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ON SOLVABILITY OF NONLINEAR BOUNDARY VALUE PROBLEMS FOR THE EQUATION $(x' + g(t, x, x'))' = f(t, x, x')$ WITH ONE-SIDED GROWTH RESTRICTIONS ON f

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Abstract. We consider boundary value problems for second order differential equations of the form $(x' + g(t, x, x'))' = f(t, x, x')$ with the boundary conditions $r(x(0), x'(0), x(T)) + \varphi(x) = 0$, $w(x(0), x(T), x'(T)) + \psi(x) = 0$, where g, r, w are continuous functions, f satisfies the local Carathéodory conditions and φ, ψ are continuous and nondecreasing functionals. Existence results are proved by the method of lower and upper functions and applying the degree theory for α-condensing operators.

1. Introduction, notation

Let $J = [0, T]$ be a compact interval and let $\|x\| = \max\{|x(t)| : t \in J\}$, $\|x\|_{L_1} = \int_0^T |x(t)|\,dt$ and $\|(x, a, b)\|_0 = \|x\| + |a| + \frac{1}{T}|b|$ be the norm in the Banach spaces $C^0(J)$, $L_1(J)$ and $C^0(J) \times \mathbb{R}^2$, respectively.

Denote by $\mathcal{C}$ the set of all functionals $\varphi : C^0(J) \to \mathbb{R}$ which are

a) continuous and
b) nondecreasing, that is, $x, y \in C^0(J)$, $x(t) \le y(t)$ for $t \in J \Rightarrow \varphi(x) \le \varphi(y)$.

Consider the boundary value problem (BVP for short)

$$(x'(t) + g(t, x(t), x'(t)))' = f(t, x(t), x'(t))\,, \tag{1}$$

$$r(x(0), x'(0), x(T)) + \varphi(x) = 0\,, \tag{2}$$

$$w(x(0), x(T), x'(T)) + \psi(x) = 0\,, \tag{3}$$

where $g \in C^0(J \times \mathbb{R}^2)$, f satisfies the local Carathéodory conditions on $J \times \mathbb{R}^2$, $r, w \in C^0(\mathbb{R}^3)$ and $\varphi, \psi \in \mathcal{C}$.

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We say that $x \in C^1(J)$ is *a solution of BVP* (1)–(3) if $x'(t) + g(t, x(t), x'(t))$ is absolutely continuous on J ($AC(J)$ for short), (1) is satisfied for a.e. $t \in J$ and x satisfies the boundary conditions (2) and (3).

There are many papers devoted to the consideration of existence results for second order differential equations and functional differential equations with fully nonlinear two-point boundary conditions (see, e.g., [1], [3]-[8] and references therein). Existence results are proved by a combination of the method of upper and lower functions and methods for a priori bounds on the derivative of solutions. A priori bounds on x' follow for instance if f satisfies either Bernstein-Nagumo growth conditions with respect to the third variable ([1], [7], [8]) or one sided growth restrictions ([3]-[5]) or only sign conditions ([6]). We observe that in [3]-[5], [7] and [8] BVPs were also considered with boundary conditions $(x(0), x'(0)) \in \Omega_0$, $(x(T), x'(T)) \in \Omega_1$, where Ω_0, Ω_1 are closed connected subsets of the plane.

In this paper we give existence results for BVP (1), (2). We shall show that the existence of lower and upper functions of BVP (1), (2) together with some conditions on g, r, w and one-sided growth restrictions on f guarantee a priori estimates for the derivative of solutions. Existence results are then proved by the Borsuk antipodal theorem and the Leray-Schauder degree for α-condensing operators (see [2]). In our case α-condensing operators can be written in the form $\mathcal{K} + \mathcal{L}$, where $\mathcal{K}$ is a compact operator and $\mathcal{L}$ is a strict contraction.

A function $\alpha \in C^1(J)$ is said to be *a lower function of BVP* (1)–(3) if $\alpha'(t) + g(t, \alpha(t), \alpha'(t)) \in AC(J)$,

$$(\alpha'(t) + g(t, \alpha(t), \alpha'(t)))' \geq f(t, \alpha(t), \alpha'(t)) \quad \text{for a.e. } t \in J$$

and

$$r(\alpha(0), \alpha'(0), \alpha(T)) + \varphi(\alpha) \geq 0, \quad w(\alpha(0), \alpha(T), \alpha'(T)) + \psi(\alpha) \geq 0.$$

Similarly, a function $\beta \in C^1(J)$ is *an upper function of BVP* (1)–(3) if $\beta'(t) + g(t, \beta(t), \beta'(t)) \in AC(J)$,

$$(\beta'(t) + g(t, \beta(t), \beta'(t)))' \leq f(t, \beta(t), \beta'(t)) \quad \text{for a.e. } t \in J$$

and

$$r(\beta(0), \beta'(0), \beta(T)) + \varphi(\beta) \leq 0, \quad w(\beta(0), \beta(T), \beta'(T)) + \psi(\beta) \leq 0.$$

For each $\alpha, \beta \in C^0(J)$, $\alpha(t) \leq \beta(t)$ on J and each positive constant S, define subsets $\mathcal{A}_1^S(\alpha, \beta)$ and $\mathcal{A}_2^S(\alpha, \beta)$ of $\mathbb{R}^3$ by

$$\mathcal{A}_1^S(\alpha, \beta) = \{(t, x, y) : (t, x, y) \in J \times [\alpha(t), \beta(t)] \times [S, \infty)\},$$
$$\mathcal{A}_2^S(\alpha, \beta) = \{(t, x, y) : (t, x, y) \in J \times [\alpha(t), \beta(t)] \times (-\infty, -S]\}.$$

We say that $\omega \in C^0(J)$ is the *Nagumo-type function* if

(i) $\omega(u) > 0$ for $u \in \mathbb{R}$,
(ii) $\omega(-u) = \omega(u)$ for $u \in \mathbb{R}$ and ω is nondecreasing on $[0, \infty)$,
(iii) $\displaystyle\int_0^\infty \frac{du}{\omega(u)} = \infty$.

Throughout the paper we shall assume that the functions g, f, r, w and the functionals $\varphi, \psi \in \mathcal{C}$ in (1)–(3) satisfy some of the following assumptions.

(H_1) There exists a lower function α and an upper function β of BVP (1) – (3) and

$$\alpha(t) \le \beta(t) \quad \text{for } t \in J;$$

(H_2) $g(t,\alpha(t),\alpha'(t))$, $g(t,\beta(t),\beta'(t)) \in AC(J)$;

(H_3) There exist nonnegative constants m and k such that

$$mT + k < 1$$

and

$$|g(t,x_1,y_1) - g(t,x_2,y_2)| \le m|x_1 - x_2| + k|y_1 - y_2|$$

for $(t,x_i,y_i) \in J \times [\alpha(t),\beta(t)] \times \mathbb{R}$ $(i = 1,2)$;

(H_4) r is nondecreasing in the second and third variables, w is nondecreasing in the first variable and nonincreasing in the third one and there exists a positive constant S,

$$S \ge \max\{\|\alpha'\|, \|\beta'\|\}$$

such that

$$r(x,-S,y) + \varphi(z) < 0, \quad r(x,S,y) + \varphi(z) > 0,$$

$$w(x,y,-S) + \psi(z) > 0, \quad w(x,y,S) + \psi(z) < 0$$

for all $x, y \in \mathbb{R}$, $|x| \le \Lambda$, $|y| \le \Lambda$ and $z \in C^0(J)$, $\alpha(t) \le z(t) \le \beta(t)$ on J, where

$$\Lambda = \max\{\|\alpha\|, \|\beta\|\}\,; \tag{4}$$

(H_5) There exist $\sigma_j \in \{-1,1\}$ $(j = 1,2)$, a Nagumo-type function ω and a nonnegative function $h \in L_1(J)$ such that

$$\sigma_j f(t,x,y) \le (h(t) + |y|)\omega(y)$$

for $(t,x,y) \in \mathcal{A}_j^S(\alpha,\beta)$ and $j = 1,2$.

By our assumptions, the function f satisfies the local Carathéodory conditions on $J \times \mathbb{R}^2$, and so from assumption (H_2) it follows that there exists a positive function $\chi \in L_1(J)$ such that

$$\left| g(t_1,\alpha(t_1),\alpha'(t_1)) - g(t_2,\alpha(t_2),\alpha'(t_2)) + \int_{t_1}^{t_2} f(s,\alpha(s),\alpha'(s))\,ds \right| \le \int_{t_1}^{t_2} \chi(s)\,ds,$$

$$\left| g(t_1,\beta(t_1),\beta'(t_1)) - g(t_2,\beta(t_2),\beta'(t_2)) + \int_{t_1}^{t_2} f(s,\beta(s),\beta'(s))\,ds \right| \le \int_{t_1}^{t_2} \chi(s)\,ds$$

for $0 \le t_1 \le t_2 \le T$.

