ARCHIVUM MATHEMATICUM (BRNO)
Tomus 38 (2002), 101 – 118

ON A TWO POINT LINEAR BOUNDARY VALUE PROBLEM FOR SYSTEM OF ODES WITH DEVIATING ARGUMENTS

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Abstract. Two point boundary value problem for the linear system of ordinary differential equations with deviating arguments

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(\tau_{11}(t)) + \mathbf{B}(t)\mathbf{u}(\tau_{12}(t)) + \mathbf{q}_1(t)\,,$$
$$\mathbf{u}'(t) = \mathbf{C}(t)\mathbf{x}(\tau_{21}(t)) + \mathbf{D}(t)\mathbf{u}(\tau_{22}(t)) + \mathbf{q}_2(t)\,,$$
$$\alpha_{11}\mathbf{x}(0) + \alpha_{12}\mathbf{u}(0) = \mathbf{c}_0\,, \quad \alpha_{21}\mathbf{x}(T) + \alpha_{22}\mathbf{u}(T) = \mathbf{c}_T$$

is considered. For this problem the sufficient condition for existence and uniqueness of solution is obtained. The same approach as in [2], [3] is applied.

Notation

$\Re = (-\infty, +\infty)$ is the set of real numbers.

$\Re_+ = \langle 0, +\infty)$ is the set of real nonnegative numbers.

$I = \langle 0, T\rangle$ is the finite segment of $\Re$, $T > 0$.

$\Re^{m\times n}$ is the space of $m\times n$ matrices $\mathbf{X} = (x_{ij})_{i,j=1}^{m,n}$ with elements $x_{ij} \in \Re$ $(i = 1,\dots,m)$, $(j = 1,\dots,n)$ and the norm

$$\|\mathbf{X}\| = \sum_{i=1}^{m}\sum_{j=1}^{n}|x_{ij}| \quad \text{and} \quad |\mathbf{X}| = (|x_{ij}|)_{i,j=1}^{m,n}\,.$$

$\Re_+^{m\times n}$ is the space of $m\times n$ matrices $\mathbf{X} = (x_{ij})_{i,j=1}^{m,n}$ with elements $x_{ij} \in \Re_+$, $(i = 1,\dots,m)$, $(j = 1,\dots,n)$.

If $\mathbf{X}$, $\mathbf{Y}$ are $m\times n$ matrices, then

$$\mathbf{X} \geq \mathbf{Y} \quad \text{if and only if} \quad \mathbf{X} - \mathbf{Y} \in \Re_+^{m\times n}\,.$$

$\mathbf{E}_n$ is the unit $n\times n$ matrix, $\mathbf{\Theta}_n$ is the zero $n\times n$ matrix.

2000 *Mathematics Subject Classification*: 34B10, 34B05, 34K10.

Key words and phrases: existence and uniqueness of solution, two point linear boundary value problem, linear system of ordinary differential equations, deviating argument, delay.

Received September 25, 2000.

If $\mathbf{X}$, $\mathbf{Y}$ are $n \times n$ matrices, then $[\mathbf{X}, \mathbf{Y}]$ is the $n \times 2n$ matrix

$$\begin{pmatrix} x_{11} & . & . & . & x_{1n} & y_{11} & . & . & . & y_{1n} \\ . & . & & & . & . & . & & & . \\ . & & . & & . & . & & . & & . \\ . & & & . & . & . & & & . & . \\ x_{n1} & . & . & . & x_{nn} & y_{n1} & . & . & . & y_{nn} \end{pmatrix}.$$

If $\mathbf{X} = (x_{ij})_{i,j=1}^{n,n} \in \Re^{n\times n}$ then $\mathbf{X}^{-1}$ is the inverse matrix to $\mathbf{X} \in \Re^{n\times n}$, $\det(\mathbf{X})$ is the determinant of the matrix $\mathbf{X} \in \Re^{n\times n}$ and $r(\mathbf{X})$ is the spectral radius of the matrix $\mathbf{X} \in \Re^{n\times n}$.

$C(I; \Re^{m\times n})$ is the space of continous matrix functions $\mathbf{X} : I \to \Re^{m\times n}$ with the norm

$$\|\mathbf{X}\|_C = \sup\{\|\mathbf{X}(t)\| : t \in I\} .$$

$\tilde{C}(I; \Re^{m\times n})$ is the space of absolutely continous matrix functions $\mathbf{X} : I \to \Re^{m\times n}$ with the norm

$$\|\mathbf{X}\|_{\tilde{C}} = \|\mathbf{X}(0)\| + \int_I \|\mathbf{X}'(t)\| \, dt .$$

$\tilde{C}_{\mathrm{loc}}(I; \Re^{m\times n})$ is the set of absolutely continous matrix functions $\mathbf{X} : I \to \Re^{m\times n}$ on arbitrary subinterval of interval I.

$L^\alpha(I; \Re^{m\times n})$, where $1 \le \alpha < +\infty$ is the space of matrix functions $\mathbf{X} : I \to \Re^{m\times n}$ integrable in the $\alpha-$th power with the norm

$$\|\mathbf{X}\|_{L^\alpha} = \left(\int_I \|\mathbf{X}(t)\|^\alpha \, dt\right)^{\frac{1}{\alpha}} ,$$
$$L(I; \Re^{m\times n}) = L^1(I; \Re^{m\times n}) .$$

$L^{+\infty}(I; \Re^{m\times n})$ is the space of mesurable and bounded matrix functions $\mathbf{X} : I \to \Re^{m\times n}$ with the norm

$$\|\mathbf{X}\|_{L^{+\infty}} = \operatorname{ess\ sup}\{\|\mathbf{X}(t)\| : t \in I\} .$$

$\Re^n = \Re^{n\times 1}$ is the space of n dimensional column vectors $\mathbf{x} = (x_i)_{i=1}^n$ with elements $x_i \in \Re$ $(i = 1, \dots, n)$.

$\boldsymbol{\theta}_n$ is the zero n vector.

$C(I; \Re^n) = C(I; \Re^{n\times 1})$, $\tilde{C}(I; \Re^n) = \tilde{C}(I; \Re^{n\times 1})$, $\tilde{C}_{\mathrm{loc}}(I; \Re^n) = \tilde{C}_{\mathrm{loc}}(I; \Re^{n\times 1})$ and $L^\alpha(I; \Re^n) = L^\alpha(I; \Re^{n\times 1})$.

Formulation of problem

We consider the system of ordinary differential equations with deviating arguments defined on interval $I = \langle 0,\ T\rangle$

$$\begin{aligned} \mathbf{x}'(t) &= \mathbf{A}(t)\mathbf{x}(\tau_{11}(t)) + \mathbf{B}(t)\mathbf{u}(\tau_{12}(t)) + \mathbf{q}_1(t) , \\ \mathbf{u}'(t) &= \mathbf{C}(t)\mathbf{x}(\tau_{21}(t)) + \mathbf{D}(t)\mathbf{u}(\tau_{22}(t)) + \mathbf{q}_2(t) , \end{aligned} \tag{1}$$

where $\mathbf{q}_1,\ \mathbf{q}_2 \in L(I;\Re^n)$, $\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D} \in L(I;\Re^{n\times n})$ and τ_{ij} is mesurable for $i,j \in \{1,2\}$. In addition, let

$$\mathbf{x}(s) = \varphi(s), \quad \mathbf{u}(s) = \psi(s)\,, \quad \text{for } s \in \Re \setminus I\,, \tag{2}$$

where $\varphi,\ \psi : \Re \setminus I \to \Re^n$ are the continous and bounded functions. By the pair $(\mathbf{x},\ \mathbf{u})$ an absolutely continous n-dimensional vector functions which satisfies (1) almost everywhere on I and satisfies (2), we understand a solution of (1), (2). Denote

$$\mathbf{c}_0 = (c_{0i})_{i=1}^n \in \Re^n\,, \quad \mathbf{c}_T = (c_{Ti})_{i=1}^n \in \Re^n$$

and consider the boundary value conditions for the solution of system (1), (2)

$$\alpha_{11}\mathbf{x}(0) + \alpha_{12}\mathbf{u}(0) = \mathbf{c}_0\,, \quad \alpha_{21}\mathbf{x}(T) + \alpha_{22}\mathbf{u}(T) = \mathbf{c}_T\,, \tag{3}$$

where $\alpha_{ij} \in \Re$ for $i,j \in \{1,2\}$. Define $\tau_{ij}^0 : I \to I,\ \chi_I : C(I;\Re) \to \{0,1\}$,

$$\tau_{ij}^0(t) = \begin{cases} 0 & \tau_{ij}(t) < 0 \\ \tau_{ij}(t) & \tau_{ij}(t) \in I \\ T & \tau_{ij}(t) > T \end{cases} ; \quad t \in I,\ i,j \in \{1,2\} \tag{4}$$

$$\chi_I(\tau(t)) = \begin{cases} 1 & \tau(t) \in I \\ 0 & \tau(t) \notin I \end{cases} ; \quad t \in I\,. \tag{5}$$

