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ON F–DIFFERENTIABLE FREDHOLM OPERATORS OF NONSTATIONARY INITIAL–BOUNDARY VALUE PROBLEMS

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Abstract. We are dealing with Dirichlet, Neumann and Newton type initial-boundary value problems for a general second order nonlinear evolution equation. Using the Fredholm operator theory we establish some sufficient conditions for Fréchet differentiability of associated operators to the given problems. With help of these results the generic properties, existence and continuous dependency of solutions for initial-boundary value problems are studied.

Introduction

The questions of existence, uniqueness and qualitative properties for different initial-boundary value problems of the parabolic type were and are the study object of many authors (see [1], [4], [5], [7], [8], [9], [10], [11]). Moreover to these standard questions quantitative and qualitative set properties for the solutions of operator equations are investigated (see [3], [12], [15]).

In the present paper we are interested in Dirichlet and Newton (or Neumann) type problems for a broad class of quasilinear dynamic equations (not necessarily of the parabolic type). Using the general Fredholm differentiable operators theory from V. Šeda [15] and some results from author's paper [6] we study the generic properties of above mentioned problems to which Fréchet differentiable Fredholm operators are associated. The derived results can be applied to the diffusional and heated models, to the reaction - diffusional equations, shock waves as well as to different technical models (dynamic deformations, vibrations).

1. General results on Fredholm operators

In our considerations we use some general results of Fredholm operators which we introduce in this part. They are presented by V. Šeda in [15] as consequences of

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the Nikolskij decomposition theorem from [17], p.233, of the Ambrosetti theorem from [2], p.216 and of the Smale-Quinn theorem from [14], p.862 and [13].

To the formulations of these results we use the following conditions:

Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces and let $A : X \to Y, B : X \to Y$ be operators satisfying the assumptions:

(i) A is a linear bounded Fredholm operator of zero index;

(ii) B is completely continuous;

(iii) $F = A + B$ is coercive;

(iv) B is a continuously Fréchet differentiable operator.

Proposition 1.1 ([15], Lemma 3.2, Remark 3.2). *Let X and Y be both real or complex Banach spaces and let the assumptions* (i), (ii), *and* (iv) *hold. Then the following statements are true:*

(j) *The mapping $F = A+B \in C^1(X, Y)$ (continuously Fréchet differentiable) is a Fredholm operator of zero index.*

(jj) *The point $u \in X$ is singular of the Fredholm operator F if and only if the equation $F'(u)h = 0$ has a nonzero solution $h \neq 0$, $h \in X$ (i.e. if and only if u is a critical point of F).*

(jjj) *The set S of all singular points of F is equal to the set of all critical points, i.e.*

$$S = \{u \in X; F'(u)h = 0 \quad \textit{has solution} \quad h \neq 0, h \in X\}.$$

Also the set $R_F = Y - F(S)$ of all regular values F is dense in Y.

(jv) *The set $\Sigma \subset X$ of all not locally invertible points is a subset of S, i.e. $\Sigma \subset S$.*

Here $F'(u)$ is the Fréchet derivative of F at the point u. Recall, that $u_0 \in X$ is a regular point of F *if $F'(u_0)$ is linear homeomorphism X onto Y. If u_0 is not a regular point then we call it* a singular point of F.

Proposition 1.2 ([15], Theorem 3.3). *Let X and Y be both real or complex Banach spaces and let the hypotheses* (i), (ii), (iii) *and* (iv) *hold. Then*

(vj) *The operator F is proper C^1-Fredholm of zero index;*

(vjj) *The cardinal number* card $F^{-1}(\{g\})$ *is constant and finite (it may be zero) on each connected component of the open and dense subset $R_F = Y - F(S)$ (for S see* (jjj) *from Proposition 1.1);*

(vjjj) *For each $u \in X - S$ (an open set), the operator F is a local C^1-diffeomorphism at u;*

(jx) *If $S = \emptyset$, then $F : X \to Y$ is a C^1-diffeomorphism;*

(xj) *The set $F(S)$ of all singular values of F is closed and nowhere dense in Y.*

Recall, that an operator $F : X \to Y$ is proper *if for each compact $K \subset Y$ the set $F^{-1}(K)$ is compact. We say that F is* a local C^1-diffeomorphism at u *if there exists a neighbourhood $U_1(u)$ of u and $U_2(F(u))$ of $F(u)$ such that F bijectively maps $U_1(u)$ onto $U_2(F(u))$ and both F and F^{-1} are C^1-maps.*

Proposition 1.3 ([15], Corollary 3.5). *Let X and Y be both real or complex Banach space with* $\dim Y \geq 3$. *Further, let the conditions* (i), (ii), (iii) *and* (iv) *hold together with*

(vi) *Each point $u \in X$ is either a regular point or an isolated critical point of F.*

Then F is a homeomorphism of X onto Y.

2. The formulation of problem and preliminary results

Through this paper we assume that the set $\Omega \subset R^n$ for $n \in N$ is a bounded domain with the sufficiently smooth boundary $\partial\Omega$. The real number T is positive and $Q := (0,T] \times \Omega, \Gamma := [0,T] \times \partial\Omega$.

We use the distinction D_t for $\partial/\partial t$ and D_i for $\partial/\partial x_i$ and D_{ij} for $\partial^2/\partial x_i \partial x_j$ where $i,j = 1,\ldots,n$ and $D_0 u$ for u. The symbol $\operatorname{cl} M$ means the closure of the set M in R^n.

We consider the nonlinear differential equation (not necessarily of parabolic type)

$$D_t u - A(t,x,D_x)u + f(t,x,u,D_1 u,\ldots,D_n u) = g(t,x) \tag{2.1}$$

for $(t,x) \in Q$, where the coefficients a_{ij}, a_i, a_0 for $i,j = 1,\ldots,n$ of the operator

$$A(t,x,D_x)u = \sum_{i,j=1}^{n} a_{ij}(t,x) D_{ij} u + \sum_{i=1}^{n} a_i(t,x) D_i u + a_0(t,x) u$$

are continuous functions from the space $C(\operatorname{cl} Q, R)$. The function f is continuous from the space $C(\operatorname{cl} Q \times R^{n+1}, R)$ and $g \in C(\operatorname{cl} Q, R)$.

