ARCHIVUM MATHEMATICUM (BRNO)
Tomus 38 (2002), 219 – 226

COMMON FIXED POINT THEOREMS FOR FUZZY MAPPINGS

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ABSTRACT. In this paper, we prove common fixed point theorems for fuzzy mappings satisfying a new inequality initiated by Constantin [6] in complete metric spaces.

1. INTRODUCTION

In 1922, the Polish mathematician, Banach, proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach's fixed point theorem or the Banach contraction principle. This theorem is also applied to show the existence and uniqueness of the solutions of differential equations, integral equations and many other applied mathematics. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways. Markin [11] and Nadler [12] initiated independently the extension of the Banach contraction principle to multivalued mappings. In the last few decades several results on fixed points of contractive-type multivalued mappings have appeared in Beg and Azam [2], Iseki [9], Popa [15] and Singh and Whitfield [20] and others.

The theory of fuzzy sets was introduced by Zadeh in 1965 [22]. Heilpern [7] first introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy mappings. Since then, many fixed point theorems for fuzzy mappings have been obtained by many authors (see, e.g., [1, 3-5, 10, 13, 14, 16-19, 21]).

Motivated and inspired by the works of Arora and Sharma [1], Constantin [6] and Park and Jeong [13], the purpose of this paper is to prove some common fixed point theorems for fuzzy mappings satisfying new contractive-type condition of Constantin [6] in complete metric linear spaces. Our results are the fuzzy extensions of some theorems in [2, 9, 15, 20]. Also, our results generalize the results of Arora and Sharma [1], Heilpern [7] and Park and Jeong [13].

2000 *Mathematics Subject Classification*: 47H10, 54A40.
Key words and phrases: fuzzy sets, fuzzy mappings, fixed point.
Received December 13, 2000.

2. Preliminaries

We cite briefly some definitions and terminologies from [1, 7, 10, 13, 16, 17, 19]. Let (X, d) be a metric linear space. A fuzzy set in X is a function with domain X and values in $[0, 1]$. If A is a fuzzy set and $x \in X$, then the function values $A(x)$ is called the grade of membership of x in A. The α-level set of A, denoted by A_α, is defined by

$$\begin{aligned} A_\alpha &= \{x : A(x) \geq \alpha\} \quad \text{if} \quad \alpha \in (0, 1]\,, \\ A_0 &= \overline{\{x : A(x) \geq 0\}}\,, \end{aligned}$$

where $\overline{B}$ denotes the closure of the set B.

Definition 2.1. A fuzzy set A in X is said to be an approximate quantity iff A_α is compact and convex in X for each $\alpha \in [0, 1]$ and $\sup_{x\in X} A(x) = 1$.

Let $\Im(X)$ be the collection of all fuzzy sets in X and $W(X)$ be a subcollection of all approximate quantities.

Definition 2.2. Let $A, B \in W(X), \alpha \in [0, 1]$. Then

$$\begin{aligned} p_\alpha(A, B) &= \inf_{x\in A_\alpha, y\in B_\alpha} d(x, y)\,, \\ \delta_\alpha(A, B) &= \sup_{x\in A_\alpha, y\in B_\alpha} d(x, y)\,, \\ D_\alpha(A, B) &= H(A_\alpha, B_\alpha)\,, \end{aligned}$$

where H is the Hausdorff distance metric in the collection $CP(X)$ of all nonempty compact subsets of X,

$$\begin{aligned} p(A, B) &= \sup_\alpha p_\alpha(A, B)\,, \\ \delta(A, B) &= \sup_\alpha \delta_\alpha(A, B) \end{aligned}$$

and

$$D(A, B) = \sup_\alpha D_\alpha(A, B)\,.$$

It is noted that p_α is nondecreasing function of α.

Definition 2.3. Let $A, B \in W(X)$. Then A is said to be more accurate than B (or B includes A), denoted by $A \subset B$, iff $A(x) \leq B(x)$ for each $x \in X$.

The relation $\subset$ induces a partial order on $W(X)$.

Definition 2.4. Let X be an arbitrary set and Y be any metric linear space. F is said to be a fuzzy mapping iff F is a mapping from the set X into $W(Y)$, i.e., $F(x) \in W(Y)$ for each $x \in X$.

A fuzzy mapping F is a fuzzy subset on $X \times Y$ with membership function $F(x)(y)$. The function $F(x)(y)$ is the grade of membership of y in $F(x)$.