By (2), (4) and (5) we can the system (1) rewrite as follows

$$\begin{aligned} \mathbf{x}'(t) &= \chi_I(\tau_{11}(t))\mathbf{A}(t)\mathbf{x}(\tau_{11}^0(t)) + \chi_I(\tau_{12}(t))\mathbf{B}(t)\mathbf{u}(\tau_{12}^0(t)) \\ &\quad + [1 - \chi_I(\tau_{11}(t))]\,\mathbf{A}(t)\varphi(\tau_{11}(t)) \\ &\quad + [1 - \chi_I(\tau_{12}(t))]\,\mathbf{B}(t)\psi(\tau_{12}(t)) + \mathbf{q}_1(t)\,, \\ \mathbf{u}'(t) &= \chi_I(\tau_{21}(t))\mathbf{C}(t)\mathbf{x}(\tau_{21}^0(t)) + \chi_I(\tau_{22}(t))\mathbf{D}(t)\mathbf{u}(\tau_{22}^0(t)) \\ &\quad + [1 - \chi_I(\tau_{21}(t))]\,\mathbf{C}(t)\varphi(\tau_{21}(t)) \\ &\quad + [1 - \chi_I(\tau_{22}(t))]\,\mathbf{D}(t)\psi(\tau_{22}(t)) + \mathbf{q}_2(t)\,. \end{aligned} \tag{6}$$

Remark 1. The optimization theory is the motivation for the study of the boundary condition (3). The existence and uniqueness of the solution of a problem similar to (1), (2), (3) is studied for example by I. Öztürk in [6], L. Jodar in [7], [8] or I. Kiguradze and B. Půža in [9]. The junction of optimization theory and theory of boundary value problems is considered in [10], [11] and [12].

Existence and uniqueness of the solution

Along with system (6) we have to consider the corresponding homogeneous system

$$\begin{aligned} \mathbf{x}'(t) &= \chi_I(\tau_{11}(t))\mathbf{A}(t)\mathbf{x}(\tau_{11}^0(t)) + \chi_I(\tau_{12}(t))\mathbf{B}(t)\mathbf{u}(\tau_{12}^0(t))\,, \\ \mathbf{u}'(t) &= \chi_I(\tau_{21}(t))\mathbf{C}(t)\mathbf{x}(\tau_{21}^0(t)) + \chi_I(\tau_{22}(t))\mathbf{D}(t)\mathbf{u}(\tau_{22}^0(t)) \end{aligned} \tag{6_0}$$

and in accordance with conditions (3) we have to consider the homogeneous boundary conditions

$$\alpha_{11}\mathbf{x}(0)+\alpha_{12}\mathbf{u}(0)=\theta_n\,,\quad \alpha_{21}\mathbf{x}(T)+\alpha_{22}\mathbf{u}(T)=\theta_n\,. \tag{3_0}$$

Theorem 1. *The problem* (6), (3) *is uniquely solvable if and only if the corresponding homogeneous problem* (6_0), (3_0) *has only the trivial solution.*

Proof See [1] p. 345. □

The pair ($\mathbf{x}$, $\mathbf{u}$) an absolutely continous n-dimensional vector functions is the solution of the problem (6_0), (3_0) if and only if ($\mathbf{x}$, $\mathbf{u}$) is the solution of following integral equation on I

$$\begin{aligned}\mathbf{x}(t)&=\mathbf{x}(0)+\int_0^t\left[\chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))+\chi_I(\tau_{12}(s))\mathbf{B}(s)\mathbf{u}(\tau_{12}^0(s))\right]\,ds\,,\\ \mathbf{u}(t)&=\mathbf{u}(0)+\int_0^t\left[\chi_I(\tau_{21}(s))\mathbf{C}(s)\mathbf{x}(\tau_{21}^0(s))+\chi_I(\tau_{22}(s))\mathbf{D}(s)\mathbf{u}(\tau_{22}^0(s))\right]\,ds\,.\end{aligned} \tag{7}$$

Clearly

$$\begin{aligned}\mathbf{x}(\tau_{i1}^0(t))&=\mathbf{x}(0)+\int_0^{\tau_{i1}^0(t)}\mathbf{x}'(s)\,ds\,,\\ \mathbf{u}(\tau_{i2}^0(t))&=\mathbf{u}(0)+\int_0^{\tau_{i2}^0(t)}\mathbf{u}'(s)\,ds\,,\quad (i=1,2)\,.\end{aligned} \tag{8}$$

Denote

$$A^0(t)=\mathbf{E}_n\,,\quad B^0(t)=\mathbf{\Theta}_n\,,\quad C^0(t)=\mathbf{\Theta}_n\,,\quad D^0(t)=\mathbf{E}_n\,, \tag{9}$$

$$\begin{aligned}A^i(t)&=\int_0^t\left[\chi_I(\tau_{11}(s))\mathbf{A}(s)A^{i-1}(\tau_{11}^0(s))+\chi_I(\tau_{12}(s))\mathbf{B}(s)C^{i-1}(\tau_{12}^0(s))\right]\,ds\,,\\ B^i(t)&=\int_0^t\left[\chi_I(\tau_{11}(s))\mathbf{A}(s)B^{i-1}(\tau_{11}^0(s))+\chi_I(\tau_{12}(s))\mathbf{B}(s)D^{i-1}(\tau_{12}^0(s))\right]\,ds\,,\\ C^i(t)&=\int_0^t\left[\chi_I(\tau_{21}(s))\mathbf{C}(s)A^{i-1}(\tau_{21}^0(s))+\chi_I(\tau_{22}(s))\mathbf{D}(s)C^{i-1}(\tau_{22}^0(s))\right]\,ds\,,\\ D^i(t)&=\int_0^t\left[\chi_I(\tau_{21}(s))\mathbf{C}(s)B^{i-1}(\tau_{21}^0(s))+\chi_I(\tau_{22}(s))\mathbf{D}(s)D^{i-1}(\tau_{22}^0(s))\right]\,ds\,,\\ &\qquad (i=1,2,\dots)\end{aligned}$$

and

$$
\begin{aligned}
&(AB)^1(\mathbf{x},\mathbf{u})^T(t)\\
&= \int_0^t \left[\chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s)) + \chi_I(\tau_{12}(s))\mathbf{B}(s)\mathbf{u}(\tau_{12}^0(s))\right]\, ds\,,\\
&(CD)^1(\mathbf{x},\mathbf{u})^T(t)\\
&= \int_0^t \left[\chi_I(\tau_{21}(s))\mathbf{C}(s)\mathbf{x}(\tau_{21}^0(s)) + \chi_I(\tau_{22}(s))\mathbf{D}(s)\mathbf{u}(\tau_{22}^0(s))\right]\, ds\,,
\end{aligned}
$$

$$
\begin{aligned}
&\qquad (i = 1, 2, \ldots)\,,\\
&(AB)^i(\mathbf{x},\mathbf{u})^T(t) = \int_0^t \left[\chi_I(\tau_{11}(s))\mathbf{A}(s)(AB)^{i-1}(\mathbf{x},\mathbf{u})^T(\tau_{11}^0(s))\right.\\
&\qquad\qquad \left. +\chi_I(\tau_{12}(s))\mathbf{B}(s)(CD)^{i-1}(\mathbf{x},\mathbf{u})^T(\tau_{12}^0(s))\right]\, ds\,,\\
&(CD)^i(\mathbf{x},\mathbf{u})^T(t) = \int_0^t \left[\chi_I(\tau_{21}(s))\mathbf{C}(s)(AB)^{i-1}(\mathbf{x},\mathbf{u})^T(\tau_{21}^0(s))\right.\\
&\qquad\qquad \left. +\chi_I(\tau_{22}(s))\mathbf{D}(s)(CD)^{i-1}(\mathbf{x},\mathbf{u})^T(\tau_{22}^0(s))\right]\, ds\,,
\end{aligned} \tag{10}
$$

