} = 0$, then there exists a point $t \in Y$ such that $ft \in Tt$.*

Proof. By virtue of Theorem 7, it suffices to show that there exists an asymptotically T-regular sequence $\{x_n\}$ in Y with respect to f.

Pick $x_0 \in X$ and consider a sequence $\{x_n\} \subset Y$ such that $fx_n \in Tx_{n-1}$ for all $n \in N$. By (3),

$$\begin{aligned} d^r(fx_n,Tx_n) &\le H^r(Tx_{n-1},Tx_n) \\ &\le \alpha_1(d(fx_{n-1},Tx_{n-1}))d^r(fx_{n-1},Tx_{n-1}) \\ &\quad + \alpha_2(d(fx_n,Tx_n))d^r(fx_n,Tx_n) \end{aligned}$$

i.e.

$$d^r(fx_n,Tx_n) \le \frac{\alpha_1(d(fx_{n-1},Tx_{n-1}))}{1-\alpha_2(d(fx_n,Tx_n))}d^r(fx_{n-1},Tx_{n-1}) \le d^r(fx_{n-1},Tx_{n-1})\,.$$

It follows from the above inequality that the sequence $\{d(fx_n,Tx_n)\}$ is decreasing. Therefore, $d(fx_n,Tx_n) \to \inf\{d(fx_n,Tx_n) : n \in N\}$, and so $d(fx_n,Tx_n) \to 0$. Hence $\{x_n\}$ is asymptotically T-regular with respect to f. This completes the proof. □

Remark 3. If $Y = X$ and $f = \mathrm{id}_x$ (the identity map of (X)), then we conclude from the above theorem that T has a fixed point in X. Thus, our Theorem 8 is a proper generalization of corresponding Theorem 3.3 of Pathak, Kang and Cho [6].

Finally, we conclude that our Theorems 6, 7 and 8 also generalize the main results of Shiau, Tan and Wong [9], and Beg and Azam [1, 2]. In these theorems, we have dropped the hypothesis of compactness of Tx (cf. Theorem 1 in [9]). Moreover, we have consider the domain of our mappings an arbitrary set rather than a metric space. This shows the very general nature of our results in contrast to other known results in the literature.

Moreover, the following example give an insight view of our results and applicable superiority over those of [6]. We call the sequence $\{fx_n\}$ (resp. $\{Tx_n\}$) constructed in the proof of Theorem 5 as orbit of f with respect to T (resp. the orbit of T with respect to f) (see also, [10] for the motivation of the following examples).

Example 1. Let $Y = X = \{x : 0 \leq x \leq 1,\ x \in Q\}$ be endowed with the usual metric. Let $Tx = \{0, 1\}$, $fx = 1 - x$, $x \in X$, then $T(X) = \{0, 1\} \subset f(X) = X$. It is easy to see that all the hypothesis of Theorem 2 are satisfied and $ft \in Tt$, $t = 0, 1$. However, Theorem 1 can not be applied as X is not complete. Note that $T(X) = \{0, 1\}$ is complete.

The multiple functions play significant role in several branches of mathematics. In particular, the doubling function D (cf. below) finds significance in chaotic dynamical theory (see, e.g. Devaney [11, p. 24]). We now introduce an auxillary function T to make the orbit of D with respect to T behave nicely (refer to Theorem 5).

Example 2. Let $X = [0, 1]$ be endowed with the usual metric. Let $D, T : X \to X$ be defined by

$$Dx = 2x \bmod 1 \text{ and } Tx = \begin{cases} \frac{x}{4} & \text{if } \ 0 \leq x \leq \frac{1}{2} \\ \frac{x}{4} - \frac{1}{8} & \text{if } \ \frac{1}{2} \leq x \leq 1 \end{cases}.$$

Then $T(X) = \left[0, \frac{1}{8}\right] \subset D(X) = X$, and $|Tx - Ty| = \left(\frac{1}{8}\right)|Dx - Dy|$ for all $x, y \in X$. Clearly, Theorem 5 applies and 0 is the unique common fixed point of T and D. Moreover, the orbit of D with respect to T for any $x_0 \in X$; i.e., the sequence $\{Dx_n : Dx_n = Tx_{n-1}, n = 1, 2, \ldots\}$ converges to 0. In particular, if $0 \leq x_0 < \frac{1}{2}$, then $Dx_n = \frac{2x_0}{8^n}$, $n = 1, 2, \ldots$. However, it is interesting to note that Theorem 4 cannot be applied to T and $f = D$, since both the mappings are discontinuous.

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