For each $\gamma, \delta \in C^0(J)$, $\gamma(t) \le \delta(t)$ on J, $a \in J$ and $n \in \mathbb{N}$, define the truncation operator $\Delta_{\gamma\delta} : C^0(J) \to C^0(J)$, the penalty operator $p^n_{\gamma\delta} : C^0(J) \to L_1(J)$ and the function $\Psi^a_{\gamma\delta} : \mathbb{R} \to \mathbb{R}$ by the formulas

$$(\Delta_{\gamma\delta} x)(t) = \begin{cases} \delta(t) & \text{if } \; x(t) > \delta(t) \\ x(t) & \text{if } \; \gamma(t) \le x(t) \le \delta(t) \\ \gamma(t) & \text{if } \; x(t) < \gamma(t), \end{cases}$$

$$(p^n_{\gamma\delta}x)(t) = \begin{cases} \chi(t) & \text{if} \quad x(t) > \delta(t) + \frac{1}{n} \\ n\chi(t)(x(t) - \delta(t)) & \text{if} \quad \delta(t) \le x(t) \le \delta(t) + \frac{1}{n} \\ 0 & \text{if} \quad \gamma(t) \le x(t) < \delta(t) \\ n\chi(t)(x(t) - \gamma(t)) & \text{if} \quad \gamma(t) - \frac{1}{n} \le x(t) < \gamma(t) \\ -\chi(t) & \text{if} \quad x(t) < \gamma(t) - \frac{1}{n} \end{cases}$$

and

$$\Psi^a_{\gamma\delta}(u) = \begin{cases} \delta(a) & \text{if} \quad u > \delta(a) \\ u & \text{if} \quad \gamma(a) \le u \le \delta(a) \\ \gamma(a) & \text{if} \quad u < \gamma(a)\,. \end{cases} \tag{5}$$

If $\gamma, \delta \in C^1(J)$ then $\Delta_{\gamma\delta} : C^1(J) \to AC(J)$ and $\lim_{n\to\infty} (\Delta_{\gamma\delta}x_n)'(t) = (\Delta_{\gamma\delta}x)'(t)$ for a.e. $t \in J$ whenever $x_n, x \in C^1(J)$ and $\lim_{n\to\infty} x_n = x$ in $C^1(J)$ (see [9, Lemma 2]).

2. Lemmas

Let assumptions (H_1) and (H_2) be satisfied. Consider the auxiliary BVP

$$\begin{aligned} \Big(x'(t) + \lambda g\Big(t, (\Delta_{\alpha\beta}x)(t), x'(t)\Big)\Big)' &= \lambda f\Big(t, (\Delta_{\alpha\beta}x)(t), (\Delta_{\alpha\beta}x)'(t)\Big) \\ &+ \Big(1 - \lambda + \frac{1}{n}\Big)(p^n_{\alpha\beta}x)(t), \end{aligned} \tag{$6)^n_\lambda$}$$

$$x(0) = \Psi^0_{\alpha-\frac{1}{n},\beta+\frac{1}{n}}\Big(x(0) + r\Big(\Psi^0_{\alpha\beta}(x(0)), x'(0), \Psi^T_{\alpha\beta}(x(T))\Big) + \varphi(\Delta_{\alpha\beta}x)\Big), \tag{$7)^n$}$$

$$x(T) = \Psi^T_{\alpha-\frac{1}{n},\beta+\frac{1}{n}}\Big(x(T) + w\Big(\Psi^0_{\alpha\beta}(x(0)), \Psi^T_{\alpha\beta}(x(T)), x'(T)\Big) + \psi(\Delta_{\alpha\beta}x)\Big) \tag{$8)^n$}$$

depending on the parameters $\lambda \in [0,1]$ and $n \in \mathbb{N}$.

Lemma 1. *Let assumptions $(H_1) - (H_4)$ be satisfied and let $x(t)$ be a solution of BVP $(6)^n_\lambda - (8)^n$ for some $\lambda \in (0,1]$ and $n \in \mathbb{N}$. Then*

$$\alpha(t) - \frac{1}{n} \le x(t) \le \beta(t) + \frac{1}{n} \quad \textit{for} \quad t \in J \tag{9}$$

and

$$|x'(0)| \le S\,, \quad |x'(T)| \le S\,. \tag{10}$$

Proof. By $(7)^n$, $(8)^n$ and the definition of the functions $\Psi^0_{\alpha\beta}$ and $\Psi^T_{\alpha\beta}$,

$$\alpha(0) - \frac{1}{n} \le x(0) \le \beta(0) + \frac{1}{n}\,, \quad \alpha(T) - \frac{1}{n} \le x(T) \le \beta(T) + \frac{1}{n}\,. \tag{11}$$

Let

$$\max\{x(t) - \beta(t) : t \in J\} = x(\xi) - \beta(\xi) > \frac{1}{n}\,. \tag{12}$$

Then (cf. (11)) $\xi \in (0,T)$, and so $x'(\xi) = \beta'(\xi)$. In addition, $x(t) - \beta(t) > \frac{1}{n}$ on some interval $[\xi, \xi+\varepsilon]$ ($\subset J$). Then $(\Delta_{\alpha\beta}x)(t) = \beta(t)$, $(\Delta_{\alpha\beta}x)'(t) = \beta'(t)$ and $(p^n_{\alpha\beta}x)(t) = \chi(t)$ for $t \in [\xi, \xi+\varepsilon]$. Hence (for $t \in (\xi, \xi+\varepsilon]$)

$$\begin{aligned}
x'(t) - \beta'(t) &= x'(t) - x'(\xi) - \beta'(t) + \beta'(\xi) \\
&\geq (\lambda - 1)\Big(g(\xi, \beta(\xi), \beta'(\xi)) - g(t, \beta(t), \beta'(t)) + \int_\xi^t f(s, \beta(s), \beta'(s))\, ds\Big) \\
&\quad + \lambda\Big(g(t, \beta(t), \beta'(t)) - g(t, \beta(t), x'(t))\Big) + \Big(1 - \lambda + \frac{1}{n}\Big) \int_\xi^t (p^n_{\alpha\beta}x)(s)\, ds \\
&\geq -(1-\lambda) \int_\xi^t \chi(s)\, ds - \lambda k|\beta'(t) - x'(t)| + \Big(1 - \lambda + \frac{1}{n}\Big) \int_\xi^t \chi(s)\, ds \\
&> -\lambda k|\beta'(t) - x'(t)|
\end{aligned}$$

and

$$x'(t) - \beta'(t) + \lambda k|\beta'(t) - x'(t)| > 0 \quad \text{for } t \in (\xi, \xi+\varepsilon].$$

From the last inequality we deduce that $x'(t) > \beta'(t)$ for $t \in (\xi, \xi+\varepsilon]$, contrary to (12).

Assume that

$$\min\{x(t) - \alpha(t) : t \in J\} = x(\tau) - \alpha(\tau) < -\frac{1}{n}\,. \tag{13}$$

Then (cf. (11)) $\tau \in (0,T)$, and so $x'(\tau) = \alpha'(\tau)$. Moreover, there exists $\nu \in (0, T-\tau]$ such that $x(t) - \alpha(t) < -\frac{1}{n}$ for $t \in [\tau, \tau+\nu]$. Hence $(\Delta_{\alpha\beta}x)(t) = \alpha(t)$, $(\Delta_{\alpha\beta}x)'(t) = \alpha'(t)$ and $(p^n_{\alpha\beta}x)(t) = -\chi(t)$ for $t \in [\tau, \tau+\nu]$, and consequently (for $t \in (\tau, \tau+\nu]$)

$$\begin{aligned}
\alpha'(t) - x'(t) &= \alpha'(t) - \alpha'(\tau) - x'(t) + x'(\tau) \\
&\geq (1 - \lambda)\Big(g(\tau, \alpha(\tau), \alpha'(\tau)) - g(t, \alpha(t), \alpha'(t)) + \int_\tau^t f(s, \alpha(s), \alpha'(s))\, ds\Big) \\
&\quad + \lambda\Big(g(t, \alpha(t), x'(t)) - g(t, \alpha(t), \alpha'(t))\Big) - \Big(1 - \lambda + \frac{1}{n}\Big) \int_\tau^t (p^n_{\alpha\beta}x)(s)\, ds \\
&\geq -(1-\lambda) \int_\xi^t \chi(s)\, ds - \lambda k|\alpha'(t) - x'(t)| + \Big(1 - \lambda + \frac{1}{n}\Big) \int_\tau^t \chi(s)\, ds \\
&> -\lambda k|\alpha'(t) - x'(t)|.
\end{aligned}$$

From the inequality $\alpha'(t) - x'(t) + \lambda k|\alpha'(t) - x'(t)| > 0$ we conclude that $\alpha'(t) > x'(t)$ for $t \in (\tau, \tau+\nu]$, contrary to (13).