Note 1. For an arbitrary matrix function $\mathbf{Z} \in C(I;\Re^{2n\times 2n})$ with column $\mathbf{z}_1,\ldots,\mathbf{z}_{2n}$ we understand by $(AB)^i(\mathbf{Z})$, $(CD)^i(\mathbf{Z})$ the matrix with column $(AB)^i(\mathbf{z}_1),\ldots,(AB)^i(\mathbf{z}_{2n})$, $(CD)^i(\mathbf{z}_1),\ldots,(CD)^i(\mathbf{z}_{2n})$, respectively. $(AB)^k(\mathbf{E})(t) = \left[A^k(t), B^k(t)\right]$, $(CD)^k(\mathbf{E})(t) = \left[C^k(t), D^k(t)\right]$, $(k = 1, 2, \ldots)$.

According to (9) we denote

$$
|A|^0(t) = \mathbf{E}_n, \quad |B|^0(t) = \mathbf{\Theta}_n\,, \quad |C|^0(t) = \mathbf{\Theta}_n, \quad |D|^0(t) = \mathbf{E}_n\,, \tag{9a}
$$

$$
\begin{aligned}
|A|^i(t) &= \int_0^t \left[\chi_I(\tau_{11}(s))|\mathbf{A}(s)||A|^{i-1}(\tau_{11}^0(s)) + \chi_I(\tau_{12}(s))|\mathbf{B}(s)||C|^{i-1}(\tau_{12}^0(s))\right]\, ds\,,\\
|B|^i(t) &= \int_0^t \left[\chi_I(\tau_{11}(s))|\mathbf{A}(s)||B|^{i-1}(\tau_{11}^0(s)) + \chi_I(\tau_{12}(s))|\mathbf{B}(s)||D|^{i-1}(\tau_{12}^0(s))\right]\, ds\,,\\
|C|^i(t) &= \int_0^t \left[\chi_I(\tau_{21}(s))|\mathbf{C}(s)||A|^{i-1}(\tau_{21}^0(s)) + \chi_I(\tau_{22}(s))|\mathbf{D}(s)||C|^{i-1}(\tau_{22}^0(s))\right]\, ds\,,\\
|D|^i(t) &= \int_0^t \left[\chi_I(\tau_{21}(s))|\mathbf{C}(s)||B|^{i-1}(\tau_{21}^0(s)) + \chi_I(\tau_{22}(s))|\mathbf{D}(s)||D|^{i-1}(\tau_{22}^0(s))\right]\, ds\,,\\
&\qquad (i = 1, 2, \ldots)\,.
\end{aligned}
$$

So, by (10) we can rewrite (8) as follows

$$
\begin{aligned}
\mathbf{x}(\tau_{i1}^0(t)) &= \mathbf{x}(0) + (AB)^1(\mathbf{x},\mathbf{u})^T(\tau_{i1}^0(t))\,,\\
\mathbf{u}(\tau_{i2}^0(t)) &= \mathbf{u}(0) + (CD)^1(\mathbf{x},\mathbf{u})^T(\tau_{i2}^0(t))\,, \quad (i = 1, 2)\,.
\end{aligned}
$$

If we use last equalities in (7) then we obtain

$$\begin{aligned}
\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t & \left[\chi_I(\tau_{11}(s))\mathbf{A}(s)\left[\mathbf{x}(0) + (AB)^1(\mathbf{x},\mathbf{u})^T(\tau_{11}^0(s))\right]\right. \\
& \left. + \chi_I(\tau_{12}(s))\mathbf{B}(s)\left[\mathbf{u}(0) + (CD)^1(\mathbf{x},\mathbf{u})^T(\tau_{12}^0(s))\right]\right] ds \\
= \mathbf{x}(0) & + \left[\int_0^t \chi_I(\tau_{11}(s))\mathbf{A}(s)\,ds\right]\mathbf{x}(0) \\
& + \left[\int_0^t \chi_I(\tau_{12}(s))\mathbf{B}(s)\,ds\right]\mathbf{u}(0) + (AB)^2(\mathbf{x},\mathbf{u})^T(t) \\
= \left[A^0(t) \right. & \left. + A^1(t)\right]\mathbf{x}(0) + B^1(t)\mathbf{u}(0) + (AB)^2(\mathbf{x},\mathbf{u})^T(t)\,, \\
\mathbf{u}(t) = \mathbf{u}(0) + \int_0^t & \left[\chi_I(\tau_{21}(s))\mathbf{C}(s)\left[\mathbf{x}(0) + (AB)^1(\mathbf{x},\mathbf{u})^T(\tau_{21}^0(s))\right]\right. \\
& \left. + \chi_I(\tau_{22}(s))\mathbf{D}(s)\left[\mathbf{u}(0) + (CD)^1(\mathbf{x},\mathbf{u})^T(\tau_{22}^0(s))\right]\right] ds \\
= \left[\int_0^t \right. & \left. \chi_I(\tau_{21}(s))\mathbf{C}(s)\,ds\right]\mathbf{x}(0) \\
& + \left[\mathbf{E}_n + \int_0^t \chi_I(\tau_{22}(s))\mathbf{D}(s)\,ds\right]\mathbf{u}(0) + (CD)^2(\mathbf{x},\mathbf{u})^T(t) \\
= C^1(t) & \mathbf{x}(0) + \left[D^0(t) + D^1(t)\right]\mathbf{u}(0) + (CD)^2(\mathbf{x},\mathbf{u})^T(t)
\end{aligned}$$

and by the same way

$$\begin{aligned}
\mathbf{x}(t) &= \left[A^0(t) + A^1(t) + A^2(t)\right]\mathbf{x}(0) + \left[B^1(t) + B^2(t)\right]\mathbf{u}(0) + (AB)^3(\mathbf{x},\mathbf{u})^T(t)\,, \\
\mathbf{u}(t) &= \left[C^1(t) + C^2(t)\right]\mathbf{x}(0) + \left[D^0(t) + D^1(t) + D^2(t)\right]\mathbf{u}(0) + (CD)^3(\mathbf{x},\mathbf{u})^T(t)\,.
\end{aligned}$$

If we continue this process for $k = 1, 2, \dots$ then we obtain

$$\begin{aligned}
\mathbf{x}(t) &= \left[\sum_{i=0}^{k-1} A^i(t)\right]\mathbf{x}(0) + \left[\sum_{i=0}^{k-1} B^i(t)\right]\mathbf{u}(0) + (AB)^k(\mathbf{x},\mathbf{u})^T(t)\,, \\
\mathbf{u}(t) &= \left[\sum_{i=0}^{k-1} C^i(t)\right]\mathbf{x}(0) + \left[\sum_{i=0}^{k-1} D^i(t)\right]\mathbf{u}(0) + (CD)^k(\mathbf{x},\mathbf{u})^T(t)\,.
\end{aligned} \tag{11}$$