Lemma 2.1 [7]. *Let $x \in X, A \in W(X)$, and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$. Then $\{x\} \subset A$ if and only if $p_\alpha(x, A) = 0$ for each $\alpha \in [0,1]$.*

Lemma 2.2 [7]. *$p_\alpha(x, A) \le d(x, y) + p_\alpha(y, A)$ for any $x, y \in X$.*

Lemma 2.3 [7]. *If $\{x_0\} \subset A$ then $p_\alpha(x_0, B) \le D_\alpha(A, B)$ for each $B \in W(X)$.*

Proposition 2.1 [10]. *Let (X, d) be a complete metric linear space and $F : X \to W(X)$ be a fuzzy mapping and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.*

Proposition 2.2 [12]. *If $A, B \in CP(X)$ and $a \in A$, then there exists $b \in B$ such that*

$$d(a, b) \le H(A, B)\,.$$

3. Fixed point theorems

Following Constantin [6], we consider the set G of all continuous functions $g : [0,\infty)^5 \to [0,\infty)$ with the following properties:

(i) g is nondecreasing in the 2^{nd}, 3^{th}, 4^{th} and 5^{th} variable,

(ii) if $u, v \in [0,\infty)$ are such that $u \le g(v, v, u, u+v, 0)$ or $u \le g(v, u, v, 0, u+v)$ then $u \le hv$ where $0 < h < 1$ is a given constant,

(iii) if $u \in [0,\infty)$ is such that $u \le g(u, 0, 0, u, u)$, then $u = 0$.

Now, we are ready to prove our main theorems.

Theorem 3.1. *Let X be a complete metric linear space and let F_1 and F_2 be fuzzy mappings from X into $W(X)$. If there is a $g \in G$ such that for all $x, y \in X$*

$$D(F_1(x), F_2(y)) \le g(d(x,y), p(x, F_1(x)), p(y, F_2(y)), p(x, F_2(y)), p(y, F_1(x)))\,,$$

then there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$.

Proof. Let $x_0 \in X$. Then by Proposition 2.1 there exists an $x_1 \in X$ such that $\{x_1\} \subset F_1(x_0)$. From Proposition 2.1, there exists $x_2 \in (F_2(x_1))_1$. Since

$$(F_1(x_0))_1, (F_2(x_1))_1 \in CP(X)\,,$$

then by Proposition 2.2 we obtain

$$\begin{aligned} d(x_1, x_2) &\le D_1(F_1(x_0), F_2(x_1)) \le D(F_1(x_0), F_2(x_1)) \\ &\le g(d(x_0, x_1), p(x_0, F_1(x_0)), p(x_1, F_2(x_1)), p(x_0, F_2(x_1)), p(x_1, F_1(x_0))) \\ &\le g(d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_1) + d(x_1, x_2), 0)\,, \end{aligned}$$

and hence $d(x_1, x_2) \le h d(x_0, x_1)$. Again

$$d(x_2, x_3) \le h d(x_1, x_2)\,.$$

By induction, we produce a sequence (x_n) of points of X such that for $k \geq 0$,

$$\{x_{2k+1}\} \subset F_1(x_{2k})\,, \quad \{x_{2k+2}\} \subset F_2(x_{2k+1})$$

and

$$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n) \leq \cdots \leq h^n d(x_0, x_1)\,.$$

Furthermore, for $m > n$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \{h^n + h^{n+1} + \cdots + h^{m-1}\} d(x_0, x_1) \leq \frac{h^n}{1-h} d(x_0, x_1)\,. \end{aligned}$$

It follows that (x_n) is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Next, we show that $\{z\} \subset F_i(z), i = 1, 2$. Now by Lemma 2.2,

$$p_0(z, F_2(z)) \leq d(z, x_{2n+1}) + p_0(x_{2n+1}, F_2(z))\,.$$

Then by Lemma 2.3,

$$\begin{aligned} p(z, F_2(z)) &\leq d(z, x_{2n+1}) + D(F_1(x_{2n}), F_2(z)) \\ &\leq d(z, x_{2n+1}) + g\big(d(x_{2n}, z), p(x_{2n}, F_1(x_{2n})), \\ &\quad p(z, F_2(z)), p(x_{2n}, F_2(z)), p(z, F_1(x_{2n}))\big) \\ &\leq d(z, x_{2n+1}) + g\big(d(x_{2n}, z), d(x_{2n}, x_{2n+1}), \\ &\quad p(z, F_2(z)), p(x_{2n}, F_2(z)), d(z, x_{2n+1})\big) \\ &\leq d(z, x_{2n+1}) + g\big(d(x_{2n}, z), d(x_{2n}, x_{2n+1}), \\ &\quad p(z, F_2(z)), p(x_{2n}, F_2(z)), d(z, x_{2n+1})\big)\,. \end{aligned}$$