It remains to verify (10). Assume that $x'(0) < -S$ Then (cf. (H_4))

$$r\Big(\Psi^0_{\alpha\beta}(x(0)), x'(0), \Psi^T_{\alpha\beta}(x(T))\Big) + \varphi(\Delta_{\alpha\beta}x) < 0\,, \tag{14}$$

and consequently (cf. $(7)^n$) $x(0)=\alpha(0)-\frac{1}{n}$. Then (cf. (9)) $x'(0)\ge\alpha'(0)$, which yields

$$0\le r(\alpha(0),\alpha'(0),\alpha(T))+\varphi(\alpha)=r\Big(\Psi^0_{\alpha\beta}(x(0)),\alpha'(0),\alpha(T)\Big)+\varphi(\alpha)$$
$$\le r\Big(\Psi^0_{\alpha\beta}(x(0)),x'(0),\Psi^T_{\alpha\beta}(x(T))\Big)+\varphi(\Delta_{\alpha\beta}x)\,,$$

contrary to (14). If $x'(0)>S$ then (cf. (H_4))

$$r\Big(\Psi^0_{\alpha\beta}(x(0)),x'(0),\Psi^T_{\alpha\beta}(x(T))\Big)+\varphi(\Delta_{\alpha\beta}x)>0\,, \tag{15}$$

and consequently (cf. $(7)^n$) $x(0)=\beta(0)+\frac{1}{n}$. Hence (cf. (9)) $x'(0)\le\beta'(0)$ and

$$0\ge r(\beta(0),\beta'(0),\beta(T))+\varphi(\beta)=r\Big(\Psi^0_{\alpha\beta}(x(0)),\beta'(0),\beta(T)\Big)+\varphi(\beta)$$
$$\ge r\Big(\Psi^0_{\alpha\beta}(x(0)),x'(0),\Psi^T_{\alpha\beta}(x(T))\Big)+\varphi(\Delta_{\alpha\beta}x)\,,$$

contrary to (15).

Assume $x'(T)<-S$. Then (cf. (H_4))

$$w\Big(\Psi^0_{\alpha\beta}(x(0)),\Psi^T_{\alpha\beta}(x(T)),x'(T)\Big)+\psi(\Delta_{\alpha\beta}x)>0\,. \tag{16}$$

Therefore (cf. $(8)^n$) $x(T)=\beta(T)+\frac{1}{n}$ and $x'(T)\ge\beta'(T)$ by (9). Hence

$$0\ge w(\beta(0),\beta(T),\beta'(T))+\psi(\beta)=w\Big(\beta(0),\Psi^T_{\alpha\beta}(x(T)),\beta'(T)\Big)+\psi(\beta)$$
$$\ge w\Big(\Psi^0_{\alpha\beta}(x(0)),\Psi^T_{\alpha\beta}(x(T)),x'(T)\Big)+\psi(\Delta_{\alpha\beta}x)\,,$$

contrary to (16). If $x'(T)>S$ then

$$w\Big(\Psi^0_{\alpha\beta}(x(0)),\Psi^T_{\alpha\beta}(x(T)),x'(T)\Big)+\psi(\Delta_{\alpha\beta}x)<0\,. \tag{17}$$

by(H_4) and (cf. $(8)^n$) $x(T)=\alpha(T)-\frac{1}{n}$. Then (cf. (9)) $x'(T)\le\alpha'(T)$ and

$$0\le w(\alpha(0),\alpha(T),\alpha'(T))+\psi(\alpha)=w\Big(\alpha(0),\Psi^T_{\alpha\beta}(x(T)),\alpha'(T)\Big)+\psi(\alpha)$$
$$\le w\Big(\Psi^0_{\alpha\beta}(x(0)),\Psi^T_{\alpha\beta}(x(T)),x'(T)\Big)+\psi(\Delta_{\alpha\beta}x)\,,$$

contrary to (17). □

Lemma 2. *Let assumptions $(H_1)-(H_5)$ be satisfied and $x(t)$ be a solution of BVP $(6)^n_\lambda-(8)^n$ for some $\lambda\in(0,1]$ and $n\in\mathbb{N}$. Let P be a positive constant satisfying the inequality*

$$\int\limits_{\frac{S+2A}{1-k}}^{P}\frac{du}{\omega(u)}>\frac{1}{1-k}\Big(\|h\|_{L_1}+2\Lambda+2+\frac{2\|\chi\|_{L_1}}{\omega(0)}\Big) \tag{18}$$

where

$$A=\max\{|g(t,u,v)|:(t,u,v)\in J\times[\alpha(t),\beta(t)]\times[-S,S]\}\,.$$

Then $x(t)$ satisfies (9) *and*

$$\|x'\| < P\,. \tag{19}$$

In addition, for each $\varepsilon > 0$ there exists $\delta > 0$ independent of n and λ such that

$$|x'(t_1) - x'(t_2)| < \varepsilon$$

whenever $t_1, t_2 \in J$, $|t_1 - t_2| < \delta$.

Proof. Let $\omega_* \in C^0(J)$ be defined by

$$\omega_*(u) = \begin{cases} \omega\left(\frac{u+A}{1-k}\right) & \text{for} \quad u > A \\ \omega\left(\frac{2A}{1-k}\right) & \text{for} \quad |u| \le A \\ \omega\left(\frac{u-A}{1-k}\right) & \text{for} \quad u < -A\,. \end{cases} \tag{20}$$

Then ω_* is nondecreasing on $[-A, \infty)$, $\omega_*(-u) = \omega_*(u)$ for $u \in \mathbb{R}$ and

$$\omega(u) = \omega_*((1-k)u - A\mathrm{sign}\, u) \quad \text{for} \quad u \in \mathbb{R}\,. \tag{21}$$

By Lemma 1, inequalities (9) are satisfied. Set

$$q(t) = x'(t) + \lambda g\Big(t, (\Delta_{\alpha\beta} x)(t), x'(t)\Big) \quad \text{for} \quad t \in J\,.$$

Observe that $|x'(t_0)| \le S$ for some $t_0 \in J$ implies

$$|q(t_0)| \le |x'(t_0)| + \Big|g\Big(t_0, (\Delta_{\alpha\beta} x)(t_0), x'(t_0)\Big)\Big| \le S + A\,. \tag{22}$$

From the inequalities (cf. (H_3))

$$\begin{aligned} \Big|g\Big(t, (\Delta_{\alpha\beta} x)(t), x'(t)\Big)\Big| \;&\le \Big|g\Big(t, (\Delta_{\alpha\beta} x)(t), x'(t)\Big) - g\Big(t, (\Delta_{\alpha\beta} x)(t), 0\Big)\Big| \\ &\quad + \;\Big|g\Big(t, (\Delta_{\alpha\beta} x)(t), 0\Big)\Big| \\ &\le k|x'(t)| + A \end{aligned}$$

for $t \in J$ we see that

$$q(t) \ge x'(t) - kx'(t) - A = (1-k)x'(t) - A \tag{23}$$

whenever $x'(t) > 0$ and

$$q(t) \le x'(t) - kx'(t) + A = (1-k)x'(t) + A \tag{24}$$

whenever $x'(t) < 0$. Hence

$$\omega_*((1-k)x'(t) - A) \le \omega_*(q(t)) \quad \text{if} \quad x'(t) > 0 \tag{25}$$

and

$$\omega_*((1-k)x'(t) + A) \le \omega_*(q(t)) \quad \text{if} \quad x'(t) < 0\,. \tag{26}$$

Assume that $\|x'\| = |x'(\xi)| > S$. By Lemma 1, (10) holds, and so $\xi \in (0, T)$. We first assume that $x'(\xi) > S$. Then there exist (cf. (10)) τ_1 and τ_2, $0 \le \tau_1 < \xi < \tau_2 \le T$ such that

$$x'(\tau_1) = S = x'(\tau_2)$$

and

$$S \le x'(t) \le x'(\xi) \quad \text{for } t \in [\tau_1, \tau_2]\,.$$

Hence (cf. (H_5), (21) and (25))

$$\begin{aligned}
\sigma_1 q'(t) &= \lambda\sigma_1 f\Big(t, (\Delta_{\alpha\beta}x)(t), (\Delta_{\alpha\beta}x)'(t)\Big) + \sigma_1\Big(1-\lambda+\frac{1}{n}\Big)(p^n_{\alpha\beta}x)(t) \\
&\le (h(t)+x'(t))\omega(x'(t)) + 2\chi(t) \\
&\le (h(t)+x'(t))\omega_*((1-k)x'(t)-A) + 2\chi(t) \\
&\le (h(t)+x'(t))\omega_*(q(t)) + 2\chi(t)
\end{aligned} \tag{27}$$

for a.e. $t \in [\tau_1, \tau_2]$. If $\sigma_1 = 1$ then (cf. (9) and (27))

$$\begin{aligned}
\int\limits_{q(\tau_1)}^{q(\xi)} \frac{du}{\omega_*(u)} &= \int\limits_{\tau_1}^{\xi} \frac{q'(t)}{\omega_*(q(t))}\,dt \le \int\limits_{\tau_1}^{\xi} \Big(h(t)+x'(t)+\frac{2\chi(t)}{\omega_*(q(t))}\Big)\,dt \\
&\le \|h\|_{L_1} + 2\Lambda + 2 + \frac{2\|\chi\|_{L_1}}{\omega_*(0)}
\end{aligned}$$

and since $|q(\tau_1)| \le S + A$ by (22) with $t_0 = \tau_1$ and $q(\xi) \ge (1-k)\|x'\| - A$ by (23) with $t = \xi$, we have

$$\int\limits_{S+A}^{(1-k)\|x'\|-A} \frac{du}{\omega_*(u)} \le \int\limits_{q(\tau_1)}^{q(\xi)} \frac{du}{\omega_*(u)} \le \|h\|_{L_1} + 2\Lambda + 2 + \frac{2\|\chi\|_{L_1}}{\omega_*(0)}\,. \tag{28}$$