From (3_0) and (11)

$$\begin{aligned}
\theta_{2n} &= \begin{pmatrix} \alpha_{11}\mathbf{E}_n & \alpha_{12}\mathbf{E}_n \\ \alpha_{21}\sum_{i=0}^{k-1} A^i(T) + \alpha_{22}\sum_{i=0}^{k-1} C^i(T) & \alpha_{21}\sum_{i=0}^{k-1} B^i(T) + \alpha_{22}\sum_{i=0}^{k-1} D^i(T) \end{pmatrix} \\
&\cdot \begin{pmatrix} \mathbf{x}(0) \\ \mathbf{u}(0) \end{pmatrix} + \begin{pmatrix} \theta_n \\ \alpha_{21}(AB)^k(\mathbf{x},\mathbf{u})^T(T) + \alpha_{22}(CD)^k(\mathbf{x},\mathbf{u})^T(T) \end{pmatrix}
\end{aligned} \tag{12}$$

and if we assume the invertibility of the matrix

$$\begin{aligned}\mathbf{S}_k = {}& \alpha_{12}\left(\alpha_{21}\sum_{i=0}^{k-1} A^i(T) + \alpha_{22}\sum_{i=0}^{k-1} C^i(T)\right)\\ & - \alpha_{11}\left(\alpha_{21}\sum_{i=0}^{k-1} B^i(T) + \alpha_{22}\sum_{i=0}^{k-1} D^i(T)\right)\end{aligned} \tag{13}$$

then

$$\mathbf{x}(0) = -\alpha_{12}\mathbf{R}_k\,, \quad \mathbf{u}(0) = \alpha_{11}\mathbf{R}_k \tag{14}$$

is valid, where

$$\mathbf{R}_k = \mathbf{S}_k^{-1}\left[\alpha_{21}(AB)^k(\mathbf{x},\mathbf{u})^T(T) + \alpha_{22}(CD)^k(\mathbf{x},\mathbf{u})^T(T)\right]\,. \tag{15}$$

Using (14) in (11) we obtain

$$\begin{aligned}\mathbf{x}(t) &= -\alpha_{12}\left[\sum_{i=0}^{m-1} A^i(t)\right]\mathbf{R}_k + \alpha_{11}\left[\sum_{i=0}^{m-1} B^i(t)\right]\mathbf{R}_k + (AB)^m(\mathbf{x},\mathbf{u})^T(t)\,,\\ \mathbf{u}(t) &= -\alpha_{12}\left[\sum_{i=0}^{m-1} C^i(t)\right]\mathbf{R}_k + \alpha_{11}\left[\sum_{i=0}^{m-1} D^i(t)\right]\mathbf{R}_k + (CD)^m(\mathbf{x},\mathbf{u})^T(t)\,.\end{aligned} \tag{16}$$

Henceforth

$$\begin{aligned}\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{pmatrix} = {}& \left[\sum_{i=0}^{m-1}\begin{pmatrix}-\alpha_{12}A^i(t) & \alpha_{11}B^i(t)\\ -\alpha_{12}C^i(t) & \alpha_{11}D^i(t)\end{pmatrix}\right]\\ & \left[\begin{pmatrix}\mathbf{\Theta}_n & \mathbf{S}_k^{-1}\\ \mathbf{\Theta}_n & \mathbf{S}_k^{-1}\end{pmatrix}\begin{pmatrix}\boldsymbol{\theta}_n\\ \alpha_{21}(AB)^k(\mathbf{x},\mathbf{u})^T(T) + \alpha_{22}(CD)^k(\mathbf{x},\mathbf{u})^T(T)\end{pmatrix}\right]\\ & + \begin{pmatrix}(AB)^m(\mathbf{x},\mathbf{u})^T(t)\\ (CD)^m(\mathbf{x},\mathbf{u})^T(t)\end{pmatrix}.\end{aligned} \tag{17}$$

From (17) by the notation (9a), Note 1 and properties of the norm we obtain

$$\left|\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix}\right|_C \le \mathbf{M}_{k,m}\left|\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix}\right|_C\,, \tag{18}$$

where $|\mathbf{y}|_C = (\|y_i\|_C)_{i=1}^{2n}$ and

$$\begin{aligned}\mathbf{M}_{k,m} = {}& \left[\sum_{i=0}^{m-1}\begin{pmatrix}\alpha_{12}|A|^i(T) & \alpha_{11}|B|^i(T)\\ \alpha_{12}|C|^i(T) & \alpha_{11}|D|^i(T)\end{pmatrix}\right]\\ & \left|\begin{pmatrix}\mathbf{\Theta}_n & \mathbf{S}_k^{-1}\\ \mathbf{\Theta}_n & \mathbf{S}_k^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{\Theta}_n & \mathbf{\Theta}_n\\ \alpha_{21}A^k(T) + \alpha_{22}C^k(T) & \alpha_{21}B^k(T) + \alpha_{22}D^k(T)\end{pmatrix}\right|\\ & + \begin{pmatrix}|A|^m(T) & |B|^m(T)\\ |C|^m(T) & |D|^m(T)\end{pmatrix}\end{aligned} \tag{19}$$

and from inequality (18) evidently

$$(\mathbf{E}_{2n} - \mathbf{M}_{k,m}) \left\| \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} \right\|_C \le \theta_{2n} \,.$$

Now, if $m \in \{1, 2, \dots\}$ exists, such that

$$r(\mathbf{M}_{k,m}) < 1 \,, \tag{20}$$

then the inverse $(\mathbf{E}_{2n} - \mathbf{M}_{k,m})^{-1}$ of the matrix $(\mathbf{E}_{2n} - \mathbf{M}_{k,m})$ exists and

$$\left\| \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} \right\|_C \le (\mathbf{E}_{2n} - \mathbf{M}_{k,m})^{-1} \theta_{2n} = \theta_{2n} \,,$$

holds, so

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} \equiv \theta_{2n}$$

and the problem $(6_0), (3_0)$ has only trivial solution. According to Theorem 1, we have obtained, that the condition of invertibility of the matrix (13) for any $k = 1, 2, \dots$ and condition (20) for any $m = 1, 2, \dots$ are sufficient conditions for existence and uniqueness of the solution of the problem (1), (2), (3). By this result we can formulate the following theorem: Sufficient condition for existence and uniqueness of the solution in effective form.

Theorem 2. *Let positive integers k, m exist, such that the matrix* (13) *is invertible and the matrix* (19) *the condition* (20) *holds. Then the problem* (1), (2), (3) *is uniquely solvable.*

Consider now the differential system on I

$$\begin{aligned} \mathbf{x}'(t) &= \epsilon \mathbf{A}(t)\mathbf{x}(\tau_{11}(t)) + \epsilon \mathbf{B}(t)\mathbf{u}(\tau_{12}(t)) + \mathbf{q}_1(t) \,, \\ \mathbf{u}'(t) &= \epsilon \mathbf{C}(t)\mathbf{x}(\tau_{21}(t)) + \epsilon \mathbf{D}(t)\mathbf{u}(\tau_{22}(t)) + \mathbf{q}_2(t) \,, \end{aligned} \tag{21}$$

which is dependent on small positive parametr ϵ. Again $\mathbf{q}_1,\ \mathbf{q}_2 \in L(I; \Re^n)$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D} \in L(I; \Re^{n \times n})$ and τ_{ij} is mesurable for $i, j \in \{1, 2\}$ and consider the boundary value conditions for the solution of system (21), (2)

$$\alpha_1 \mathbf{x}(0) + \alpha_2 \mathbf{u}(0) = \mathbf{c}_0 \,, \quad \alpha_1 \mathbf{x}(T) + \alpha_2 \mathbf{u}(T) = \mathbf{c}_T \,, \tag{3$'$}$$

where $\alpha_i \in \Re$ for $i \in \{1, 2\}$. We proved, that Theorem 2 implies for problem (21), (2), (3') following corollary.

Corollary 1. *Let there exist a $j \in \{1, 2, \dots\}$ such that*

$$\begin{aligned} &A^i(T) = \mathbf{\Theta}_n \,, \quad D^i(T) = \mathbf{\Theta}_n \qquad i = 1, \dots, j-1 \\ &B^i(T) = \mathbf{\Theta}_n, \quad C^i(T) = \mathbf{\Theta}_n \qquad i = 0, \dots, j-1 \\ &\det \left[\alpha_2 \left(\alpha_1 A^j(T) + \alpha_2 C^j(T) \right) - \alpha_1 \left(\alpha_1 B^j(T) + \alpha_2 D^j(T) \right) \right] \ne 0 \,. \end{aligned} \tag{22}$$

Then $\epsilon_0 > 0$ exists, such that the problem (21), (2), (3') is uniquely solvable for arbitrary $\epsilon \in (0, \epsilon_0)$.