As $n \to \infty$, we have

$$p(z, F_2(z)) \leq g\big(0, 0, p(z, F_2(z)), p(z, F_2(z)), 0\big)\,,$$

and hence $p(z, F_2(z)) = 0$ by (ii). So by Lemma 2.1, we get $\{z\} \subset F_2(z)$. Similarly, it can be shown that $\{z\} \subset F_1(z)$. □

As corollaries of Theorem 3.1, we have the following:

Corollary 3.1 [13; Theorem 3.1]. *Let X be a complete metric linear space and let F_1 and F_2 be fuzzy mappings from X into $W(X)$. If there exists a constant α, $0 \leq \alpha < 1$, such that for each $x, y \in X$,*

$$\begin{aligned} &D(F_1(x), F_2(y)) \\ &\quad \leq \alpha \max\big\{d(x,y), p(x, F_1(x)), p(y, F_2(y)), \frac{1}{2}\big[p(x, F_2(y)) + p(y, F_1(x))\big]\big\}\,, \end{aligned}$$

then there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$.

Proof. We consider the function $g : [0,\infty)^5 \to [0,\infty)$ defined by

$$g(x_1, x_2, x_3, x_4, x_5) = \alpha \max\{x_1, x_2, x_3, \frac{1}{2}[x_4 + x_5]\}\,.$$

Since $g \in G$ we can apply Theorem 3.1 and obtain Corollary 3.1. □

Corollary 3.2 [13; Theorem 3.2]**.** *Let X be a complete metric linear space and let F_1 and F_2 be fuzzy mappings from X into $W(X)$ satisfying*

$$D(F_1(x), F_2(y)) \le k[p(x, F_1(x))p(y, F_2(y))]^{\frac{1}{2}} ,$$

for all $x, y \in X$ and $0 < k < 1$. Then there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$.

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by

$$g(x_1, x_2, x_3, x_4, x_5) = k[x_2 x_3]^{\frac{1}{2}} .$$

Since $g \in G$ we can apply Theorem 3.1 and obtain Corollary 3.2. □

Corollary 3.3 [13; Theorem 3.4]**.** *Let X be a complete metric linear space and let F_1 and F_2 be fuzzy mappings from X into $W(X)$ such that*

$$D(F_1(x), F_2(y)) \le \alpha \frac{p(y, F_1(y))[1 + p(x, F_2(x))]}{1 + d(x, y)} + \beta d(x, y) ,$$

for all $x \neq y$, $\alpha, \beta > 0$, and $\alpha + \beta < 1$. Then there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$.

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by

$$g(x_1, x_2, x_3, x_4, x_5) = \frac{\alpha x_3[1 + x_2]}{1 + x_1} + \beta x_1 .$$

Since $g \in G$ we can apply Theorem 3.1 and obtain Corollary 3.3. □

Corollary 3.4 [1; Theorem 3.2]**.** *Let X be a complete metric linear space and let F_1 and F_2 be fuzzy mappings from X into $W(X)$. If there exists a constant q, $0 \le q < 1$, such that for each $x, y \in X$,*

$$\begin{aligned} &D(F_1(x), F_2(y)) \\ &\qquad \le q \max\{d(x, y), p(x, F_1(x)), p(y, F_2(y)), p(x, F_2(y)), p(y, F_1(x))\} , \end{aligned}$$

then there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$.

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by

$$g(x_1, x_2, x_3, x_4, x_5) = q \max\{x_1, x_2, x_3, x_4, x_5\} .$$

Since $g \in G$ we can apply Theorem 3.1 and obtain Corollary 3.4. □

The following Corollary is a fuzzy version of the fixed point theorem for multi-valued mappings of Iseki [9].

Corollary 3.5. *Let X be a complete metric linear space and let F_1 and F_2 be fuzzy mappings from X into $W(X)$. If for each $x, y \in X$, such that*

$$\begin{aligned} &D(F_1(x), F_2(y)) \\ &\qquad \leq \alpha[p(x, F_1(x)) + p(y, F_2(y))] + \beta[p(x, F_2(y)) + p(y, F_1(x))] + \gamma d(x, y) \end{aligned}$$

where α, β, γ are nonnegative and $2\alpha + 2\beta + \gamma < 1$. Then there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$.