If $\sigma_1 = -1$ then (cf. (9) and (27))

$$\begin{aligned}
\int\limits_{q(\tau_2)}^{q(\xi)} \frac{du}{\omega_*(u)} &= -\int\limits_{\xi}^{\tau_2} \frac{q'(t)}{\omega_*(q(t))}\,dt \le \int\limits_{\xi}^{\tau_2} \Big(h(t)+x'(t)+\frac{2\chi(t)}{\omega_*(q(t))}\Big)\,dt \\
&\le \|h\|_{L_1} + 2\Lambda + 2 + \frac{2\|\chi\|_{L_1}}{\omega_*(0)}
\end{aligned}$$

and using the inequalities $|q(\tau_2)| \le S + A$, $q(\xi) \ge (1-k)\|x'\| - A$, we have

$$\int\limits_{S+A}^{(1-k)\|x'\|-A} \frac{du}{\omega_*(u)} \le \int\limits_{q(\tau_2)}^{q(\xi)} \frac{du}{\omega_*(u)} \le \|h\|_{L_1} + 2\Lambda + 2 + \frac{2\|\chi\|_{L_1}}{\omega_*(0)}\,. \tag{29}$$

Let $x'(\xi) < -S$. Then there exist (cf. (10)) ν_1 and ν_2, $0 \le \nu_1 < \xi < \nu_2 \le T$ such that

$$x'(\nu_1) = -S = x'(\nu_2)$$

and

$$-S \ge x'(t) \ge x'(\xi) \quad \text{for} \quad t \in [\nu_1, \nu_2]\,.$$

Hence (cf. (H_5), (21) and (26))

$$\begin{aligned}\sigma_2 q'(t) &= \lambda\sigma_2 f\Big(t, (\Delta_{\alpha\beta}x)(t), (\Delta_{\alpha\beta}x)'(t)\Big) + \sigma_2\Big(1-\lambda+\frac{1}{n}\Big)(p^n_{\alpha\beta}x)(t)\\ &\le (h(t)-x'(t))\omega(x'(t)) + 2\chi(t)\\ &\le (h(t)-x'(t))\omega_*((1-k)x'(t)+A) + 2\chi(t)\\ &\le (h(t)-x'(t))\omega_*(q(t)) + 2\chi(t)\end{aligned}$$

for a.e. $t\in[\nu_1,\nu_2]$. If $\sigma_2 = 1$ then

$$\begin{aligned}\int\limits_{q(\xi)}^{q(\nu_2)} \frac{du}{\omega_*(u)} &= \int\limits_{\xi}^{\nu_2} \frac{q'(t)}{\omega_*(q(t))}\,dt \le \int\limits_{\xi}^{\nu_2}\Big(h(t)-x'(t)+\frac{2\chi(t)}{\omega_*(q(t))}\Big)\,dt\\ &\le \|h\|_{L_1} + 2\Lambda + 2 + \frac{2\|\chi\|_{L_1}}{\omega_*(0)}\end{aligned}$$

and

$$\int\limits_{-(1-k)\|x'\|+A}^{-S-A} \frac{du}{\omega_*(u)} \le \int\limits_{q(\xi)}^{q(\nu_2)} \frac{du}{\omega_*(u)} \le \|h\|_{L_1} + 2\Lambda + 2 + \frac{2\|\chi\|_{L_1}}{\omega_*(0)} \tag{30}$$

since $|q(\nu_2)| \le S + A$ by (22) with $t_0 = \nu_2$ and (cf. (24) with $t = \xi$) $q(\xi) \le -(1-k)\|x'\| + A$. If $\sigma_2 = -1$ then

$$\int\limits_{-(1-k)\|x'\|+A}^{-S-A} \frac{du}{\omega_*(u)} \le \int\limits_{q(\xi)}^{q(\nu_1)} \frac{du}{\omega_*(u)} = -\int\limits_{\nu_1}^{\xi} \frac{q'(t)}{\omega_*(q(t))}\,dt$$

$$\le \int\limits_{\nu_1}^{\xi}\Big(h(t)-x'(t)+\frac{2\chi(t)}{\omega_*(q(t))}\Big)\,dt \le \|h\|_{L_1} + 2\Lambda + 2 + \frac{2\|\chi\|_{L_1}}{\omega_*(0)}\,. \tag{31}$$

Since

$$\int_{-(1-k)\|x'\|+A}^{-S-A} \frac{du}{\omega_*(u)} = \int_{S+A}^{(1-k)\|x'\|-A} \frac{du}{\omega_*(u)}$$

and (cf. (18) and (20))

$$\begin{aligned}\int_{S+A}^{(1-k)P-A} \frac{du}{\omega_*(u)} &= \int_{S+A}^{(1-k)P-A} \frac{du}{\omega(\frac{u+A}{1-k})} = (1-k)\int_{\frac{S+2A}{1-k}}^{P} \frac{du}{\omega(u)}\\ &> \|h\|_{L_1} + 2\Lambda + 2 + \frac{2\|\chi\|_{L_1}}{\omega(0)} \ge \|h\|_{L_1} + 2\Lambda + 2 + \frac{2\|\chi\|_{L_1}}{\omega_*(0)}\end{aligned}$$

we see that (28)-(31) imply (19).

Fix $\varepsilon > 0$ and let $\varrho \in L_1(J)$ be defined by

$$\varrho(t) = \sup\{|f(t,u,v)| : (u,v)\in[-\Lambda,\Lambda]\times[-P,P]\} \quad \text{for a.e.} \quad t\in J\,.$$

From the continuity of g, the inequality $|(p^n_{\alpha\beta}x)(t)| \le \chi(t)$ for a.e. $t \in J$ and $\varrho \in L_1(J)$ we see that there exists $\delta_1 > 0$ such that

$$|g(t_1, u, v) - g(t_2, u, v)| < \frac{\varepsilon(1-k)}{4}, \quad \Big| \int_{t_1}^{t_2} (p^n_{\alpha\beta}x)(t)\,dt \Big| < \frac{\varepsilon(1-k)}{8}$$

and

$$\Big| \int_{t_1}^{t_2} \varrho(t)\,dt \Big| < \frac{\varepsilon(1-k)}{4}$$

whenever $t_1, t_2 \in J$, $|t_1 - t_2| < \delta_1$ and $(u, v) \in [-\Lambda, \Lambda] \times [-P, P]$. Set

$$\delta = \min\Big\{ \delta_1, \frac{\varepsilon(1-k)}{4mP} \Big\}$$

(for m and k see (H_3)). Then for any $t_1, t_2 \in J$, $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}
|x'(t_1) - x'(t_2)| &\le \Big| g\Big(t_1, (\Delta_{\alpha\beta}x)(t_1), x'(t_1)\Big) - g\Big(t_2, (\Delta_{\alpha\beta}x)(t_2), x'(t_2)\Big) \Big| \\
&\quad + \Big| \int_{t_1}^{t_2} f\Big(s, (\Delta_{\alpha\beta}x)(s), (\Delta_{\alpha\beta}x)'(s)\Big)\,ds \Big| + 2\Big| \int_{t_1}^{t_2} (p^n_{\alpha\beta}x)(s)\,ds \Big| \\
&< \Big| g\Big(t_1, (\Delta_{\alpha\beta}x)(t_1), x'(t_1)\Big) - g\Big(t_2, (\Delta_{\alpha\beta}x)(t_1), x'(t_1)\Big) \Big| \\
&\quad + \Big| g\Big(t_2, (\Delta_{\alpha\beta}x)(t_1), x'(t_1)\Big) - g\Big(t_2, (\Delta_{\alpha\beta}x)(t_2), x'(t_2)\Big) \Big| \\
&\quad + \Big| \int_{t_1}^{t_2} \varrho(s)\,ds \Big| + \frac{\varepsilon(1-k)}{4} \\
&\le \frac{\varepsilon(1-k)}{4} + m\Big| (\Delta_{\alpha\beta}x)(t_1) - (\Delta_{\alpha\beta}x)(t_2) \Big| \\
&\quad + k|x'(t_1) - x'(t_2)| + \frac{\varepsilon(1-k)}{2} \\
&\le \frac{3\varepsilon(1-k)}{4} + m|x(t_1) - x(t_2)| + k|x'(t_1) - x'(t_2)| \\
&\le \frac{3\varepsilon(1-k)}{4} + mP|t_1 - t_2| + k|x'(t_1) - x'(t_2)| \\
&\le \varepsilon(1-k) + k|x'(t_1) - x'(t_2)|
\end{aligned}$$