Together with the equation (2.1) one of the following homogeneous boundary conditions is fulfilled: Either the *first boundary condition - Dirichlet condition*

$$B_1(t,x,D_x)u|_\Gamma := u|_\Gamma = 0 \tag{2.2$_1$}$$

or *the second boundary condition - Newton or for $b_0 = 0$ Neumann condition*

$$B_2(t,x,D_x)u|_\Gamma := \partial u/\partial\nu + b_0(t,x)u|_\Gamma = 0 \tag{2.2$_2$}$$

where $\nu := (0,\nu_1,\ldots,\nu_n) \colon \Gamma \to R^n$ is a vector function for which the value $\nu(t,x)$ means the inner normal vector to Γ at the point $(t,x) \in \operatorname{cl}\Gamma$ and $\partial/\partial\nu$ means derivative with respect to the normal ν. Here $b_0 \in C(\Gamma, R)$.

In addition to the boundary condition we require for the solution of (2.1) to satisfy the homogeneous initial condition

$$u|_{t=0} = 0 \quad \text{on} \quad \operatorname{cl}\Omega\,. \tag{2.3}$$

We shall use the notation

$$\langle u\rangle^{s}_{t,\mu,Q} := \sup_{\substack{(t,x),\,(s,x)\in \mathrm{cl}\,Q\\ t\neq s}} \frac{|u(t,x)-u(s,x)|}{|t-s|^{\mu}}$$

$$\langle u\rangle^{y}_{x,\nu,Q} := \sup_{\substack{(t,x),\,(t,y)\in \mathrm{cl}\,Q\\ x\neq y}} \frac{|u(t,x)-u(t,y)|}{|x-y|^{\nu}}$$

$$\langle f\rangle^{s,y,v}_{t,x,u} := |f(t,x,u_0,u_1,\dots,u_n)-f(s,y,v_0,v_1,\dots v_n)|$$

$$\langle f\rangle^{s,y,v(s,y)}_{t,x,u(t,x)} := |f[t,x,u(t,x),D_1u(t,x),\dots,D_nu(t,x)] \\ -f[s,y,v(s,y),D_1v(s,y),\dots,D_nv(s,y)]|$$

where $t,s>0, x=(x_1,\dots,x_n), y=(y_1,\dots,y_n)$ are from R^n and $|x-y|=\left[\sum_{i=1}^{n}(x_i-y_i)^2\right]^{1/2}$ and $\mu,\nu\in R$ and we define the following Hölder spaces (see [4], p.147):

Definition 2.1. Let $\alpha\in(0,1)$.

1. By the symbol $C^{\alpha/2,\alpha}_{t,x}(\mathrm{cl}\,Q,R)$ we shall denote the vector space of continuous functions $u\colon \mathrm{cl}\ Q\to R$ which have the finite norm

$$\|u\|_{\alpha/2,\alpha,Q} = \sup_{(t,x)\in \mathrm{cl}\,Q}|u(t,x)| + \langle u\rangle^{s}_{t,\alpha/2,Q} + \langle u\rangle^{y}_{x,\alpha,Q} \tag{2.4}$$

2. By the symbol $C^{(1+\alpha)/2,1+\alpha}_{t,x}(\mathrm{cl}\,Q,R)$ we denote the vector space of continuous functions $u\colon \mathrm{cl}\,Q\to R$ which have continuous derivatives D_iu for $i=1,\dots,n$ on $\mathrm{cl}\,Q$ and the norm

$$\|u\|_{(1+\alpha)/2,1+\alpha,Q} := \sum_{i=0}^{n}\sup_{(t,x)\in \mathrm{cl}\,Q}|D_iu(t,x)| + \langle u\rangle^{s}_{t,(1+\alpha)/2,Q} \\ + \sum_{i=1}^{n}\langle D_iu\rangle^{s}_{t,\alpha/2,Q} + \sum_{i=1}^{n}\langle D_iu\rangle^{y}_{x,\alpha,Q} \tag{2.5}$$

is finite.

3. The symbol $C^{(2+\alpha)/2,2+\alpha}_{(t,x)}(\mathrm{cl}\ Q,R)$ means the space of continuous functions $u\colon \mathrm{cl}\ Q\to R$ for which there exist continuous derivatives $D_tu, D_iu, D_{ij}u$ on $\mathrm{cl}\,Q,\ i,j=1,\dots n$ and the norm

$$\|u\|_{(2+\alpha)/2,2+\alpha,Q} := \sum_{i=0}^{n}\sup_{(t,x)\in \mathrm{cl}\,Q}|D_iu(t,x)| + \sup_{(t,x)\in \mathrm{cl}\,Q}|D_tu(t,x)| \\ + \sum_{i,j=1}^{n}\sup_{(t,x)\in \mathrm{cl}\,Q}|D_{ij}u(t,x)| + \sum_{i=1}^{n}\langle D_iu\rangle^{s}_{t,(1+\alpha)/2,Q} + \langle D_tu\rangle^{s}_{t,\alpha/2,Q} \\ + \sum_{i,j=1}^{n}\langle D_{ij}u\rangle^{s}_{t,\alpha/2,Q} + \langle D_tu\rangle^{y}_{x,\alpha,Q} + \sum_{i,j=1}^{n}\langle D_{ij}u\rangle^{y}_{x,\alpha,Q} \tag{2.6}$$

is finite.