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by

$$g(x_1, x_2, x_3, x_4, x_5) = \alpha[x_2 + x_3] + \beta[x_4 + x_5] + \gamma x_1 .$$

Since $g \in G$ we can apply Theorem 3.1 and obtain Corollary 3.5. □

The following Corollary is a fuzzy version of the fixed point theorem for multi-valued mappings of Singh and Whitfield [20].

Corollary 3.6. *Let X be a complete metric linear space and let F_1 and F_2 be fuzzy mappings from X into $W(X)$. If there exists a constant α, $0 \leq \alpha < 1$, such that for each $x, y \in X$,*

$$\begin{aligned} &D(F_1(x), F_2(y)) \\ &\quad \leq \alpha \max\{d(x, y), \frac{1}{2}[p(x, F_1(x)) + p(y, F_2(y))], \frac{1}{2}[p(x, F_2(y)) + p(y, F_1(x))]\} , \end{aligned}$$

then there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$.

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by

$$g(x_1, x_2, x_3, x_4, x_5) = \alpha \max\{x_1, \frac{1}{2}[x_2 + x_3], \frac{1}{2}[x_4 + x_5]\} .$$

Since $g \in G$ we can apply Theorem 3.1 and obtain Corollary 3.6. □

Remark 3.1. If there is a $g \in G$ such that for all $x, y \in X$

$$\delta(F_1(x), F_2(y)) \leq g\big(d(x, y), p(x, F_1(x)), p(y, F_2(y)), p(x, F_2(y)), p(y, F_1(x))\big) ,$$

then the conclusion of Theorem 3.1 remains valid. This result is considered as a special case of Theorem 3.1 because $D(F_1(x), F_2(y)) \leq \delta(F_1(x), F_2(y))$ [8; page 414]. Moreover, this result generalize Theorem 3.3 of Jeong and Park [13].

The following theorem extends Theorem 3.1 to a sequence of fuzzy mappings:

Theorem 3.2. *Let X be a complete metric linear space and let $\{F_n : n \in Z^+\}$ be fuzzy mappings from X into $W(X)$. If there is a $g \in G$ such that for all $x, y \in X$*

$$D(F_0(x), F_n(y)) \leq g\big(d(x, y), p(x, F_0(x)), p(y, F_n(y)), p(x, F_n(y)), p(y, F_0(x))\big) ,$$

then there exists a common fixed point of the family $\{F_n : n \in Z^+\}$.

Proof. From Theorem 3.1, we get a common fixed point x_i, $i = 1, 2, \ldots$, for each pair (F_0, F_i), $i = 1, 2, \ldots$.

Applying Lemma 2.2, one can have that

$$p_\alpha(x_i, F_0 x_i) = P_\alpha(x_i, F_i(x_i) = 0\,,$$

for all $i = 1, 2, \ldots$. Thus one can deduce from Lemma 2.3, for $i \neq j$, that

$$\begin{aligned} d(x_i, x_j) &= p_\alpha(x_i, F_j(x_j) \le D_\alpha(F_i(x_i), F_j(x_j)) \le D(F_i(x_i), F_j(x_j)) \\ &\le g\big(d(x_i, x_j), p(x_i, F_i(x_i)), p(x_j, F_j(x_j)), p(x_i, F_j(x_j)), p(x_j, F_i(x_i))\big) \\ &= g\big(d(x_i, x_j), 0, 0, d(x_i, x_j), d(x_i, x_j)\big)\,, \end{aligned}$$

and hence $d(x_i, x_j) = 0$ by (iii), i.e., $x_i = x_j$ for all $i \neq j$. □

Corollary 3.7 [1; Theorem 3.4]**.** *Let X be a complete metric linear space and let $\{F_n : n \in Z^+\}$ be fuzzy mappings from X into $W(X)$. If for each $x, y \in X$, $q \in \left(0, \frac{1}{2}\right)$, $n = 1, 2, \ldots$, such that*

$$\begin{aligned} &D(F_0(x), F_i(y)) \\ &\qquad \le q \max\{d(x, y), p(x, F_0(x)), p(y, F_i(y)), p(x, F_i(y)), p(y, F_0(x))\}\,. \end{aligned}$$

Then there exists a common fixed point of the family $\{F_n : n \in Z^+\}$.

Acknowledgement. We wish to thank Prof. B. E. Rhoades at Indiana University (U.S.A.) for his critical reading of the manuscript and his valuable comments.

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