which yields $|x'(t_1) - x'(t_2)| < \varepsilon$. □

Lemma 3. *Let assumptions* (H_1) *and* (H_2) *be satisfied,* $n \in \mathbb{N}$ *and* Λ *be defined by* (4). *Set*

$$\begin{aligned}
\Omega_0 = \Big\{ (x, a, b) :\ &(x, a, b) \in C^0(J) \times \mathbb{R}^2,\ \|x\| < E + 8\|\chi\|_{L_1} + C, \\
&|a| < E + 4\|\chi\|_{L_1} + D,\ |b| < \Lambda + 2 \Big\},
\end{aligned} \tag{32}$$

where

$$E = \frac{1}{T} \max\Big\{ |\alpha(T) - \beta(0) - 2|,\ |\beta(T) - \alpha(0) + 2| \Big\} \tag{33}$$

and C, D are positive constants. Let

$$\mathcal{K} = \overline{\Omega}_0 \to C^0(J) \times \mathbb{R}^2 ,$$

$$\begin{aligned}
\mathcal{K}(x,a,b) = \Bigg[& a + \Big(1+\frac{1}{n}\Big) \int_0^t \Big(p^n_{\alpha\beta} \Big(\int_0^\nu x(\tau)\, d\tau + b \Big)\Big)(s) ds, \\
& a + b - \Psi^0_{\alpha-\frac{1}{n},\beta+\frac{1}{n}} \Bigg(b + r\Big(\Psi^0_{\alpha\beta}(b), x(0), \Psi^T_{\alpha\beta}\Big(\int_0^T x(s)\, ds + b\Big)\Big) \\
& \qquad + \varphi\Big(\Delta_{\alpha\beta}\Big(\int_0^t x(s)\, ds + b\Big)\Big)\Bigg), \\
& 2b + \int_0^T x(s)\, ds - \Psi^T_{\alpha-\frac{1}{n},\beta+\frac{1}{n}} \Bigg(\int_0^T x(s)\, ds + b \\
& + w\Big(\Psi^0_{\alpha\beta}(b), \Psi^T_{\alpha\beta}\Big(\int_0^T x(s)\, ds + b\Big), x(T)\Big) + \psi\Big(\Delta_{\alpha\beta}\Big(\int_0^t x(s)\, ds + b\Big)\Big)\Bigg)\Bigg].
\end{aligned} \tag{34}$$

Then

$$\mathrm{D}(I - \mathcal{K}, \Omega_0, 0) \neq 0 , \tag{35}$$

where "D" stands for the Leray-Schauder degree and I is the identity operator on $C^0(J) \times \mathbb{R}^2$.

Proof. Ω_0 is an open, bounded and symmetric with respect to 0 subset of the Banach space $C^0(J) \times \mathbb{R}^2$. Let

$$\mathcal{Z} : [0,1] \times \overline{\Omega}_0 \to C^0(J) \times \mathbb{R}^2 ,$$

$$\begin{aligned}
\mathcal{Z}(\lambda, x, a, b) = \Bigg[& a + \Big(1+\frac{1}{n}\Big) \int_0^t \Big(p^n_{\alpha\beta} \Big(\int_0^\nu x(\tau)\, d\tau + b \Big)\Big)(s)\, ds \\
& - (1-\lambda)\Big(1+\frac{1}{n}\Big) \int_0^t \Big(p^n_{\alpha\beta} \Big(-\int_0^\nu x(\tau)\, d\tau - b \Big)\Big)(s)\, ds, \\
& a + b - \lambda\Psi^0_{\alpha-\frac{1}{n},\beta+\frac{1}{n}} \Bigg(b + r\Big(\Psi^0_{\alpha\beta}(b), x(0), \Psi^T_{\alpha\beta}\Big(\int_0^T x(s)\, ds + b\Big)\Big) \\
& + \varphi\Big(\Delta_{\alpha\beta}\Big(\int_0^t x(s)\, ds + b\Big)\Big)\Bigg),\ 2b + \int_0^T x(s)\, ds - \lambda\Psi^T_{\alpha-\frac{1}{n},\beta+\frac{1}{n}} \Bigg(\int_0^T x(s)\, ds + b \\
& + w\Big(\Psi^0_{\alpha\beta}(b), \Psi^T_{\alpha\beta}\Big(\int_0^T x(s)\, ds + b\Big), x(T)\Big) + \psi\Big(\Delta_{\alpha\beta}\Big(\int_0^t x(s)\, ds + b\Big)\Big)\Bigg)\Bigg].
\end{aligned}$$

Then $\mathcal{Z}(0,-x,-a,-b) = -\mathcal{Z}(0,x,a,b)$ for $(x,a,b) \in \overline{\Omega}_0$, and so $\mathcal{Z}(0,\cdot)$ is an odd operator.

$\mathcal{Z}$ is easily checked to be continuous. Let $\{(\lambda_j, x_j, a_j, b_j)\} \subset [0,1] \times \overline{\Omega}_0$. Then

$$\Big| a_j + \Big(1+\frac{1}{n}\Big) \int_0^t \Big(p^n_{\alpha\beta}\Big(\int_0^\nu x_j(\tau)\,d\tau + b_j\Big)\Big)(s)\,ds$$

$$- \ (1-\lambda_j)\Big(1+\frac{1}{n}\Big) \int_0^t \Big(p^n_{\alpha\beta}\Big(-\int_0^\nu x_j(\tau)\,d\tau - b_j\Big)\Big)(s)\,ds\Big| \le E + 8\|\chi\|_{L_1} + D\,,$$

$$\Big|\int_{t_1}^{t_2}\Big(p^n_{\alpha\beta}\Big(\int_0^\nu x_j(\tau)\,d\tau + b_j\Big)\Big)(s)\,ds - (1-\lambda_j)\int_{t_1}^{t_2}\Big(p^n_{\alpha\beta}\Big(-\int_0^\nu x_j(\tau)\,d\tau - b_j\Big)\Big)(s)\,ds\Big|$$

$$\le 2\Big|\int_{t_1}^{t_2}\chi(s)\,ds\Big|$$

for $t, t_1, t_2 \in J$, $j \in \mathbb{N}$ and

$$\Bigg| a_j + b_j - \lambda_j \Psi^0_{\alpha-\frac{1}{n},\beta+\frac{1}{n}}\Bigg(b_j + r\Big(\Psi^0_{\alpha\beta}(b_j), x_j(0), \Psi^T_{\alpha\beta}\Big(\int_0^T x_j(s)\,ds + b_j\Big)\Big)$$

$$+\varphi\Big(\Delta_{\alpha\beta}\Big(\int_0^t x_j(s)\,ds + b_j\Big)\Big)\Bigg)\Bigg| \le E + 4\|\chi\|_{L_1} + 2\Lambda + D + 3\,,$$

$$\Bigg| 2b_j + \int_0^T x_j(s)\,ds - \lambda_j \Psi^T_{\alpha-\frac{1}{n},\beta+\frac{1}{n}}\Bigg(\int_0^T x_j(s)\,ds + b_j + w\Big(\Psi^0_{\alpha\beta}(b_j)\,,$$

$$\Psi^T_{\alpha\beta}\Big(\int_0^T x_j(s)\,ds + b_j\Big), x_j(T)\Big) + \psi\Big(\Delta_{\alpha\beta}\Big(\int_0^t x_j(s)\,ds + b_j\Big)\Big)\Bigg)\Bigg|$$

$$\le 3\Lambda + T(E + 8\|\chi\|_{L_1} + C) + 5$$

for $j \in \mathbb{N}$. By the Arzelà-Ascoli theorem and the Bolzano-Weierstrass theorem $\{\mathcal{Z}(\lambda_j, x_j, a_j, b_j)\}$ is relatively compact. Hence $\mathcal{Z}$ is a compact operator.