Proof. We apply Theorem 2. Let $k = j + 1,\ m = 1$. In addition we denote

$$\mathbf{A}(t;\epsilon) = \epsilon\mathbf{A}(t),\ \mathbf{B}(t;\epsilon) = \epsilon\mathbf{B}(t),\ \mathbf{C}(t;\epsilon) = \epsilon\mathbf{C}(t),\ \mathbf{D}(t;\epsilon) = \epsilon\mathbf{D}(t),$$

$$\begin{aligned}\mathbf{S}_k(\epsilon) = \alpha_2\left(\alpha_1\sum_{i=0}^{k-1}A^i(T;\epsilon) + \alpha_2\sum_{i=0}^{k-1}C^i(T;\epsilon)\right)\\ - \alpha_1\left(\alpha_1\sum_{i=0}^{k-1}B^i(T;\epsilon) + \alpha_2\sum_{i=0}^{k-1}D^i(T;\epsilon)\right),\end{aligned}$$

$$\begin{aligned}\mathbf{M}_{k,1}(\epsilon) = &\begin{pmatrix}\alpha_2\mathbf{E}_n & \mathbf{\Theta}_n\\ \mathbf{\Theta}_n & \alpha_1\mathbf{E}_n\end{pmatrix}\\ &\left|\begin{pmatrix}\mathbf{\Theta}_n & \mathbf{S}_k^{-1}(\epsilon)\\ \mathbf{\Theta}_n & \mathbf{S}_k^{-1}(\epsilon)\end{pmatrix}\begin{pmatrix}\mathbf{\Theta}_n & \mathbf{\Theta}_n\\ \alpha_1A^k(T;\epsilon) + \alpha_2C^k(T;\epsilon) & \alpha_1B^k(T;\epsilon) + \alpha_2D^k(T;\epsilon)\end{pmatrix}\right|\\ &\quad + \begin{pmatrix}|A|^1(T;\epsilon) & |B|^1(T;\epsilon)\\ |C|^1(T;\epsilon) & |D|^1(T;\epsilon)\end{pmatrix},\end{aligned}$$

where matrix functions $A^i(.;\epsilon),\ B^i(.;\epsilon),\ C^i(.;\epsilon),\ D^i(.;\epsilon)$ are defined analogously as in (9). Evidently

$$\begin{aligned}&A^i(t;\epsilon) = \epsilon^iA^i(t), \quad B^i(t;\epsilon) = \epsilon^iB^i(t),\\ &C^i(t;\epsilon) = \epsilon^iC^i(t), \quad D^i(t;\epsilon) = \epsilon^iD^i(t), \qquad (i = 0, 1, 2, \dots)\end{aligned}$$

is valid. Hence

$$\begin{aligned}\mathbf{S}_k(\epsilon) &= \alpha_2\left(\alpha_1\sum_{i=0}^{k-1}\epsilon^iA^i(T) + \alpha_2\sum_{i=0}^{k-1}\epsilon^iC^i(T)\right)\\ &\quad - \alpha_1\left(\alpha_1\sum_{i=0}^{k-1}\epsilon^iB^i(T) + \alpha_2\sum_{i=0}^{k-1}\epsilon^iD^i(T)\right)\\ &= \epsilon^{k-1}\left[\alpha_2\left(\alpha_1A^{k-1}(T) + \alpha_2C^{k-1}(T)\right) - \alpha_1\left(\alpha_1B^{k-1}(T) + \alpha_2D^{k-1}(T)\right)\right]\\ &\quad + (\alpha_2\alpha_1 - \alpha_1\alpha_2)\mathbf{E}_n\end{aligned}$$

and in accordance with $k = j + 1$, (22) implies

$$\det\left(\mathbf{S}_k(\epsilon)\right) \neq 0.$$

So

$$\mathbf{M}_{k,1}(\epsilon) = \begin{pmatrix} \alpha_2 \mathbf{E}_n & \mathbf{\Theta}_n \\ \mathbf{\Theta}_n & \alpha_1 \mathbf{E}_n \end{pmatrix}$$

$$\left| \frac{1}{\epsilon^{k-1}} \begin{pmatrix} \mathbf{\Theta}_n & \mathbf{S}_k^{-1} \\ \mathbf{\Theta}_n & \mathbf{S}_k^{-1} \end{pmatrix} \epsilon^k \begin{pmatrix} \mathbf{\Theta}_n & \mathbf{\Theta}_n \\ \alpha_1 A^k(T) + \alpha_2 C^k(T) & \alpha_1 B^k(T) + \alpha_2 D^k(T) \end{pmatrix} \right|$$

$$+ \epsilon \begin{pmatrix} |A|^1(T) & |B|^1(T) \\ |C|^1(T) & |D|^1(T) \end{pmatrix} = \epsilon \mathbf{M}_{k,1} .$$

If we get

$$\epsilon_0 = \frac{1}{r\left(\mathbf{M}_{k,1}\right)} ,$$

then $r\left(\mathbf{M}_{k,1}(\epsilon)\right) < 1$ is valid for arbitrary $\epsilon \in (0, \epsilon_0)$. So, the assumptions of Theorem 2 are valid and in this way Corollary 1 is proved. □

Let now $\alpha_{11} = \alpha_{22} = 1$ and $\alpha_{12} = \alpha_{21} = 0$. Then the problem (1), (2), (3) call Cauchy-Nicoletti. By Theorem 2 we can formulated the following corollary for this problem.

Corollary 2. *Let $\alpha_{11} = \alpha_{22} = 1$, $\alpha_{12} = \alpha_{21} = 0$ and positive integers k, m exist, such that the matrix*

$$-\left[\sum_{i=0}^{k-1} D^i(T)\right]$$

is invertible and the matrix

$$\mathbf{M}_{k,m} = \left[\sum_{i=0}^{m-1} \begin{pmatrix} \mathbf{\Theta}_n & |B|^i(T) \\ \mathbf{\Theta}_n & |D|^i(T) \end{pmatrix}\right]$$

$$\left[\left| \begin{pmatrix} \Theta_n & \left[\sum_{i=0}^{k-1} D^i(T)\right]^{-1} \\ \Theta_n & \left[\sum_{i=0}^{k-1} D^i(T)\right]^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{\Theta}_n & \mathbf{\Theta}_n \\ C^k(T) & D^k(T) \end{pmatrix} \right|\right]$$

$$+ \begin{pmatrix} |A|^m(T) & |B|^m(T) \\ |C|^m(T) & |D|^m(T) \end{pmatrix}$$

the condition (20) *holds. Then the Cauchy-Nicoletti problem* (1), (2), (3) *is uniquely solvable.*

Remark 2. I. Öztürk study the similar problem (but with different boundary condition) in [6], where $\tau_{ij}(t) = t_{ij}$, $t_{ij} \in I$, $i, j \in \{1, 2\}$ ("the problem of interpolation"). The question of existence and uniqueness of problem's solution is solvable by Theorem 2 too.

Let now

$$(23) \qquad \left(\tau_{ij}^0(t)\right)' \quad \text{exists almost everywhere on} \quad I, \qquad i, j \in \{1, 2\}$$

and

$$0 < \inf_I \left\{ \left| \left(\tau_{ij}^0(t)\right)' \right| \right\} , \qquad i,j \in \{1,2\} . \tag{24}$$

Denote

$$\tau_{ij}^* = \inf_I \left\{ \left| \left(\tau_{ij}^0(t)\right)' \right| \right\} , \qquad i,j \in \{1,2\} . \tag{25}$$

Lemma 1 (Hölder inequality - see [4], p. 462). *Let $p,\ q \in \langle 1, +\infty\rangle$ such that $\frac{1}{p}+\frac{1}{q}=1$ and $f \in L^p(I;\Re^{n_1\times n_2})$, $g \in L^q(I;\Re^{n_2\times n_3})$. Then*

$$\int_I \|f(s)g(s)\|\, ds = \left(\int_I \|f(s)\|^p\, ds\right)^{\frac{1}{p}} \left(\int_I \|g(s)\|^q\, ds\right)^{\frac{1}{q}}$$

is valid.

