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RANDOM FIXED POINTS OF MULTIVALUED MAPS IN FRECHET SPACES

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ABSTRACT. In this paper we prove a general random fixed point theorem for multivalued maps in Frechet spaces. We apply our main result to obtain some common random fixed point theorems. Our main result unifies and extends the work due to Benavides, Acedo and Xu [4], Itoh [8], Lin [12], Liu [13], Tan and Yuan [20], Xu [23], etc.

1. Introduction

Random fixed point theorems are stochastic generalizations of (classical or deterministic) fixed point theorems. During the last three decades several results regarding random fixed points of various types of random maps have been given, and a number of their applications have been obtained. A great deal of the existing work in Banach space random fixed point theory has been motivated by the survey article of Bharucha-Reid [5]. In 1979, Itoh [8] derived several random fixed point theorems for nonexpansive multivalued mappings in Banach spaces. Recently, Lin [12] obtained a random fixed point theorem for 1-set-contractive single-valued self-mappings in a separable Banach space. Related (but different) problems were also studied by Beg and Shahzad [1-3], Shahzad [17], Shahzad and Latif [19], Tan and Yuan [20, 21] and Xu [23]. More recently, Benavides, Acedo and Xu [4] established some random fixed point theorems for multivalued maps in Banach spaces. The extension of random fixed point theorems to Frechet spaces [18] leads one to wonder what further analogous results can be obtained in this more general setting. The aim of this paper is to provide an answer to that question. We prove a random fixed point theorem which unifies and extends many well known random fixed point results due to Benavides, Acedo and Xu [4], Itoh [8], Lin [12], Liu [13], Tan and Yuan [20], Xu [23], etc. We apply our main result to derive some common random fixed point theorems.

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2. Preliminaries

Throughout this paper, (Ω, Σ) denotes a measurable space. Let S be a subset of a Frechet space X. Let 2^S be the family of all subsets of S, $CB(S)$ all nonempty closed bounded subsets of S and $K(S)$ all nonempty compact subsets of S, respectively. A mapping $F : \Omega \longrightarrow 2^S \setminus \{\phi\}$ is called measurable if, for any open subset B of S, $F^{-1}(B) = \{\omega \in \Omega : F(\omega) \cap B \neq \phi\} \in \Sigma$. A mapping $\xi : \Omega \longrightarrow S$ is called a measurable selector of a measurable mapping $F : \Omega \longrightarrow 2^S \setminus \{\phi\}$ if ξ is measurable and $\xi(\omega) \in F(\omega)$ for each $\omega \in \Omega$. A mapping $f : \Omega \times S \longrightarrow X$ (resp. $F : \Omega \times S \longrightarrow CB(X)$) is called a random operator if, for each $x \in S$, $f(., x)$ (resp. $F(., x)$) is measurable. A mapping $\xi : \Omega \longrightarrow S$ is called a random fixed point of a random operator $f : \Omega \times S \longrightarrow X$ (resp. $F : \Omega \times S \longrightarrow CB(X)$) if ξ is measurable and, for each $\omega \in \Omega$, $f(\omega, \xi(\omega)) = \xi(\omega)$ (resp. $\xi(\omega) \in F(\omega, \xi(\omega))$). The mapping ξ is said to be a deterministic fixed point of f (resp. F) if, for each $\omega \in \Omega$, $f(\omega, \xi(\omega)) = \xi(\omega)$ (resp. $\xi(\omega) \in F(\omega, \xi(\omega))$).

It is well known that a locally convex topological vector space (always assumed Hausdorff) X is metrizable if and only if X has a countable base of absolutely convex neighbourhoods of zero or, equivalently X has a countable family of seminorms $\{p_n\}$ that generates the locally convex topology on X. We can always assume that $p_n \leq p_{n+1}$, $n \geq 1$. In this case the topology may be defined by a translation invariant metric d, that is, $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$. In particular, $d(x, y) = d(x - y, 0)$. A function $d : X \times X \longrightarrow [0, \infty)$ given by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{c_n p_n(x - y)}{1 + p_n(x - y)}$$

for $x \in X$, where $c_n > 0$ and $\sum_{n=1}^{\infty} c_n < \infty$, defines a translation invariant metric on X.

A mapping $f : S \longrightarrow S$ is called asymptotically regular if, for any $x \in S$, $d(f^n(x), f^{n+1}(x)) \longrightarrow 0$ as $n \longrightarrow \infty$. The mapping f is called nonexpansive if, for every $x, y \in S$, $d(f(x), f(y)) \leq d(x, y)$. The mapping f is said to commute with a mapping $F : S \longrightarrow CB(S)$ if, for each $x \in S$, $f(F(x)) \subset F(f(x))$. A random operator $f : \Omega \times S \longrightarrow X$ (resp. $F : \Omega \times S \longrightarrow CB(X)$) is called continuous (nonexpansive, etc.) if for each $\omega \in \Omega$, $f(\omega, \cdot)$ (resp. $F(\omega, \cdot)$) is continuous (nonexpansive, etc.).