Assume that $\mathcal{Z}(\lambda_0, x_0, a_0, b_0) = (x_0, a_0, b_0)$ for some $(\lambda_0, x_0, a_0, b_0) \in [0,1] \times \partial\Omega_0$. Then

$$\begin{aligned} x_0(t) = a_0 &+ \Big(1+\frac{1}{n}\Big)\int_0^t \Big(p^n_{\alpha\beta}\Big(\int_0^\nu x_0(\tau)\,d\tau + b_0\Big)\Big)(s)\,ds \\ &- (1-\lambda_0)\Big(1+\frac{1}{n}\Big)\int_0^t\Big(p^n_{\alpha\beta}\Big(-\int_0^\nu x_0(\tau)\,d\tau - b_0\Big)\Big)(s)\,ds \end{aligned} \tag{36}$$

for $t \in J$,

$$\begin{aligned} b_0 = \lambda_0 \Psi^0_{\alpha-\frac{1}{n},\beta+\frac{1}{n}}\Bigg(& b_0 + r\Big(\Psi^0_{\alpha\beta}(b_0), x_0(0), \Psi^T_{\alpha\beta}\Big(\int_0^T x_0(s)\,ds + b_0\Big)\Big) \\ &+ \varphi\Big(\Delta_{\alpha\beta}\Big(\int_0^t x_0(s)\,ds + b_0\Big)\Big)\Bigg), \end{aligned} \tag{37}$$

$$
\begin{aligned}
b_0 + \int_0^T x_0(s)\,ds = \lambda_0 \Psi^T_{\alpha-\frac{1}{n},\beta+\frac{1}{n}} \Bigg(& \int_0^T x_0(s)\,ds + b_0 \\
& + w\Big(\Psi^0_{\alpha\beta}(b_0), \Psi^T_{\alpha\beta}\Big(\int_0^T x_0(s)\,ds + b_0\Big), x_0(T)\Big) \\
& + \psi\Big(\Delta_{\alpha\beta}\Big(\int_0^t x_0(s)\,ds + b_0\Big)\Big)\Bigg) .
\end{aligned} \tag{38}
$$

From the definition of $\Psi^c_{\alpha-\frac{1}{n},\beta+\frac{1}{n}}$, (37) and (38) it follows that

$$
\begin{aligned}
& \alpha(0) - \frac{1}{n} \le b_0 \le \beta(0) + \frac{1}{n}\,, \\
& \alpha(T) - \frac{1}{n} \le b_0 + \int_0^T x_0(s)\,ds \le \beta(T) + \frac{1}{n}\,,
\end{aligned} \tag{39}
$$

and consequently

$$
\alpha(T)-\beta(0)-\frac{2}{n} \le \alpha(T)-b_0-\frac{1}{n} \le \int_0^T x_0(s)\,ds \le \beta(T)-b_0+\frac{1}{n} \le \beta(T)-\alpha(0)+\frac{2}{n}\,.
$$

Since $\int_0^T x_0(s)\,ds = x_0(\varepsilon)T$ for some $\varepsilon \in J$, we have

$$
\frac{\alpha(T)-\beta(0)-2}{T} \le x_0(\varepsilon) \le \frac{\beta(T)-\alpha(0)+2}{T}
$$

and $|x_0(\varepsilon)| \le E$. Applying the last inequality and the inequality $|(p^n_{\alpha\beta} z)(t)| \le \chi(t)$ which is satisfied for each $z \in C^0(J)$ and a.e. $t \in J$ to (36), we have

$$
|a_0| \le |x_0(\varepsilon)| + 4\|\chi\|_{L_1} \le E + 4\|\chi\|_{L_1}\,, \tag{40}
$$

and consequently

$$
|x_0(t)| \le |a_0| + 4\|\chi\|_{L_1} \le E + 8\|\chi\|_{L_1} \quad \text{for } t \in J\,. \tag{41}
$$

From (39) – (41) we deduce that $(x_0, a_0, b_0) \notin \partial\Omega_0$, a contradiction.

Hence $\mathrm{D}(I - \mathcal{Z}(0,\cdot), \Omega_0, 0) \ne 0$ by the antipodal Borsuk theorem and

$$
\mathrm{D}(I - \mathcal{Z}(1,\cdot), \Omega_0, 0) = \mathrm{D}(I - \mathcal{Z}(0,\cdot), \Omega_0, 0)\,,
$$

by the homotopy. (35) now follows from the equality $\mathcal{Z}(1,\cdot) = \mathcal{K}$. ☐

Lemma 4. *Let assumptions $(H_1) - (H_5)$ be satisfied. Then for each $n \in \mathbb{N}$, BVP $(6)^n_1 - (8)^n$ has a solution $x(t)$ satisfying inequalities* (9) *and* (19).

Proof. Fix $n \in \mathbb{N}$. Let P be a positive constant satisfying inequality (18) and the constant E be given by (33). Set

$$
U = \max\{|g(t,x,y)| : (t,x,y) \in J \times [-\Lambda,\Lambda] \times [-P,P]\},
$$

$$
\begin{aligned}
\Omega \;\; = \;\; \Big\{ & (x,a,b) : (x,a,b) \in C^0(J) \times \mathbb{R}^2, \|x\| < E + 8\|\chi\|_{L_1} + P\,, \\
& |a| < E + 4\|\chi\|_{L_1} + P + U\,, |b| < \Lambda + 2 \Big\}.
\end{aligned}
$$

Let the operators

$$\mathcal{W},\, \mathcal{S} : [0,1] \times \overline{\Omega} \to C^0(J) \times \mathbb{R}^2$$

be defined by the formulas

$$\begin{aligned}
&\mathcal{W}(\lambda, x, a, b) = \\
&= \Bigg[a + \lambda \int_0^t f\Bigg(s, \Big(\Delta_{\alpha\beta}\Big(\int_0^\nu x(\tau)\, d\tau + b\Big)\Big)(s), \Big(\Delta_{\alpha\beta}\Big(\int_0^\nu x(\tau)\, d\tau + b\Big)\Big)'(s)\Bigg) ds \\
&\qquad + \Big(1 - \lambda + \frac{1}{n}\Big) \int_0^t \Big(p^n_{\alpha\beta}\Big(\int_0^\nu x(\tau)\, d\tau + b\Big)\Big)(s)\, ds, \\
&\qquad a + b - \Psi^0_{\alpha-\frac{1}{n},\beta+\frac{1}{n}}\Bigg(b + r\Big(\Psi^0_{\alpha\beta}(b), x(0), \Psi^T_{\alpha\beta}\Big(\int_0^T x(s)\, ds + b\Big)\Big) \\
&\qquad + \varphi\Big(\Delta_{\alpha\beta}\Big(\int_0^t x(s)\, ds + b\Big)\Big)\Bigg),\ 2b + \int_0^T x(s)\, ds \\
&\qquad - \Psi^T_{\alpha-\frac{1}{n},\beta+\frac{1}{n}}\Bigg(\int_0^T x(s)\, ds + b + w\Big(\Psi^0_{\alpha\beta}(b), \Psi^T_{\alpha\beta}\Big(\int_0^T x(s)\, ds + b\Big), x(T)\Big) \\
&\qquad + \psi\Big(\Delta_{\alpha\beta}\Big(\int_0^t x(s)\, ds + b\Big)\Big)\Bigg)\Bigg]
\end{aligned}$$

and

$$\mathcal{S}(\lambda, x, a, b) = \lambda\Bigg(-g\Big(t, \Big(\Delta_{\alpha\beta}\Big(\int_0^\nu x(\tau)\, d\tau + b\Big)\Big)(t), x(t)\Big), 0, 0\Bigg).$$

Then $\mathcal{W}(0,\cdot)+\mathcal{S}(0,\cdot) = \mathcal{K}$, where $\mathcal{K}$ is defined by (34) (with $C = P$ and $D = P+U$ in Ω_0), and so

$$\mathrm{D}(I - \mathcal{W}(0,\cdot) - \mathcal{S}(0,\cdot), \Omega, 0) \neq 0$$

by (35). If we verify that

(j) $\mathcal{W}$ is a compact operator,

(jj) there exists $\mu \in [0,1)$ such that

$$\|\mathcal{S}(\lambda, x_1, a_1, b_1) - \mathcal{S}(\lambda, x_2, a_2, b_2)\|_0 \le \mu \|(x_1, a_1, b_1) - (x_2, a_2, b_2)\|_0$$

for $(\lambda, x_i, a_i, b_i) \in [0,1] \times \overline{\Omega}$ $(i = 1, 2)$, and

(jjj) $\mathcal{W}(\lambda, x, a, b) + \mathcal{S}(\lambda, x, a, b) \neq (x, a, b)$ for $(\lambda, x, a, b) \in [0,1] \times \partial\Omega$

then, by the homotopy theory for α-condensing operators,

$$\mathrm{D}(I - \mathcal{W}(1,\cdot) - \mathcal{S}(1,\cdot), \Omega, 0) \neq 0\,. \tag{42}$$

It is easy to check that $\mathcal{W}$ is a continuous operator. To prove that $\mathcal{W}([0,1] \times \overline{\Omega})$ is a relatively compact subset of the Banach space $C^0(J) \times \mathbb{R}^2$, let $\{(\lambda_j, x_j, a_j, b_j)\} \subset [0,1] \times \overline{\Omega}$. Set

$$Q = \max\{E + 8\|\chi\|_{L_1} + P,\ \|\alpha'\|,\ \|\beta'\|\}$$

and

$$\varrho(t) = \sup\{|f(t, x, y)| :\ (x, y) \in [-\Lambda, \Lambda] \times [-Q, Q]\} \quad \text{for a.e.} \quad t \in J\,.$$