4. The symbol $C^{(3+\alpha)/2,3+\alpha}_{t,x}(\mathrm{cl}\,Q,R)$ means the vector space of the continuous functions $u\colon \mathrm{cl}\,Q\to R$ for which the derivatives D_t, D_iu, D_tD_iu, $D_{ij}u$, $D_{ijk}u$, $i,j,k=1,\dots,n$ are continuous on $\mathrm{cl}\,Q$ and the norm

$$
\begin{aligned}
\|u\|_{(3+\alpha)/2,3+\alpha,Q} :=& \sum_{i=0}^{n}\sup_{(t,x)\in \mathrm{cl}\,Q}|D_iu(t,x)| + \sum_{i,j=1}^{n}\sup_{(t,x)\in \mathrm{cl}\,Q}|D_{ij}u(t,x)| \\
&+ \sum_{i=0}^{n}\sup_{(t,x)\in \mathrm{cl}\,Q}|D_tD_iu(t,x)| + \sum_{i,j,k=1}^{n}\sup_{(t,x)\in \mathrm{cl}\,Q}|D_{ijk}u(t,x)| \\
&+ \langle D_tu\rangle^{s}_{t,(1+\alpha)/2,Q} + \sum_{i,j=1}^{n}\langle D_{ij}u\rangle^{s}_{t,(1+\alpha)/2,Q} \\
&+ \sum_{i=1}^{n}\langle D_tD_iu\rangle^{s}_{t,\alpha/2,Q} + \sum_{i,j,k=1}^{n}\langle D_{ijk}u\rangle^{s}_{t,\alpha/2,Q} \\
&+ \sum_{i=1}^{n}\langle D_tD_iu\rangle^{y}_{x,\alpha,Q} + \sum_{i,j,k=1}^{n}\langle D_{ijk}u\rangle^{y}_{x,\alpha,Q}
\end{aligned} \tag{2.7}
$$

is finite.

The previous norm spaces are Banach ones.

For $i=1,2$ we define two linear mappings $A_i\colon X_i\to Y_i$ by the equation (see (2.1))

$$A_iu = D_tu - A(t,x,D_x)u, \quad u\in X_i \tag{2.8}$$

where

$$
\begin{aligned}
X_i &= (D(A_i),\|\cdot\|_{(i+1+\alpha)/2,i+1+\alpha,Q})\\
Y_i &= (H(A_i),\|\cdot\|_{(i-1+\alpha)/2,i-1+\alpha,Q})
\end{aligned}
$$

and

$$
\begin{aligned}
D(A_i) &= \Big\{u\in C^{(i+1+\alpha)/2,i+1+\alpha}_{t,x}(\mathrm{cl}\,Q,R);\, B_i(t,x,D_x)u|_\Gamma = 0,\, u|_{t=0}=0 \ \text{ on } \ \mathrm{cl}\,Q\Big\}\\
H(A_i) &= \Big\{v\in C^{(i-1+\alpha)/2,i-1+\alpha}_{t,x}(\mathrm{cl}\,Q,R);\, B_i(t,x,D_x)v(t,x)|_{t=0,x\in\partial\Omega}=0\Big\}
\end{aligned}
$$

Moreover, we define the two Nemitskij operators $N_i\colon X_i\to Y_i$ for $i=1,2$ with the values

$$(N_iu)(t,x) = f[t,x,u(t,x),D_1u(t,x),\dots,D_nu(t,x)] \tag{2.9}$$

for $u\in X_i$ and $(t,x)\in \mathrm{cl}\,Q$, where $f\colon (\mathrm{cl}\,Q)\times R^{n+1}\to R$ is the nonlinear part of equation (2.1).

For simplicity we shall formulate some introductory assumptions.

Definition 2.2. Let $f := f(t, x, u_0, u_1, \dots, u_n)\colon (\mathrm{cl}\, Q) \times R^{n+1} \to R,\ \alpha \in (0,1)$.

1. Fredholm conditions.

$(A_i.1)$ The operator $A_i\colon X_i \to Y_i$ from (2.8) satisfies the smoothness condition $(S_i^{i-1+\alpha})$ for $\alpha \in (0,1)$ and $i = 1, 2$. That is: We say that the differential operator $A(t, x, D_x)$ from (2.1) and $B_i(t, x, D_x)$ from (2.2_i), respectively satisfies *the smoothness condition* $(S_i^{i-1+\alpha})$ if

(i) the coefficients a_{ij}, a_i, a_0 of the operator $A(t, x, D_x)$ for $i, j = 1, \dots, n$ belong to the space $C_{t,x}^{(i-1+\alpha)/2, i-1+\alpha}(\mathrm{cl}\, Q, R)$ and $\partial\Omega \in C^{i+1+\alpha}$ and

(ii) the coefficient b_0 from (2.2_2) belongs to the space $C_{t,x}^{(2+\alpha)/2, 2+\alpha}(\mathrm{cl}\, \Gamma, R)$.

$(A_i.2)$ There exists a linear homeomorphism $C_i\colon X_i$ onto Y_i, where

$$C_i u := D_t u - C(t, x, D_x)u, \quad u \in X_i$$

and

$$C(t, x, D_x)u := \sum_{i,j=1}^{n} c_{ij}(t, x) D_{ij} u + \sum_{i=1}^{n} c_i(t, x) D_i u + c_0(t, x) u$$

such that the smoothness condition $(S_i^{i-1+\alpha})$ holds (for $a_{ij} = c_{ij}, a_i = c_i, a_0 = c_0$).

2. Local Hölder conditions.

Let for any compact subset D of $(\mathrm{cl}\, Q) \times R^{n+1}$ there exist nonnegative constants (depending on D) $p, q, p_r, r = 0, 1, \dots, n$ with the following property:

$(N_1.1)$ Let $f\colon (\mathrm{cl}\, Q) \times R^{n+1} \to R$ be a locally Hölder continuous function on $(\mathrm{cl}\, Q) \times R^{n+1}$ such that the inequality

$$\langle f \rangle_{t,x,u}^{s,y,v} \le p|t-s|^{\alpha/2} + q|x-y|^{\alpha} + \sum_{r=0}^{n} p_r |u_r - v_r|$$

holds on any D.

$(N_2.1)$ Let $f \in C^1(\mathrm{cl}\, Q \times R^{n+1}, R)$ and let the first derivatives $\partial f/\partial x_i,\ \partial f/\partial u_j$ be locally Hölder continuous on $\mathrm{cl}\, Q \times R^{n+1}$ such that

$$\left.\begin{matrix} \langle \partial f/\partial x_i \rangle_{t,x,u}^{s,y,v} \\ \langle \partial f/\partial u_j \rangle_{t,x,u}^{s,y,v} \end{matrix}\right\} \quad \le \quad p|t-s|^{\alpha/2} + q|x-y|^{\alpha} + \sum_{r=0}^{n} p_r |u_r - v_r|$$

for $i = 1, \dots, n$ and $j = 0, 1, \dots, n$ and any D.