A Banach space X is said to satisfy Opial's condition (cf. Opial [14] and Lami Dozo [11]) if the following holds; if $\{x_n\}$ converges weakly to x_0 and $x \neq x_0$, then

$$\liminf_{n \to \infty} \|x_n - x\| > \liminf_{n \to \infty} \|x_n - x_0\| .$$

A mapping $F : S \longrightarrow CB(X)$ is called demiclosed if $\{x_n\} \subset S$ and $y_n \in F(x_n)$ are sequences such that $\{x_n\}$ converges weakly to x_0 and $\{y_n\}$ converges to y_0 in X, then $y_0 \in F(x_0)$.

3. Main Results

Theorem 3.1. *Let S be a nonempty weakly compact convex subset of a separable Frechet space X, and let $F : \Omega \times S \longrightarrow CB(X)$ be a continuous random operator* *such that $I - F(\omega, \cdot)$ is demiclosed at zero for each $\omega \in \Omega$, where I is the identity map on S. If F has a deterministic fixed point, then F has a random fixed point.*

Proof. Since F has a deterministic fixed point, the set $G(\omega) = \{x \in S : x \in F(\omega, x)\}$ is nonempty for each $\omega \in \Omega$. Motivated by Itoh [9], for each integer $n \geq 1$, define mappings $h_n : \Omega \times S \longrightarrow I\!R$ and $G_n : \Omega \longrightarrow 2^S$ by $h_n(\omega, x) = d(x, F(\omega, x)) - \frac{1}{n}$ and $G_n(\omega) = \{x \in S : h_n(\omega, x) < 0\}$. Then, by Proposition 3 of Itoh [9], each G_n is measurable. Since $G(\omega) \subset G_n(\omega)$, $G_n(\omega)$ is nonempty for each $\omega \in \Omega$. Thus, for each $n \geq 1$, the mapping $H_n : \Omega \longrightarrow CB(S)$ defined by $H_n(\omega) = cl(G_n(\omega))$, $\omega \in \Omega$, is measurable and so by the Kuratowski and Ryll-Nardzewski selection theorem [10], each H_n has a measurable selector ξ_n. Further, for this ξ_n, we have $d(\xi_n(\omega), F(\omega, \xi_n(\omega))) \leq \frac{1}{n}$. For each $n \geq 1$, define $L_n : \Omega \longrightarrow WK(S)$ by $L_n(\omega) = \text{w-cl}\,\{\xi_i(\omega) : i \geq n\}$, where w-cl denotes the weak closure and $WK(S)$ represents the family of all nonempty weakly compact subsets of S. Let $L : \Omega \longrightarrow WK(S)$ be a mapping defined by $L(\omega) = \bigcap_{n=1}^{\infty} L_n(\omega)$. Since the weak topology on S is a metric topology (see Rudin [16, p.86]), L is w-measurable by Himmelberg [7, Theorem 4.1], that is, L is measurable with respect to the weak topology on S. Again, by the Kuratowski and Ryll-Nardzewski selection theorem [10], there is a w-measurable selector ξ of L. For any $x^* \in X^*$ (the dual space of X), the numerically-valued function $x^*(\xi(\cdot))$ is measurable. Since X is separable, by Thomas [22, Theorem 1], ξ is measurable. We show that ξ is a random fixed point of F. Indeed, for any fixed $\omega \in \Omega$, some subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ converges weakly to $\xi(\omega)$. For each m, there is an element $u_m \in F(\omega, \xi_m(\omega))$ such that $d(\xi_m(\omega) - u_m, 0) = d(\xi_m(\omega), u_m) < \frac{2}{m}$. Since $\xi_m(\omega) - u_m \in (I - F(\omega, \cdot))(\xi_m(\omega))$, $\{\xi_m(\omega))\}$ converges weakly to $\xi(\omega)$ and $\{\xi_m(\omega) - u_m\}$ converges strongly to 0, it follows by the demiclosedness of $I - F(\omega, \cdot)$ at zero that $\xi(\omega) \in F(\omega, \xi(\omega))$. □

When F is single-valued, we obtain the following result.

Theorem 3.2. *Let S be a nonempty weakly compact convex subset of a separable Frechet space X, and let $f : \Omega \times S \longrightarrow X$ be a continuous random operator such that $I - f(\omega, \cdot)$ is demiclosed at zero for each $\omega \in \Omega$. If f has a deterministic fixed point, then f has a random fixed point.*

Remark 3.1.