Lemma 2 (Wirtinger inequality - see [5], p. 99). *Let $u : \langle a,b\rangle \to \Re$ is absolutely continous function, such that its derivation in second power is integrable. If exists $t_0 \in \langle a,b\rangle$ such that*

$$u(t_0) = 0 ,$$

then

$$\int_a^b u^2(t)\, dt \le \frac{4(b-a)^2}{\pi^2} \int_a^b \left(u'(t)\right)^2\, dt .$$

holds.

So, from (7)

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \left[\chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s)) + \chi_I(\tau_{12}(s))\mathbf{B}(s)\mathbf{u}(\tau_{12}^0(s))\right]\, ds$$

and

$$\begin{aligned} \|\mathbf{x}\|_{L^\beta} \le\ & \|\mathbf{x}(0)\|_{L^\beta} \\ & + \left\| \int_0^t \left[\chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s)) + \chi_I(\tau_{12}(s))\mathbf{B}(s)\mathbf{u}(\tau_{12}^0(s))\right]\, ds \right\|_{L^\beta} . \end{aligned} \tag{26}$$

Now

$$\begin{aligned} \left\|(AB)^1(\mathbf{x},\mathbf{u})^T(t)\right\|_{L^\beta} \le & \left(\int_0^T \left(\int_0^t \left\|\chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))\right\|\, ds\right)^\beta dt\right)^{\frac{1}{\beta}} \\ & + \left(\int_0^T \left(\int_0^t \left\|\chi_I(\tau_{12}(s))\mathbf{B}(s)\mathbf{u}(\tau_{12}^0(s))\right\|\, ds\right)^\beta dt\right)^{\frac{1}{\beta}} . \end{aligned}$$

By Lemma 1, where $f = \chi_I(\tau_{11})\mathbf{A},\ \chi_I(\tau_{12})\mathbf{B}$ and $g = \mathbf{x}(\tau_{11}^0),\ \mathbf{u}(\tau_{12}^0)$, respectively and $\frac{1}{\alpha} + \frac{2}{\beta} = 1$ we obtain

$$\left\|(AB)^1(\mathbf{x},\mathbf{u})^T(t)\right\|_{L^\beta} \le \|\chi_I(\tau_{11})\mathbf{A}\|_{L^\alpha} \left[\int_0^T \left(\int_0^t \left\|\mathbf{x}(\tau_{11}^0(s))\right\|^{\frac{\beta}{2}} ds\right)^2 dt\right]^{\frac{1}{\beta}} + \|\chi_I(\tau_{12})\mathbf{B}\|_{L^\alpha} \left[\int_0^T \left(\int_0^t \left\|\mathbf{u}(\tau_{12}^0(s))\right\|^{\frac{\beta}{2}} ds\right)^2 dt\right]^{\frac{1}{\beta}} .$$

Let $\xi_{1j} = \tau_{1j}^0(s),\ j \in \{1,2\}$. Then, by (25)

$$d\xi_{1j} = \left(\tau_{1j}^0(s)\right)' ds\,, \quad ds = \frac{d\xi_{1j}}{\left(\tau_{1j}^0(s)\right)'} \le \frac{d\xi_{1j}}{\tau_{1j}^*} \qquad j \in \{1,2\}\,.$$

Hence

$$\left\|(AB)^1(\mathbf{x},\mathbf{u})^T\ (t)\right\|_{L^\beta} \le \|\chi_I(\tau_{11})\mathbf{A}\|_{L^\alpha} \left(\frac{1}{\tau_{11}^*}\right)^{\frac{2}{\beta}} \left[\int_0^T \left(\int_{\tau_{11}^0(0)}^{\tau_{11}^0(t)} \|\mathbf{x}(\xi_{11})\|^{\frac{\beta}{2}} d\xi_{11}\right)^2 dt\right]^{\frac{1}{\beta}} + \|\chi_I(\tau_{12})\mathbf{B}\|_{L^\alpha} \left(\frac{1}{\tau_{12}^*}\right)^{\frac{2}{\beta}} \left[\int_0^T \left(\int_{\tau_{12}^0(0)}^{\tau_{12}^0(t)} \|\mathbf{u}(\xi_{12})\|^{\frac{\beta}{2}} d\xi_{12}\right)^2 dt\right]^{\frac{1}{\beta}} .$$

From Lemma 2, where $u = \int_{\tau_{11}^0(0)}^{\tau_{11}^0(t)} \|\mathbf{x}(\xi_{11})\|^{\frac{\beta}{2}} d\xi_{11}\,,\ \int_{\tau_{12}^0(0)}^{\tau_{12}^0(t)} \|\mathbf{u}(\xi_{12})\|^{\frac{\beta}{2}} d\xi_{12}$, respectively, we obtain

$$\left\|(AB)^1(\mathbf{x},\mathbf{u})^T(t)\right\|_{L^\beta} \le \|\chi_I(\tau_{11})\mathbf{A}\|_{L^\alpha} \left(\frac{2T}{\pi\tau_{11}^*}\right)^{\frac{2}{\beta}} \left[\int_0^T \left\|\mathbf{x}(\tau_{11}^0(t))\right\|^\beta dt\right]^{\frac{1}{\beta}} + \|\chi_I(\tau_{12})\mathbf{B}\|_{L^\alpha} \left(\frac{2T}{\pi\tau_{12}^*}\right)^{\frac{2}{\beta}} \left[\int_0^T \left\|\mathbf{u}(\tau_{12}^0(t))\right\|^\beta dt\right]^{\frac{1}{\beta}}$$

and by the same transformation

$$\left\|(AB)^1(\mathbf{x},\mathbf{u})^T(t)\right\|_{L^\beta} \le \|\chi_I(\tau_{11})\mathbf{A}\|_{L^\alpha} \left(\frac{2T}{\pi\tau_{11}^*}\right)^{\frac{2}{\beta}} \left(\frac{1}{\tau_{11}^*}\right)^{\frac{1}{\beta}} \|\mathbf{x}\|_{L^\beta} + \|\chi_I(\tau_{12})\mathbf{B}\|_{L^\alpha} \left(\frac{2T}{\pi\tau_{12}^*}\right)^{\frac{2}{\beta}} \left(\frac{1}{\tau_{12}^*}\right)^{\frac{1}{\beta}} \|\mathbf{u}\|_{L^\beta} . \tag{27}$$

Let k=2. By (13), (14) and (15)

$$\mathbf{x}(0) = -\alpha_{12}\mathbf{S}_2^{-1}\left[\alpha_{21}(AB)^2(\mathbf{x},\mathbf{u})^T(T) + \alpha_{22}(CD)^2(\mathbf{x},\mathbf{u})^T(T)\right] .$$

So, by properties of the norm

$$\|\mathbf{x}(0)\|_{L^\beta} \le |\alpha_{12}| T^{\frac{1}{\beta}} \left\|\mathbf{S}_2^{-1}\right\| \left\|\alpha_{21}(AB)^2(\mathbf{x},\mathbf{u})^T(T) + \alpha_{22}(CD)^2(\mathbf{x},\mathbf{u})^T(T)\right\| . \tag{28}$$