$(N_1.2)$ Let $f \in C^2(\mathrm{cl}\, Q \times R^{n+1}, R)$ and let the local growth conditions for the second derivatives of f satisfy on any D:

$$\left.\begin{matrix} (2.10) \qquad \left\langle \partial^2 f/\partial\tau\partial u_0 \right\rangle_{t,x,u}^{t,x,v} \\ (2.11) \qquad \left\langle \partial^2 f/\partial x_i \partial u_0 \right\rangle_{t,x,u}^{t,x,v} \\ (2.12) \qquad \left\langle \partial^2 f/\partial u_0^2 \right\rangle_{t,x,u}^{t,x,v} \end{matrix}\right\} \quad \le p_0 |u_0 - v_0|^{\beta_0}$$

where $\beta_0 > 0$ and $i = 1, \dots, n$.

(N_2.2) Let $f \in C^3(\operatorname{cl} Q \times R^{n+1}, R)$ and let the local growth conditions for the third derivatives of f hold on any D:

$$\left.\begin{array}{ll}(2.13) & \langle \partial^3 f/\partial\tau\partial x_i\partial u_j\rangle_{t,x,u}^{t,x,v} \\ (2.14) & \langle \partial^3 f/\partial\tau\partial u_j\partial u_k\rangle_{t,x,u}^{t,x,v} \\ (2.15) & \langle \partial^3 f/\partial x_i\partial x_l\partial u_j\rangle_{t,x,u}^{t,x,v} \\ (2.16) & \langle \partial^3 f/\partial x_i\partial u_j\partial u_k\rangle_{t,x,u}^{t,x,v} \\ (2.17) & \langle \partial^3 f/\partial u_j\partial u_k\partial u_r\rangle_{t,x,u}^{t,x,v}\end{array}\right\} \leq \sum_{s=0}^{n} p_s|u_s - v_s|^{\beta_s}$$

where $\beta_s > 0$ for $s = 0, 1, \dots, n$ and $i, l = 1, \dots, n;\ j, k, r = 0, 1, \dots, n$.

3. Coercivity conditions. Let $i = 1, 2$ and let for any bounded set $M_i \subset Y_i$ there exist a number $K > 0$ such that for all solutions $u \in X_i$ of the problem (2.1), (2.2i), (2.3) with the right hand side $g \in M_i$, the following alternatives hold:

(F_1.1) Either

(a_1) $\|u\|_{\alpha/2,\alpha,Q} \leq K,\ f := f(t, x, u_0)\colon \operatorname{cl} Q \times R \to R$ and the coefficients of the operators A_1 and C_1 satisfy the equations

$$a_{ij} = c_{ij},\ a_i = c_i \quad \text{for} \quad i, j = 1, \dots, n,\ a_0 \neq c_0 \quad \text{on} \quad \operatorname{cl} Q$$

or

(b_1) $\|u\|_{(1+\alpha)/2,1+\alpha,Q} \leq K,\ f = f(t, x, u_0, u_1, \dots, u_n)\colon \operatorname{cl} Q \times R^{n+1} \to R$ and the coefficients of the operators A_1 and C_1 satisfy the equations

$$a_{ij} = c_{ij} \quad \text{for} \quad i, j = 1, \dots, n \quad \text{and} \quad a_i \neq c_i$$

for at least one $i = 1, \dots, n$ on $\operatorname{cl} Q$.

(F_2.1) Either

(a_2) $\|u\|_{(1+\alpha)/2,1+\alpha,Q} \leq K,\ f := f(t, x, u_0)\colon \operatorname{cl} Q \times R \to R$ and the coefficients of the operators A_2 and C_2 satisfy the equations

$$a_{ij} = c_{ij},\ a_i = c_i \quad \text{for} \quad i, j = 1, \dots, n,\ a_0 \neq c_0 \quad \text{on} \quad \operatorname{cl} Q$$

or

(b_2) $\|u\|_{(2+\alpha)/2,2+\alpha,Q} \leq K$

$$f\colon f(t, x, u_0, u_1, \dots, u_n)\colon \operatorname{cl} Q \times R^{n+1} \to R$$

and the coefficients of the operators A_2 and C_2 satisfy the relations

$$a_{ij} = c_{ij} \quad \text{for} \quad i, j = 1, \dots, n \quad \text{and} \quad a_i \neq c_i$$

for at least one $i = 1, \dots, n$ on $\operatorname{cl} Q$.

Remark 2.1. 1. Especially, the condition $(A_i.2)$ is satisfied for the diffusion operator

$$C_i u = D_t u - \Delta u\,, \qquad u \in X_i$$

or for any uniformly parabolic operator C_i (see [10], p. 12) with sufficiently smooth coefficients.

However, the operator C_i is not necessarily uniformly parabolic.

2. The local Hölder conditions are sufficiently strong to establish the complete continuity of N_i for $i = 1, 2$ but also they admit sufficiently strong growths of f in the last variables $u_0, u_1, \dots, u_n$. For example, they include exponential, power and other type growths, as we can see by Lagrange mean value theorem.

3. The nonparabolicity of the operator $D_t - A(t, x, D_x)$ and the nonlinearity of f from (2.1) allowe us to consider the problems of the type (2.1), (2.2_i), (2.3) which are not uniquely solvable.

4. The alternatives $(F_i.1)$ for $i = 1, 2$ from Definition 2.2 we shall call "almost coercive conditions" for the problem (2.1), (2.2_i), (2.3).

Now we are prepared to formulate F-CC-C–lemmas for $i = 1, 2$.

Lemma F_i ([6], Corollary 2.2). *Let the condition $(A_i.1)$ and $(A_i.2)$ hold. Then the operator A_i is a linear bounded Fredholm operator of the zero index for $i = 1, 2$.*

Lemma CC_i ([6], Lemma 2.2). *Let the assumptions $(N_i.1)$ and*

$$(\mathrm{N}_i.3) \qquad B_i(t, x, D_x) f(t, x, 0, \dots, 0)|_{t=0, x\in\partial\Omega} = 0$$

be satisfied. Then the Nemitskij operator $N_i : X_i \to Y_i$ from (2.9) is completely continuous for $i = 1, 2$.