1. Theorem 3.1 extends Theorem 3.1-Theorem 3.4 of Liu [13] and Theorem 3.1 of Benavides, Acedo and Xu [4]. We further remark that in the Banach space setting Theorem 3.1 remains valid for the case when S is separable instead of X being separable.
2. Theorem 3.2 contains Theorem 2.1-Corollary 2.2 of Lin [12] estabished by him for the case when $f(\omega, S) \subset S$ for each $\omega \in \Omega$. Theorems 1(ii) and 4 of Xu [23] and Theorem 3.3-Corollary 3.5 of Tan and Yuan [20] follow also from our Theorem 3.2.

Corollary 3.1. *Let S be a nonempty weakly compact convex subset of a separable Banach space X satisfying Opial's condition, and let $F : \Omega \times S \longrightarrow K(X)$ be a nonexpansive random operator. If F has a deterministic fixed point, then F has a random fixed point.*

Proof. Since, by Lami Dozo [11], $I - F(\omega, \cdot)$ is demiclosed at zero for each $\omega \in \Omega$, the corollary follows at once from Theorem 3.1. $\square$

Corollary 3.2. *Let S be a nonempty weakly compact convex subset of a separable Banach space X satisfying Opial's condition, and let $F : \Omega \times S \longrightarrow K(S)$ be a nonexpansive random operator. Then F has a random fixed point.*

Proof. Since, by Lami Dozo [11], F has a deterministic fixed point, Corollary 3.1 further implies that F has a random fixed point. $\square$

Remark 3.2. It is worth mentioning that, under the hypotheses of Corollary 3.2, the fixed point set function $G : \Omega \longrightarrow 2^S \setminus \{\phi\}$ defined by $G(\omega) = \{x \in S : x \in F(\omega, x)\}$ is measurable and the existence of a random fixed point for F follows from the well-known selection theorem. For more details, we refer to [24].

Theorem 3.3. *Let S be a nonempty weakly compact convex subset of a separable Banach space X, $f : \Omega \times S \longrightarrow S$ a continuous asymptotically regular random operator, $F : \Omega \times S \longrightarrow CB(S)$ a continuous random operator such that $I - f(\omega, \cdot)$ and $I - F(\omega, \cdot)$ are demiclosed at zero for each $\omega \in \Omega$, and f and F commute. If F has a deterministic fixed point, then there exists a common random fixed point ξ of f and F, that is, for each $\omega \in \Omega$, $f(\omega, \xi(\omega)) = \xi(\omega) \in F(\omega, \xi(\omega))$.*

Proof. Since F has a deterministic fixed point, by Theorem 3.1, F has a random fixed point ξ_1. Further the mapping $\xi_2 : \Omega \longrightarrow S$ defined by $\xi_2(\omega) = f(\omega, \xi_1(\omega))$ is measurable by Himmelberg [7]. It follows from commutativity of f and F that ξ_2 is a random fixed point of F. Furthermore, the sequence $\{\xi_n\}$ of mappings $\xi_n : \Omega \longrightarrow S$ defined by $\xi_{n+1}(\omega) = f(\omega, \xi_n(\omega))$ $(\omega \in \Omega, n = 1, 2, \ldots)$ are random fixed points of F. For each n, define $L_n : \Omega \longrightarrow WK(S)$ by $L_n(\omega) = \text{w-cl}\{\xi_i(\omega) : i \geq n\}$. Let $L : \Omega \longrightarrow WK(S)$ be defined by $L(\omega) = \bigcap_{n=1}^{\infty} L_n(\omega)$. Then, as in the proof of Theorem 3.1, L is w-measurable and has a measurable selector ξ. We show that ξ is a common random fixed point of f and F. Indeed, for any fixed $\omega \in \Omega$, some subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ converges weakly to $\xi(\omega)$. Since $f(\omega, \cdot)$ is asymptotically regular, $\{\xi_{m+1}(\omega)\}$ also converges weakly to $\xi(\omega)$. Since $I - f(\omega, \cdot)$ and $I - F(\omega, \cdot)$ are demiclosed at zero, it follows that $f(\omega, \xi(\omega)) = \xi(\omega) \in F(\omega, \xi(\omega))$. $\square$

Corollary 3.3. *Let S be a nonempty closed bounded convex subset of a separable uniformly convex Banach space X, $f : \Omega \times S \longrightarrow S$ a nonexpansive asymptotically regular random operator, $F : \Omega \times S \longrightarrow CB(S)$ a continuous random operator such that $I - F(\omega, \cdot)$ is demiclosed at zero for each $\omega \in \Omega$, and f and F commute. If F has a deterministic fixed point, then there exists a common random fixed point ξ of f and F.*

Proof. By Theorem 3 of Browder [6], $I - f(\omega, \cdot)$ is demiclosed for each $\omega \in \Omega$. The corollary follows immediately from Theorem 3.3. $\square$

Remark 3.3. Theorem 3.7-Corollary 3.8 of [8] can be viewed as special cases of Theorem 3.3.

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