From (10) evidently

$$\begin{aligned}
(AB)^2(\mathbf{x},\mathbf{u})^T(T) = &\int_0^T \left[\chi_I(\tau_{11}(t))\mathbf{A}(t)\left(\int_0^{\tau_{11}^0(t)} \chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))\,ds\right)\right] dt \\
&+ \int_0^T \left[\chi_I(\tau_{11}(t))\mathbf{A}(t)\left(\int_0^{\tau_{11}^0(t)} \chi_I(\tau_{12}(s))\mathbf{B}(s)\mathbf{u}(\tau_{12}^0(s))\,ds\right)\right] dt \\
&+ \int_0^T \left[\chi_I(\tau_{12}(t))\mathbf{B}(t)\left(\int_0^{\tau_{12}^0(t)} \chi_I(\tau_{21}(s))\mathbf{C}(s)\mathbf{x}(\tau_{21}^0(s))\,ds\right)\right] dt \\
&+ \int_0^T \left[\chi_I(\tau_{12}(t))\mathbf{B}(t)\left(\int_0^{\tau_{12}^0(t)} \chi_I(\tau_{22}(s))\mathbf{D}(s)\mathbf{u}(\tau_{11}^0(s))\,ds\right)\right] dt\,.
\end{aligned} \tag{29}$$

holds. Clearly, by properties of the norm

$$\begin{aligned}
&\left\|\int_0^T \left[\chi_I(\tau_{11}(t))\mathbf{A}(t)\left(\int_0^{\tau_{11}^0(t)} \chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))\,ds\right)\right] dt\right\| \\
&\le \int_0^T \left\|\chi_I(\tau_{11}(t))\mathbf{A}(t)\left(\int_0^{\tau_{11}^0(t)} \chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))\,ds\right)\right\| dt \\
&\le \int_0^T \left[\|\chi_I(\tau_{11}(t))\mathbf{A}(t)\| \left\|\int_0^{\tau_{11}^0(t)} \chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))\,ds\right\|\right] dt \\
&\le \int_0^T \left[\|\chi_I(\tau_{11}(t))\mathbf{A}(t)\| \left(\int_0^{\tau_{11}^0(t)} \left\|\chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))\right\|\,ds\right)\right] dt
\end{aligned} \tag{30}$$

is valid. Hence, by Lemma 1, where $f = \chi_I(\tau_{11})\mathbf{A}$, $g = \mathbf{x}(\tau_{11}^0)$ and $\frac{1}{\alpha^*} + \frac{1}{\beta} = 1$

$$\int_0^{\tau_{11}^0(t)} \left\|\chi_I(\tau_{11}(s))\mathbf{A}(s)\mathbf{x}(\tau_{11}^0(s))\right\|\,ds \le \|\chi_I(\tau_{11})\mathbf{A}\|_{L^{\alpha^*}} \|\mathbf{x}\|_{L^\beta}\,. \tag{31}$$

Denote

$$\|A\|^1(t) = \int_0^t \|\chi_I(\tau_{11}(s))\mathbf{A}(s)\|\,ds \tag{32}$$

and $\|B\|^1(t)$, $\|C\|^1(t)$, $\|D\|^1(t)$ analogously. By the same way

$$\begin{aligned}
\left\|(AB)^2(\mathbf{x},\mathbf{u})^T(T)\right\| \le &\left[\|A\|^1(T)\,\|\chi_I(\tau_{11})\mathbf{A}\|_{L^{\alpha^*}}\right. \\
&\left.+\|B\|^1(T)\,\|\chi_I(\tau_{21})\mathbf{C}\|_{L^{\alpha^*}}\right] \|\mathbf{x}\|_{L^\beta} \\
&+\left[\|A\|^1(T)\,\|\chi_I(\tau_{12})\mathbf{B}\|_{L^{\alpha^*}}\right. \\
&\left.+\|B\|^1(T)\,\|\chi_I(\tau_{22})\mathbf{D}\|_{L^{\alpha^*}}\right] \|\mathbf{u}\|_{L^\beta}\,.
\end{aligned} \tag{33}$$

Analogously

$$\left\|(CD)^2(\mathbf{x},\mathbf{u})^T(T)\right\| \le \big[\|C\|^1(T)\,\|\chi_I(\tau_{11})\mathbf{A}\|_{L^{\alpha^*}} + \|D\|^1(T)\,\|\chi_I(\tau_{21})\mathbf{C}\|_{L^{\alpha^*}}\big]\,\|\mathbf{x}\|_{L^\beta} + \big[\|C\|^1(T)\,\|\chi_I(\tau_{12})\mathbf{B}\|_{L^{\alpha^*}} + \|D\|^1(T)\,\|\chi_I(\tau_{22})\mathbf{D}\|_{L^{\alpha^*}}\big]\,\|\mathbf{u}\|_{L^\beta}\;. \tag{34}$$

If we denote

$$\begin{aligned} H_1 &= \big[\big(|\alpha_{21}|\cdot\|A\|^1(T)+|\alpha_{22}|\cdot\|C\|^1(T)\big)\,\|\chi_I(\tau_{11})\mathbf{A}\|_{L^{\alpha^*}} \\ &\quad + \big(|\alpha_{21}|\cdot\|B\|^1(T)+|\alpha_{22}|\cdot\|D\|^1(T)\big)\,\|\chi_I(\tau_{21})\mathbf{C}\|_{L^{\alpha^*}}\big] \\ H_2 &= \big[\big(|\alpha_{21}|\cdot\|A\|^1(T)+|\alpha_{22}|\cdot\|C\|^1(T)\big)\,\|\chi_I(\tau_{12})\mathbf{B}\|_{L^{\alpha^*}} \\ &\quad + \big(|\alpha_{21}|\cdot\|B\|^1(T)+|\alpha_{22}|\cdot\|D\|^1(T)\big)\,\|\chi_I(\tau_{22})\mathbf{D}\|_{L^{\alpha^*}}\big]\;. \end{aligned} \tag{35}$$

then from (28), (33), (34) and (35)

$$\|\mathbf{x}(0)\|_{L^\beta} \le |\alpha_{12}|T^{\frac{1}{\beta}}\left\|\mathbf{S}_2^{-1}\right\| H_1\,\|\mathbf{x}\|_{L^\beta} + |\alpha_{12}|T^{\frac{1}{\beta}}\left\|\mathbf{S}_2^{-1}\right\| H_2\,\|\mathbf{u}\|_{L^\beta}\,. \tag{36}$$

By (26), (27), (35) and (36)

$$\begin{aligned} \|\mathbf{x}\|_{L^\beta} \le &\left\{\|\chi_I(\tau_{11})\mathbf{A}\|_{L^\alpha}\left(\frac{2T}{\pi\tau_{11}^*}\right)^{\frac{2}{\beta}}\left(\frac{1}{\tau_{11}^*}\right)^{\frac{1}{\beta}} + |\alpha_{12}|T^{\frac{1}{\beta}}\left\|\mathbf{S}_2^{-1}\right\| H_1\right\}\|\mathbf{x}\|_{L^\beta} \\ &+ \left\{\|\chi_I(\tau_{12})\mathbf{B}\|_{L^\alpha}\left(\frac{2T}{\pi\tau_{12}^*}\right)^{\frac{2}{\beta}}\left(\frac{1}{\tau_{12}^*}\right)^{\frac{1}{\beta}} + |\alpha_{12}|T^{\frac{1}{\beta}}\left\|\mathbf{S}_2^{-1}\right\| H_2\right\}\|\mathbf{u}\|_{L^\beta}\,. \end{aligned} \tag{37}$$

From (7) analogously

$$\begin{aligned} \|\mathbf{u}\|_{L^\beta} \le &\left\{\|\chi_I(\tau_{21})\mathbf{C}\|_{L^\alpha}\left(\frac{2T}{\pi\tau_{21}^*}\right)^{\frac{2}{\beta}}\left(\frac{1}{\tau_{21}^*}\right)^{\frac{1}{\beta}} + |\alpha_{11}|T^{\frac{1}{\beta}}\left\|\mathbf{S}_2^{-1}\right\| H_1\right\}\|\mathbf{x}\|_{L^\beta} \\ &+ \left\{\|\chi_I(\tau_{22})\mathbf{D}\|_{L^\alpha}\left(\frac{2T}{\pi\tau_{22}^*}\right)^{\frac{2}{\beta}}\left(\frac{1}{\tau_{22}^*}\right)^{\frac{1}{\beta}} + |\alpha_{11}|T^{\frac{1}{\beta}}\left\|\mathbf{S}_2^{-1}\right\| H_2\right\}\|\mathbf{u}\|_{L^\beta}\;. \end{aligned} \tag{38}$$