Lemma C_i ([6], Lemma 2.3). *Let the assumptions $(A_i.1)$, $(A_i.2)$, $(N_i.1)$, $(N_i.3)$ and $(F_i.1)$ hold. Then the operator $F_i := A_i + N_i : X_i \to Y_i$ is coercive for $i = 1, 2$.*

Just presented lemmas offer simple applicable sufficient conditions under which the operator $F_i = A_i + N_i : X_i \to Y_i$ associated to the problem $(2.1), (2.2_i), (2.3)$ keeps hypotheses (i), (ii), (iii) from the first part of this paper.

Let us define the bifurcation point.

Definition 2.3. 1. A couple $(u, g) \in X_i \times Y_i$ for $i = 1, 2$ will be called *the bifurcation point of the initial-boundary value problem* $(2.1), (2.2_i), (2.3)$ if u is a solution of this problem and there exists a sequence $\{g_k\} \subset Y_i$ such that $g_k \to g$ in Y_i as $k \to \infty$ and the problem $(2.1), (2.2_i), (2.3)$ for $g = g_k$ has at least two different solutions u_k, v_k for each $k \in N$ and $u_k \to u, v_k \to u$ in X_i as $k \to \infty$.

2. The set of all solution $u \in X_i$ of $(2.1), (2.2_i), (2.3)$ (or the set of all functions $g \in Y_i$) such that (u, g) is a bifurcation point of the initial-boundary value problem $(2.1), (2.2_i), (2.3)$ will be called *the domain of bifurcation* (*the bifurcation range*) of this problem.

Using the previous notations we can conclude this part by the following equivalence lemma.

Lemma E ([6], Lemma 3.1.). *Put $i = 1, 2$. Let $A_i : X_i \to Y_i$ be the linear operator of Lemma F_i and let $N_i : X_i \to Y_i$ be Nemitskij operator of Lemma CC_i and $F_i = A_i + N_i : X_i \to Y_i$. Then*

(E.1) *The function $u \in X_i$ is a solution of the initial-boundary value problem*(2.1), (2.2$_i$), (2.3) *for $g \in Y_i$ if and only if $F_i u = g$.*

(E.2) *The couple $(u, g) \in X_i \times Y_i$ is the bifurcation point of the problem* (2.1), (2.2$_i$), (2.3) *if and only if $F_i(u) = g$ and $u \in \Sigma_i$ (Σ_i means the set of all points of X_i at which F_i is not locally invertible).*

3. F-differentiability

In this part we establish conditions for the Fréchet differentiability of the Nemitskij operators. The case $f = f(t, x, u_0)$ is investigated in the first lemma.

Lemma 3.1. *Suppose that the Nemitskij operator $N_1 : X_1 \to Y_1$ from* (2.9) *for $i = 1$ with $f = f(t, x, u_0)\colon \operatorname{cl} Q \times R \to R$ satisfies the conditions* $(N_1.2), (N_1.3)$ *(see Definition 2.2 and Lemma CC_1). Then the operator N_1 is continuously Fréchet differentiable, i.e. $N_1 \in C^1(X_1, Y_1)$ and it is completely continuous.*

Proof. From the hypothesis $(N_1.2)$ we obtain $(N_1.1)$, from where together with $(N_1.3)$ and Lemma CC_1 we get the complete continuity of N_1.

To prove the first part of this lemma we must show that the Fréchet derivative $N_1'\colon X_1 \to L(X_1, Y_1)$ defined for $u, h \in X_1$ by the equality

$$N_1'(u)h(t,x) = \frac{\partial f}{\partial u_0}\,[t, x, u(t,x)]\,h(t,x)$$

is continuous on X_1, i.e. we need prove for any $v \in X_1$:

$$\forall\ \epsilon > 0\ \exists\ \delta(\epsilon, v) > 0\ \forall\ u \in X_1,\ \|u - v\|_{X_1} < \delta\colon \|N_1'(u) - N_1'(v)\|_{L(X_1,Y_1)} < \epsilon$$

or equivalently

$$\sup_{h \in X_1, \|h\|_{X_1} \le 1} \|[N_1'(u) - N_1'(v)]h\|_{Y_1} < \epsilon\,. \tag{3.1}$$

Employing the norms (2.4) and (2.6) we obtain for the first term of (3.1) (see (2.9) and Definition 2.1) by the condition $(N_1.2)$, using mean value theorem:

$$\sup_{(t,x) \in \operatorname{cl} Q} |[N_1'(u) - N_1'(v)]h(t,x)| \le \sup_{(t,x) \in \operatorname{cl} Q} \Big[\langle \partial f/\partial u_0 \rangle_{t,x,u(t,x)}^{t,x,v(t,x)}\, |h(t,x)|\Big] \le K\delta\,, \quad K > 0\,.$$

For the second term of (3.1) we can write by (2.10), (2.12) and by the mean value theorem for $\partial f/\partial u_0$:

$$\begin{aligned}
&\sup_{\substack{(t,x),\,(s,x)\in \mathrm{cl}\,Q\\ t\neq s}} |t-s|^{-\alpha/2}\,|[N_1'(u)-N_1'(v)]\,h(t,x)-[N_1'(u)-N_1'(v)]\,h(s,x)|\\
&\qquad\le \sup_{\substack{(t,x),\,(s,x)\in \mathrm{cl}\,Q\\ t\neq s}} |t-s|^{-\alpha/2}\left\{\left|\int_s^t \langle \partial^2 f/\partial\tau\partial u_0\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)}\,d\tau\right| |h(t,x)|\right.\\
&\qquad+\left|\int_s^t \langle \partial^2 f/\partial u_0^2\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)}\,D_\tau u(\tau,x)d\tau\right| |h(t,x)|\\
&\qquad+\left|\int_s^t \partial^2 f/\partial u_0^2[\tau,x,v(\tau,x)][D_\tau u(\tau,x)-D_\tau v(\tau,x)]d\tau\right| |h(t,x)|\\
&\qquad\left.+\,\langle \partial f/\partial u_0\rangle_{s,x,u(s,x)}^{s,x,v(s,x)}\left|\int_s^t \partial h/\partial\tau(\tau,x)d\tau\right|\right\}\le K(\delta^{\beta_0}+\delta)\,,\qquad K>0
\end{aligned}$$