Then $\varrho \in L_1(J)$ and

$$\Big|a_j + \lambda_j \int_0^t f\bigg(s, \Big(\Delta_{\alpha\beta}\Big(\int_0^\nu x_j(\tau)\,d\tau + b_j\Big)\Big)(s), \Big(\Delta_{\alpha\beta}\Big(\int_0^\nu x_j(\tau)\,d\tau + b_j\Big)\Big)'(s)\bigg)\,ds$$

$$+\Big(1-\lambda_j+\frac{1}{n}\Big)\int_0^t \Big(p^n_{\alpha\beta}\Big(\int_0^\nu x_j(\tau)\,d\tau + b_j\Big)\Big)(s)\,ds\Big| \le E + 6\|\chi\|_{L_1} + P + U + \|\varrho\|_{L_1},$$

$$\Big|\lambda_j \int_{t_1}^{t_2} f\bigg(s, \Big(\Delta_{\alpha\beta}\Big(\int_0^\nu x_j(\tau)\,d\tau + b_j\Big)\Big)(s), \Big(\Delta_{\alpha\beta}\Big(\int_0^\nu x_j(\tau)\,d\tau + b_j\Big)\Big)'(s)\bigg)\,ds$$

$$+\Big(1-\lambda_j+\frac{1}{n}\Big)\int_{t_1}^{t_2} \Big(p^n_{\alpha\beta}\Big(\int_0^\nu x_j(\tau)\,d\tau + b_j\Big)\Big)(s)\,ds\Big| \le \Big|\int_{t_1}^{t_2}\varrho(s)\,ds\Big| + 2\Big|\int_{t_1}^{t_2}\chi(s)\,ds\Big|,$$

$$\bigg|a_j + b_j - \Psi^0_{\alpha-\frac{1}{n},\beta+\frac{1}{n}}\bigg(b_j + r\Big(\Psi^0_{\alpha\beta}(b_j), x_j(0), \Psi^T_{\alpha\beta}\Big(\int_0^T x_j(s)\,ds + b_j\Big)\Big)$$

$$+\varphi\Big(\Delta_{\alpha\beta}\Big(\int_0^t x_j(s)\,ds + b_j\Big)\Big)\bigg)\bigg| \le E + 4\|\chi\|_{L_1} + 2\Lambda + P + U + 3,$$

$$\bigg|2b_j + \int_0^T x_j(s)\,ds - \Psi^T_{\alpha-\frac{1}{n},\beta+\frac{1}{n}}\bigg(\int_0^T x_j(s)\,ds + b_j + w\Big(\Psi^0_{\alpha\beta}(b_j),$$

$$\Psi^T_{\alpha\beta}\Big(\int_0^T x_j(s)\,ds + b_j\Big), x_j(T)\Big) + \psi\Big(\Delta_{\alpha\beta}\Big(\int_0^t x_j(s)\,ds + b_j\Big)\Big)\bigg)\bigg|$$

$$\le 3\Lambda + T(E + 8\|\chi\|_{L_1} + P) + 5$$

for $t, t_1, t_2 \in J$ and $n \in \mathbb{N}$. By the Arzelà-Ascoli theorem and the Bolzano-Weierstrass theorem, $\{\mathcal{W}(\lambda_j, x_j, a_j, b_j)\}$ is relatively compact, and consequently $\mathcal{W}$ is a compact operator.

Since (cf. (H_3))

$$\begin{aligned}
&\|\mathcal{S}(\lambda, x_1, a_1, b_1) - \mathcal{S}(\lambda, x_2, a_2, b_2)\|_0 \\
&\quad = |\lambda|\bigg\|\bigg(g\Big(t, \Big(\Delta_{\alpha\beta}\Big(\int_0^\nu x_1(s)\,ds + b_1\Big)\Big)(t), x_1(t)\Big) \\
&\qquad - g\Big(t, \Big(\Delta_{\alpha\beta}\Big(\int_0^\nu x_2(s)\,ds + b_2\Big)\Big)(t), x_2(t)\Big), 0, 0\bigg)\bigg\|_0 \\
&\quad \le \Big\|g\Big(t, \Big(\Delta_{\alpha\beta}\Big(\int_0^\nu x_1(s)\,ds + b_1\Big)\Big)(t), x_1(t)\Big) \\
&\qquad - g\Big(t, \Big(\Delta_{\alpha\beta}\Big(\int_0^\nu x_2(s)\,ds + b_2\Big)\Big)(t), x_2(t)\Big)\Big\| \\
&\quad \le m\Big\|\Big(\Delta_{\alpha\beta}\Big(\int_0^\nu x_1(s)\,ds + b_1\Big)\Big)(t) - \Big(\Delta_{\alpha\beta}\Big(\int_0^\nu x_2(s)\,ds + b_2\Big)\Big)(t)\Big\|
\end{aligned}$$

$$+ k\|x_1 - x_2\| \le (mT + k)\|x_1 - x_2\| + m|b_1 - b_2|$$

$$\le (mT + k)\|(x_1, a_1, b_1) - (x_2, a_2, b_2)\|_0$$

for $(\lambda, x_i, a_i, b_i) \in [0,1] \times \overline{\Omega}$ $(i = 1, 2)$, we see that (jj) is satisfied with $\mu = mT + k < 1$.

Assume that

$$\mathcal{W}(\lambda_0, x_0, a_0, b_0) + \mathcal{S}(\lambda_0, x_0, a_0, b_0) = (x_0, a_0, b_0)$$

for some $(\lambda_0, x_0, a_0, b_0) \in [0,1] \times \partial\Omega$. If $\lambda_0 = 0$ then (see the proof of Lemma 3)

$$\|x_0\| \le E + 8\|\chi\|_{L_1}, \quad |a_0| \le E + 4\|\chi\|_{L_1}, \quad |b_0| \le \Lambda + 1\,.$$

Let $\lambda_0 \in (0,1]$. Then the function

$$u_0(t) = \int_0^t x_0(s)\,ds + b_0 \quad \text{for} \quad t \in J$$

is a solution of BVP $(6)^n_{\lambda_0} - (8)^n$ since $u_0(0) = b_0$, $u_0(T) = \int_0^T x_0(s)\,ds + b_0$, $u_0'(0) = x_0(0)$ and $u_0'(T) = x_0(T)$. By Lemma 2,

$$\alpha(t) - \frac{1}{n} \le u_0(t) \le \beta(t) + \frac{1}{n} \quad \text{for} \quad t \in J$$

and $\|x_0\| < P$. Hence

$$|b_0| = |u_0(0)| \le \Lambda + 1\,,$$

$$|a_0| = |x_0(0) + \lambda_0 g(0, \Psi^0_{\alpha\beta}(b_0), x_0(0))| < P + U\,,$$

and consequently $(x_0, a_0, b_0) \notin \partial\Omega$, a contradiction. We have verified that conditions $(j) - (jjj)$ are satisfied. By (42), there exists a fixed point of the operator $\mathcal{W}(1,\cdot) + \mathcal{S}(1,\cdot)$, say (u, a, b). Set

$$x(t) = \int_0^t u(s)\,ds + b \quad \text{for} \quad t \in J\,.$$

Then $x(0) = b$ and $x(T) = \int_0^T u(s)\,ds + b$, and so $x(t)$ is a solution of BVP $(6)^n_1 - (8)^n$. By Lemma 2, $x(t)$ satisfies inequalities (9) and (19). ☐

3. Main results

Theorem 1. *Let assumptions $(H_1) - (H_5)$ be satisfied. Then there exists a solution $x(t)$ of BVP (1) − (3) satisfying the inequalities*

$$\alpha(t) \le x(t) \le \beta(t) \quad \textit{for} \quad t \in J\,. \tag{43}$$

Proof. By Lemma 4, for each $n \in \mathbb{N}$ there exists a solution $x_n(t)$ of BVP $(6)^n_1 - (8)^n$ such that

$$\alpha(t) - \frac{1}{n} \le x_n(t) \le \beta(t) + \frac{1}{n} \quad \text{for} \quad t \in J\,, \tag{44}$$

$$\|x_n'\| < P\,, \tag{45}$$

where P satisfies (18). Consider the sequence $\{x_n(t)\}$. By (44) and (45), $\{x_n\}$ is bounded in $C^1(J)$ and Lemma 2 implies that $\{x_n'(t)\}$ is equicontinuous on J.