For the last term of (3.1), analogicaly to the second term, we have by (2.11), (2.12)

$$\begin{aligned}
&\langle [N_1'(u)-N_1'(v)\,]\,h\rangle_{x,\alpha,Q}^{y}\\
&\quad=\sup_{\substack{(t,x),\,(t,y)\in \mathrm{cl}\,Q\\ x\neq y}} |x-y|^{-\alpha}\,|[N_1'(u)-N_1'(v)]\,h(t,x)-[N_1'(u)-N_1'(v)]\,h(t,y)|\\
&\quad\le \sup_{\substack{(t,x),\,(t,y)\in \mathrm{cl}\,Q\\ x\neq y}} |x-y|^{-\alpha}\left\{\left|\sum_{i=1}^n\int_{y_i}^{x_i} \langle \partial^2 f/\partial z_i\partial u_0\rangle_{t,\tilde z_i,u(t,\tilde z_i)}^{t,\tilde z_i,v(t,\tilde z_i)}\,dz_i\right| |h(t,x)|\right.\\
&\quad+\left|\sum_{i=1}^n\int_{y_i}^{x_i} \langle \partial^2 f/\partial u_0^2\rangle_{t,\tilde z_i,u(t,\tilde z_i)}^{t,\tilde z_i,v(t,\tilde z_i)}\,D_i u(t,\tilde z_i)dz_i\right| |h(t,x)|\\
&\quad+\left|\sum_{i=1}^n\int_{y_i}^{x_i} \partial^2 f/\partial u_0^2[t,\tilde z_i,v(t,\tilde z_i)][D_i u(t,\tilde z_i)-D_i v(t,\tilde z_i)]dz_i\right| |h(t,x)|\\
&\quad\left.+\,\langle \partial f/\partial u_0\rangle_{t,y,u(t,y)}^{t,y,v(t,y)}\left|\sum_{i=1}^n\int_{y_i}^{x_i} \partial h/\partial z_i(t,\tilde z_i)dz_i\right|\right\}\\
&\quad\le K(\delta^{\beta_0}+\delta)\,,\qquad K>0
\end{aligned}$$

where $\tilde z_i=(y_1,\dots,y_{i-1},z_i,x_{i+1},\dots,x_n)\in R^n$ for $i=1,\dots,n$.
The proof of lemma is finished. □

Lemma 3.2. *Let the Nemitskij operator $N_2\colon X_2\to Y_2$ from (2.9) satisfy the conditions $(N_2.2),(N_2.3)$ (see Definition 2.2 and Lemma CC_2). Then the operator N_2 is continuously Fréchet differentiable, i.e. $N_2\in C^1(X_2,Y_2)$ and it is completely continuous.*

Proof. From $(N_2.2)$ we obtain $(N_2.1)$ which implies by Lemma CC_2 the complete continuity of N_2. To obtain the first part of the assertion of this lemma we need prove that the Fréchet derivative $N_2' \colon X_2 \to L(X_2, Y_2)$ defined by the equation

$$N_2'(u)h(t,x) = \sum_{j=1}^{n} \frac{\partial f}{\partial u_j}[t, x, u(t,x), D_1 u(t,x), \ldots, D_n u(t,x)] D_j h(t,x)$$

for $u, h \in X_2$ is continuous on X_2. Thus we must prove as in Lemma 3.1 for every $v \in X_2$:

$$\forall\ \epsilon > 0\ \exists\ \delta(\epsilon, v) > 0\ \forall\ u \in X_2,\ \|u - v\|_{X_2} < \delta :$$
$$\sup_{h \in X_2, \|h\|_{X_2} \le 1} \|N_2'(u) - N_2'(v)h\|_{Y_2} < \epsilon \,. \tag{3.2}$$

Using the norms (2.5), (2.7) and the estimation $\|u - v\|_{X_2} < \delta$ we have for the first term of (3.2) (see (2.9) and Definition 2.1) by the mean value theorem:

$$\begin{aligned}
&\sum_{i=0}^{n} \sup_{(t,x) \in \mathrm{cl}\, Q} |D_i[N_2'(u) - N_2'(v)]h(t,x)| \\
&\qquad \le \sum_{i,j=0}^{n} \sup_{(t,x)\in \mathrm{cl}\, Q} \Big[\big\langle \partial^2 f/\partial x_i \partial u_j \big\rangle_{t,x,u(t,x)}^{t,x,v(t,x)} |D_j h(t,x)| \\
&\qquad + \sum_{k=0}^{n} \big\langle \partial^2 f/\partial u_j \partial u_k \big\rangle_{t,x,u(t,x)}^{t,x,v(t,x)} |D_{ik} u| \cdot |D_j h|(t,x) \\
&\qquad + \sum_{k=0}^{n} \big| \partial^2 f/\partial u_j \partial u_k (t, x, v(t,x), \ldots) \big| \, |D_{ik} u - D_{ik} v| \, |D_j h| \, (t,x) \\
&\qquad + \langle \partial f/\partial u_j \rangle_{t,x,u(t,x)}^{t,x,v(t,x)} |D_{ij} h(t,x)| \Big] < K\delta \,, \qquad K > 0 \,.
\end{aligned}$$

The second term of (3.2) we estimate as follows:

$$\begin{aligned}
&\langle [N_2'(u)\ - N_2'(v)]\, h \rangle_{t,(1+\alpha)/2,Q}^{s} \\
&\quad \le \sum_{j=0}^{n} \sup_{\substack{(t,x),\,(s,x) \in \mathrm{cl}\, Q \\ t \ne s}} |t - s|^{-(1+\alpha)/2} \left[\left| \int_s^t D_\tau \langle \partial f/\partial u_j \rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)} d\tau \right| |D_j h(t,x)| \right. \\
&\quad \left. + \langle \partial f/\partial u_j \rangle_{s,x,u(s,x)}^{s,x,v(s,x)} \left| \int_s^t D_\tau D_j h(\tau, x) d\tau \right| \right] \le K\delta \,, \qquad K > 0
\end{aligned}$$

Here we have used the mean value theorem for $\partial^2 f/\partial \tau \partial u_j$, $\partial^2 f/\partial u_j \partial u_k$ and $\partial f/\partial u_j$ for $j, k = 0, 1, \ldots, n$.

The third term of (3.2) gives by (2.13), (2.14), (2.16), (2.17):

$$\begin{aligned}
&\sum_{i=1}^{n} \langle D_i \left\{ [N_2'(u)\ - N_2'(v)]\, h \right\} \rangle_{t,\alpha/2,Q}^{s} \\
&\qquad \le \sum_{i=1}^{n} \sum_{j=0}^{n} \sup_{\substack{(t,x),\,(s,x) \in \mathrm{cl}\, Q \\ t \ne s}} |t - s|^{-\alpha/2} \left\{ \left| \int_s^t D_\tau \left\langle \partial^2 f/\partial x_i \partial u_j \right\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)} d\tau \right| \right. \\
&\qquad \cdot |D_j h(t,x)| + \left\langle \partial^2 f/\partial x_i \partial u_j \right\rangle_{s,x,u(s,x)}^{s,x,v(s,x)} \left| \int_s^t D_\tau D_j h(\tau, x) d\tau \right|
\end{aligned}$$

$$+\sum_{k=0}^{n}\Bigg[\left|\int_s^t D_\tau \left\langle \partial^2 f/\partial u_j \partial u_k\right\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)} d\tau\right| |D_{ik}u||D_jh|(t,x)$$
$$+\left|\int_s^t D_\tau \left[\partial^2 f/\partial u_j \partial u_k(\tau,x,v,\dots)\right] d\tau\right| |D_{ik}u(t,x)-D_{ik}v(t,x)||D_jh(t,x)|$$
$$+\left\langle \partial^2 f/\partial u_j \partial u_k\right\rangle_{s,x,u(s,x)}^{s,x,v(s,x)} |D_{ik}u(t,x)-D_{ik}u(s,x)||D_jh(t,x)|$$
$$+\left|\partial^2 f/\partial u_j \partial u_k(s,x,v,\dots)\right| |D_{ik}u(t,x)-D_{ik}v(t,x)$$
$$-\left[D_{ik}u(s,x)-D_{ik}v(s,x)\right]| \, |D_jh(t,x)|$$
$$+\left\langle \partial^2 f/\partial u_j \partial u_k\right\rangle_{s,x,u(s,x)}^{s,x,v(s,x)} |D_{ik}u(s,x)| \left|\int_s^t D_\tau D_j h(\tau,x)d\tau\right|$$
$$+|\partial^2 f/\partial u_j \partial u_k(s,x,v,\dots)||D_{ik}u(s,x)-D_{ik}v(s,x)| \left|\int_s^t D_\tau D_j h(\tau,x)d\tau\right|$$
$$+\left|\int_s^t D_\tau \left\langle \partial f/\partial u_j\right\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)} d\tau\right| |D_{ij}h(t,x)|$$
$$+\left\langle \partial f/\partial u_j\right\rangle_{s,x,u(s,x)}^{s,x,v(s,x)} |D_{ij}h(t,x)-D_{ij}h(s,x)|\Bigg]\Bigg\}$$
$$\le K\left(\sum_{s=0}^{n}\delta^{\beta_s}+\delta\right), \qquad K>0$$

Making the corresponding changes the last term of (3.2) (see the proof of Lemma 3.1)

$$\sum_{i=1}^{n}\left\langle D_i\left\{\left[N_2'(u)-N_2'(v)\right]h\right\}\right\rangle_{x,\alpha,Q}^{y}$$

by the condition $(N_2.2)$ gives the required estimation. This finishes the proof of Lemma 3.2. □

4. Some applications to initial-boundary value problems

In the following theorems we apply the equivalence Lemma E.

Theorem 4.1. *Take $\alpha \in (0,1)$ and $i=1,2$. Assume that the hypotheses $(A_i.1)$, $(A_i.2)$ (see Definition 2.2), $(N_i.2)$, $(N_i.3)$ (see Definition 2.2 and Lemma CC_i) hold.*

Then the open set $Y_i - R_{ib}$ is dense in Y_i and thus the range of bifurcation R_{ib} of initial-boundary value problem $(2.1), (2.2_i), (2.3)$ is nowhere dense in Y_i.

Proof. Using the assumptions $(A_i.1)$ and $(A_i.2)$ we get by Lemma F_i that the operator A_i from (2.8), is a linear Fredholm operator of the zero index. The hypothesis $(N_i.2)$ implies $(N_i.1)$. Hence, with the assumption $(N_i.3)$ we obtain the complete continuity of the Nemitskij operator $N_i : X_i \to Y_i$, with respect to Lemma CC_i. According to $(N_i.2)$ Lemma 3.i ensures that $N_i \in C^1(X_i, Y_i)$. So the assumption (i), (ii) and (iv) from Proposition 1.1 are satisfied.

Denote Σ_i the set of all points $u \in X_i$ at which the Fredholm operator of the zero index $F_i = A_i + N_i : X_i \to Y_i$ is not locally invertible. Also put S_i the set of all critical points of F_i. Then by Proposition 1.1 (jj), S_i represents the set of all singular points of F_i and by (jv) of the same proposition $\Sigma_i \subset S_i$. Hence

$$Y_i - F_i(S_i) \subset Y_i - F_i(\Sigma_i) \subset Y_i - R_{ib}\,.$$

With respect to (jjj) of Proposition 1.1 we have that $Y_i - F_i(S_i)$ is dense in Y_i and from the last relations $Y_i - R_{ib}$ is dense, too.

Recall several conceptions for $i = 1, 2$.