Going if necessary to a subsequence, we can assume that $\{x_n\}$ is converging in $C^1(J)$, say $\lim_{n\to\infty} x_n = x$. Then (43) is satisfied and $\|x'\| \le P$. Taking the limit in the equalities

$$x_n'(t) = x_n'(0) - g\Big(t, (\Delta_{\alpha\beta} x_n)(t), x_n'(t)\Big) + g\Big(0, (\Delta_{\alpha\beta} x_n)(0), x_n'(0)\Big) + \int_0^t f\Big(s, (\Delta_{\alpha\beta} x_n)(s), (\Delta_{\alpha\beta} x_n)'(s)\Big)\, ds + \frac{1}{n}\int_0^t (p^n_{\alpha\beta} x_n)(s)\, ds\,,$$

$$x_n(0) = \Psi^0_{\alpha-\frac{1}{n},\beta+\frac{1}{n}}\left(x_n(0) + r\Big(\Psi^0_{\alpha\beta}(x_n(0)), x_n'(0), \Psi^T_{\alpha\beta}(x_n(T))\Big) + \varphi(\Delta_{\alpha\beta} x_n)\right),$$

$$x_n(T) = \Psi^T_{\alpha-\frac{1}{n},\beta+\frac{1}{n}}\left(x_n(T) + w\Big(\Psi^0_{\alpha\beta}(x_n(0)), \Psi^T_{\alpha\beta}(x_n(T)), x'(T)\Big) + \psi(\Delta_{\alpha\beta} x_n)\right)$$

as $n \to \infty$ we have

$$x'(t) = x'(0) - g(t, x(t), x'(t)) + g(0, x(0), x'(0)) + \int_0^t f(s, x(s), x'(s))\, ds \tag{46}$$

for $t \in J$ and

$$x(0) = \Psi^0_{\alpha\beta}\Big(x(0) + r(x(0), x'(0), x(T)) + \varphi(x)\Big)\,, \tag{47}$$

$$x(T) = \Psi^T_{\alpha\beta}\Big(x(T) + w(x(0), x(T), x'(T)) + \psi(x)\Big)\,. \tag{48}$$

We see that (cf. (46)) that $x(t)$ is a solution of (1) on J. It remains to prove that (47) and (48) imply satisfying the boundary conditions (2) and (3).

Assume that $x(0) + r(x(0), x'(0), x(T)) + \varphi(x) < \alpha(0)$. By (47), $x(0) = \alpha(0)$ and so

$$r(x(0), x'(0), x(T)) + \varphi(x) < 0\,. \tag{49}$$

From (43) we conclude that $x'(0) \ge \alpha'(0)$, $x(T) \ge \alpha(T)$ and $\varphi(x) \ge \varphi(\alpha)$. Thus (cf. (H_4))

$$0 \le r(\alpha(0), \alpha'(0), \alpha(T)) + \varphi(\alpha) \le r(x(0), x'(0), x(T)) + \varphi(x)\,,$$

contrary to (49). If $x(0) + r(x(0), x'(0), x(T)) + \varphi(x) > \beta(0)$ then (cf. (47)) $x(0) = \beta(0)$ which yields

$$r(x(0), x'(0), x(T)) + \varphi(x) > 0\,. \tag{50}$$

On the other hand $x'(0) \le \beta'(0)$, $x(T) \le \beta(T)$ and $\varphi(x) \le \varphi(\beta)$ by (43), and consequently (cf. (H_4))

$$0 \ge r(\beta(0), \beta'(0), \beta(T)) + \varphi(\beta) \ge r(x(0), x'(0), x(T)) + \varphi(x)\,,$$

contrary to (50).

Let $x(T) + w(x(0), x(T), x'(T)) + \psi(x) < \alpha(T)$. We conclude from (48) that $x(T) = \alpha(T)$; hence

$$w(x(0), x(T), x'(T)) + \psi(x) < 0\,. \tag{51}$$

In addition (see (43)), $x'(T) \le \alpha'(T)$, $x(0) \ge \alpha(0)$ and $\psi(x) \ge \psi(\alpha)$, and so

$$0 \le w(\alpha(0), \alpha(T), \alpha'(T)) + \psi(\alpha) \le w(x(0), x(T), x'(T)) + \psi(x)\,,$$

contrary to (51). If $x(T) + w(x(0), x(T), x'(T)) + \psi(x) > \beta(T)$ then $x(T) = \beta(T)$ which gives

$$w(x(0), x(T), x'(T)) + \psi(x) > 0\,. \tag{52}$$

From the inequalities $x'(T) \ge \beta'(T)$, $x(0) \le \beta(0)$ and $\psi(x) \le \psi(\beta)$ we deduce that

$$0 \ge w(\beta(0), \beta(T), \beta'(T)) + \psi(\beta) \ge w(x(0), x(T), x'(T)) + \psi(x)\,,$$

contrary to (52). We have proved that

$$\alpha(0) \le x(0) + r(x(0), x'(0), x(T)) + \varphi(x) \le \beta(0)$$

and

$$\alpha(T) \le x(T) + w(x(0), x(T), x'(T)) + \psi(x) \le \beta(T)\,.$$

(47) and (48) now show that $x(t)$ satisfies (2) and (3). □

Corollary 1. *Let assumptions $(H_1) - (H_4)$ be satisfied. Suppose that there exists a nonnegative function $h \in L_1(J)$ and a Nagumo-type function ω such that at least one of the following inequalities*

$$f(t, x, y) \le (h(t) + |y|)\omega(y)\,, \tag{53}$$

$$f(t, x, y) \ge -(h(t) + |y|)\omega(y)\,, \tag{54}$$

$$f(t, x, y)\operatorname{sign} y \le (h(t) + |y|)\omega(y)\,, \tag{55}$$

and

$$f(t, x, y)\operatorname{sign} y \ge -(h(t) + |y|)\omega(y)\,, \tag{56}$$

is satisfied for a.e. $t \in J$ and each $x \in [\alpha(t), \beta(t)]$, $y \in (-\infty, -S] \cup [S, \infty)$. Then BVP (1) − (3) has a solution.

Proof. We see that assumption (H_5) is satisfied for (53) with $\sigma_1 = \sigma_2 = 1$, for (54) with $\sigma_1 = \sigma_2 = -1$, for (55) with $\sigma_1 = -\sigma_2 = 1$ and for (56) with $\sigma_1 = -\sigma_2 = -1$. Hence Corollary 1 follows from Theorem 1. □

Example 1. Consider BVP

$$(x' + p(x) + q(x'))' = f_1(t, x, x')x^{2l-1} + f_2(t, x, x') + \sigma f_3(t, x, x')(x')^n\,, \tag{57}$$

$$x'(0) + r_1(x(0), x(T)) + \max\{x(t) : t \in J\} = 0, \tag{58}$$

$$-x'(T) + w_1(x(0), x(T)) + \int_0^T x(t)\,dt = 0, \tag{59}$$

where $p, q \in C^0(\mathbb{R})$, $f_i : J \times \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2, 3)$ satisfy the local Carathéodory conditions, l, n are positive integers, $\sigma \in \{-1, 1\}$, $r_1, w_1 \in C^0(\mathbb{R}^2)$, r_1 is nondecreasing in the second variable and w_1 is nondecreasing in the first variable.

Suppose that there exist positive constants a, b and nonnegative constants m, k such that $mT + k < 1$ and

$$a \le f_1(t,x,y) \le b(1+y^2)\,, \quad |f_2(t,x,y)| \le b(1+y^2)\,, \quad f_3(t,x,y) \ge 0\,,$$
$$|p(x_1) - p(x_2)| \le m|x_1 - x_2|, \quad |q(y_1) - q(y_2)| \le k|y_1 - y_2|$$

for $(t,x,y) \in J \times \mathbb{R}^2$, $x_1, x_2 \in \left[-\sqrt[2l-1]{\frac{b}{a}},\ \sqrt[2l-1]{\frac{b}{a}}\right]$ and $y_1, y_2 \in \mathbb{R}$. Suppose also that

$$\varepsilon r_1\left(\varepsilon \sqrt[2l-1]{\frac{b}{a}}, \varepsilon \sqrt[2l-1]{\frac{b}{a}}\right) + \sqrt[2l-1]{\frac{b}{a}} \le 0\,,$$
$$\varepsilon w_1\left(\varepsilon \sqrt[2l-1]{\frac{b}{a}}, \varepsilon \sqrt[2l-1]{\frac{b}{a}}\right) + T \sqrt[2l-1]{\frac{b}{a}} \le 0\,,$$

for $\varepsilon = \pm 1$.

We will show that Theorem 1 implies the existence of a solution of BVP (57) − (59). We first observe that the functionals $\varphi(x) = \max\{x(t) : t \in J\}$ and $\psi(x) = \int_0^T x(t)\,dt$ belong to the set $\mathcal{C}$. We next see that assumptions (H_3) is satisfied and the constant functions

$$\alpha \equiv -\sqrt[2l-1]{\frac{b}{a}} \quad \text{and} \quad \beta \equiv \sqrt[2l-1]{\frac{b}{a}}$$

are respectively upper and lower functions of BVP (57) − (59). Setting $r(u,v,x) + \varphi(z) = v + r_1(u,x) + \max\{z(t) : t \in J\}$, $w(u,v,x) + \psi(z) = -x + w_1(u,v) + \int_0^T z(t)\,dt$,

$$S_1 = \max\{|r_1(u,v)| :\ u, v \in [\alpha,\beta]\} + \sqrt[2l-1]{\frac{b}{a}}$$
$$S_2 = \max\{|w_1(u,v)| :\ u, v \in [\alpha,\beta]\} + T \sqrt[2l-1]{\frac{b}{a}}$$

then assumption (H_4) is satisfied with $S > \max\{S_1, S_2\}$. Assumption (H_5) is satisfied with $h(t) = 1$, $\omega(u) = b(\frac{b}{a} + 1)(1 + |u|)$ and

$$\sigma_j = \begin{cases} -\sigma & \text{if } n \text{ is even} \\ (-1)^j \sigma & \text{if } n \text{ is odd.} \end{cases}$$

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