The point $u \in X_i$ means *a singular* or *critical* or *regular solution of the mixed problem* (2.1), (2.2$_i$), (2.3) if it is singular or critical or regular point of the operator F_i (see Proposition 1.1), respectively.

Also we shall investigate the linear problem in $h \in X_i$ for some $u \in X_i$:

$$(4.1)\ \ A_i h(t,x) + \sum_{j=0}^{n} \frac{\partial f}{\partial u_j}\,[t, x, u(t,x), D_1 u(t,x), \ldots, D_n u(t,x)]\, D_j h(t,x) = g(t,x)$$

with the conditions (2.2$_i$), (2.3). □

Theorem 4.2. *Let $\alpha \in (0,1)$ and $i = 1, 2$. Assume that the hypotheses $(A_i.1)$, $(A_i.2)$, $(N_i.2)$, $(N_i.3)$ and $(F_i.1)$ (see Definition 2.2 and Lemma CC_i) hold. Then*

(a) *For any compact set of Y_i (of the right hand sides $g \in Y_i$ of the equation* (2.1)*) the set of all corresponding solutions of the initial-boundary value problem* (2.1), (2.2$_i$), (2.3) *is compact, too.*

(b) *The number of solutions of* (2.1), (2.2$_i$), (2.3) *is constant and finite (it may be zero) on each connected component of the open set $Y_i - F_i(S_i)$, i.e. for any g belonging to the same connected component of $Y_i - F_i(S_i)$. Here S_i means the set of all critical points of problem* (2.1), (2.2$_i$), (2.3).

(c) *Let $u_0 \in X_i$ be a regular solution of* (2.1), (2.2$_i$), (2.3) *with the right hand side $g_0 \in Y_i$. Then there exists a neighbourhood $U(g_0) \subset Y_i$ of g_0 such that for any $g \in U(g_0)$ the initial-boundary value problem* (2.1), (2.2$_i$), (2.3) *has one and only one solution $u \in X_i$. This solution continuously depends on g.*

The associated linear problem (4.1), (2.2$_i$), (2.3) *for $u = u_0$ has a unique solution $h \in X_i$ for any g from a neighbourhood $U(g_0)$ of $g_0 = F_i(u_0)$. This solution continuously depends on g.*

(d) *Denote by G_i the set of all right hand sides $g \in Y_i$ of equation* (2.1) *for which the corresponding solutions $u \in X_i$ of the problem* (2.1), (2.2$_i$), (2.3) *are its critical solutions. Then G_i is closed and nowhere dense in Y_i.*

Proof. From the given hypotheses and by Lemmas F_i, CC_i, C_i and Lemma 3.i we obtain all assumptions (i), (ii), (iii), (iv) from Proposition 1.2.

With respect to assertion (vj) of Proposition 1.2 the operator F_i is proper which implies (a).

According to (jjj) of Proposition 1.1 the set of all singular points of F_i is equal to the set of all critical points of F_i. Then the assertion (b) follows from (vjj) of Proposition 1.2.

(c) Since, $u_0 \in X_i - S_i$, where S_i is a set of singular (under our assumptions also critical) points, then according to (vjjj) of Proposition 1.2 the mapping F_i is a local homeomorphism at u_0, which proves the first part of (c).

However, F_i is a local C^1-diffeomorphism. Thus $F_i' \in C(X_i, Y_i)$, where

$$F_i'(u)h = A_i h + \sum_{j=0}^{n} \frac{\partial f}{\partial u_j}[t, x, u, D_1 u, \dots, D_n u] D_j h$$

and $(F_i^{-1})' \in C(Y_i, X_i)$, where $(F_i^{-1})'(F_i u) = [F_i'(u)]^{-1}$ for every $u \in X_i$ (see [19], p. 115). Hence the linear problem (4.1), (2.2$_i$), (2.3) for $u = u_0$ has a unique solution $h \in X_i$ for any g from a neighbourhood $U(g_0)$ of $g_0 = F_i(u_0)$. This solution continuously depends on the right hand side g. The proof of (c) is completed.

(d) In our case the set of all singular points S_i of F_i is equal to the set of all critical point F_i and $G_i = F_i(S_i)$. We get (d) from (xj) of the Proposition 1.2. □

Corollary 4.1. *Let the hypothesis of Theorem 4.2 hold and*

(vii) *the linear homogeneous problem* (4.1), (2.2$_i$), (2.3) *(for $g = 0$) has only zero solution $h = 0 \in X_i$ for any $u \in X_i$.*

Then the initial-boundary value nonlinear problem (2.1), (2.2$_i$), (2.3) *has a unique solution $u \in X_i$ for any $g \in Y_i$ and $i = 1, 2$. This solution u continuously depends on g. Moreover linear problem* (4.1), (2.2$_i$), (2.3) *has a unique solution $h \in X_i$ for any $u \in X_i$ and right hand side $g \in Y_i$ of* (4.1) *and this solution continuously depends on g.*

The proof of Corollary 4.1 follows by (c) of Theorem 4.2

Theorem 4.3. *Let $\alpha \in (0, 1)$ and $i = 1, 2$. Suppose that the hypotheses $(A_i.1)$, $(A_i.2)$, $(N_i.2)$, $(N_i.3)$ and $(F_i.1)$ (see Definition 2.2 and Lemma CC_i) hold together with the condition*

(viii) *There exists an isolated critical point $u_0 \in X_i$ of the problem 2.1, 2.2$_i$, 2.3.*

Then there exists a neighbourhood $U(g_0) \subset Y_i$ of $g_0 = F_i(u_0)$ such that for any $g \in U(g_0)$ the nonlinear problem (2.1), (2.2$_i$), (2.3) *has a unique solution $u \in U(u_0) \subset X_i$ ($U(u_0)$ is a neighbourhood of u_0). This solution continuously depends on g.*

Proof. From (viii) it follows that $u_0 \in X_i$ is an isolated critical point of F_i, i.e. the linear homogeneous problem (4.1), (2.2$_i$), (2.3) for $g = 0$ and $u = u_0$ has a nontrivial solution $h \neq 0 \in X_i$. Proposition 1.3 ensures the assertion of this theorem. □

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