So, on the whole, by (37) and (38)

$$\begin{pmatrix}\|\mathbf{x}\|_{L^\beta}\\ \|\mathbf{u}\|_{L^\beta}\end{pmatrix} \le \mathbf{M}\begin{pmatrix}\|\mathbf{x}\|_{L^\beta}\\ \|\mathbf{u}\|_{L^\beta}\end{pmatrix}, \tag{39}$$

where

$$\mathbf{M} = \begin{pmatrix}\|\chi_I(\tau_{11})\mathbf{A}\|_{L^\alpha}T_{11}^* + U_{21} & \|\chi_I(\tau_{12})\mathbf{B}\|_{L^\alpha}T_{12}^* + U_{22}\\ \|\chi_I(\tau_{21})\mathbf{C}\|_{L^\alpha}T_{21}^* + U_{11} & \|\chi_I(\tau_{22})\mathbf{D}\|_{L^\alpha}T_{22}^* + U_{12}\end{pmatrix} \tag{40}$$

and for $i,j=1,2$

$$T_{ij}^{*}=\left(\frac{2T}{\pi\tau_{ij}^{*}}\right)^{\frac{2}{\beta}}\left(\frac{1}{\tau_{ij}^{*}}\right)^{\frac{1}{\beta}}\,,\quad U_{ij}=|\alpha_{1i}|T^{\frac{1}{\beta}}\|\mathbf{S}_2^{-1}\|H_j$$

is valid and from inequality (39) evidently

$$(\mathbf{E}_2-\mathbf{M})\begin{pmatrix}\|\mathbf{x}\|_{L^\beta}\\ \|\mathbf{u}\|_{L^\beta}\end{pmatrix}\le\theta_2\,.$$

Now, if

$$r(\mathbf{M})<1 \tag{41}$$

holds then the matrix $(\mathbf{E}_2-\mathbf{M})^{-1}$ inverse to the matrix $(\mathbf{E}_2-\mathbf{M})$ exists and

$$\begin{pmatrix}\|\mathbf{x}\|_{L^\beta}\\ \|\mathbf{u}\|_{L^\beta}\end{pmatrix}\le(\mathbf{E}_2-\mathbf{M})^{-1}\,\theta_2=\theta_2$$

is valid, so

$$\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix}\equiv\theta_{2n}$$

and the problem (6_0), (3_0) has only trivial solution. According to Theorem 1, we obtained, that the condition of invertibility of the matrix (13) for $k=2$, conditions (23), (24) for $\tau_{ij}^{0}(t)$ and condition (41) are sufficient conditions for existence and uniqueness of the solution of the problem (1), (2), (3). By this result we can formulate the following theorem - sufficient condition for existence and uniqueness of the solution in effective form.

Theorem 3. *Let conditions* (23), (24) *are valid, the matrix* (13) *is invertible for* $k=2$ *and the matrix* (40) *the condition* (41) *holds, where* $1\le\alpha\le+\infty$, $\frac{1}{\alpha}+\frac{2}{\beta}=1$, $\frac{1}{\alpha^*}+\frac{1}{\beta}=1$. *Then the problem* (1), (2), (3) *is uniquely solvable.*

Note 2. Let $\alpha_{11}=\alpha_{21}=1$ and $\alpha_{12}=\alpha_{22}=0$. Then we obtain in the both of cases (Theorem 2, Theorem 3) the same results as in [3].

The following corollary for Cauchy-Nicoletti problem we can formulated.

Corollary 3. *Let* $\alpha_{11}=\alpha_{22}=1$, $\alpha_{12}=\alpha_{21}=0$, *conditions* (23), (24) *are valid, the matrix*

$$V=-\left[\mathbf{E}_{2n}+D^1(T)\right]$$

is invertible and the matrix

$$\mathbf{M}=\begin{pmatrix}\|\chi_I(\tau_{11})\mathbf{A}\|_{L^\alpha}T_{11}^{*} & \|\chi_I(\tau_{12})\mathbf{B}\|_{L^\alpha}T_{12}^{*}\\ \|\chi_I(\tau_{21})\mathbf{C}\|_{L^\alpha}T_{21}^{*} & \|\chi_I(\tau_{22})\mathbf{D}\|_{L^\alpha}T_{22}^{*}\end{pmatrix}+T^{\frac{1}{\beta}}\|V^{-1}\|$$

$$\begin{pmatrix}0 & \|C\|^1(T)\|\chi_I(\tau_{11})\mathbf{A}\|_{L^{\alpha^*}}+\|D\|^1(\tau)\|\chi_I(\tau_{21})\mathbf{C}\|_{L^{\alpha^*}}\\ 0 & \|C\|^1(T)\|\chi_I(\tau_{12})\mathbf{B}\|_{L^{\alpha^*}}+\|D\|^1(T)\|\chi_I(\tau_{22})\mathbf{D}\|_{L^{\alpha^*}}\end{pmatrix}^{\top}$$

the condition (41) *holds, where* $1 \le \alpha \le +\infty$, $\frac{1}{\alpha} + \frac{2}{\beta} = 1$, $\frac{1}{\alpha^*} + \frac{1}{\beta} = 1$. *Then the problem* (1), (2), (3) *is uniquely solvable.*

Considering the following differential system on I with delay (retarded argument)

$$\begin{aligned} \mathbf{x}'(t) &= \mathbf{A}\mathbf{x}(t-\Delta) + \mathbf{B}\mathbf{u}(t-\Delta) + \mathbf{q}_1(t)\,, \\ \mathbf{u}'(t) &= \mathbf{C}\mathbf{x}(t-\Delta) + \mathbf{D}\mathbf{u}(t-\Delta) + \mathbf{q}_2(t)\,, \end{aligned} \tag{1'}$$

with boundary conditions (2) and (3), where $\mathbf{q}_1,\ \mathbf{q}_2 \in L(I;\Re^n)$, $\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D} \in \Re^{n\times n}$, $\Delta \in \Re_+$, Theorem 3 implies

Corollary 4. *Let the matrix*

$$\mathbf{S}_2 = (\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22})\mathbf{E}_n + [\alpha_{12}(\alpha_{21}\mathbf{A} + \alpha_{22}\mathbf{C}) - \alpha_{11}(\alpha_{21}\mathbf{B} + \alpha_{22}\mathbf{D})]\max\{0; T-\Delta\}$$

is invertible and the matrix

$$\mathbf{M} = \begin{pmatrix} \|\mathbf{A}\|\left(\frac{2T}{\pi}\right)^{\frac{2}{\beta}} + W_{21} & \|\mathbf{C}\|\left(\frac{2T}{\pi}\right)^{\frac{2}{\beta}} + W_{11} \\ \|\mathbf{B}\|\left(\frac{2T}{\pi}\right)^{\frac{2}{\beta}} + W_{22} & \|\mathbf{D}\|\left(\frac{2T}{\pi}\right)^{\frac{2}{\beta}} + W_{12} \end{pmatrix} (\max\{0; T-\Delta\})^{\frac{\beta-2}{\beta}}$$

where for $i, j = 1, 2$

$$W_{ij} = |\alpha_{1i}| T^{\frac{1}{\beta}} \|\mathbf{S}_2^{-1}\| H_j \, \max{}^{\frac{2\beta-1}{\beta-2}}\{0; T-\Delta\}$$

and

$$\begin{aligned} H_1 &= [(|\alpha_{21}|\|\mathbf{A}\| + |\alpha_{22}|\|\mathbf{C}\|)\,\|\mathbf{A}\| + (|\alpha_{21}|\|\mathbf{B}\| + |\alpha_{22}|\|\mathbf{D}\|)\,\|\mathbf{C}\|] \\ H_2 &= [(|\alpha_{21}|\|\mathbf{A}\| + |\alpha_{22}|\|\mathbf{C}\|)\,\|\mathbf{B}\| + (|\alpha_{21}|\|\mathbf{B}\| + |\alpha_{22}|\|\mathbf{D}\|)\,\|\mathbf{D}\|] \end{aligned}$$

the condition (41) *holds, where* $2 \le \beta \le +\infty$. *Then the problem* (1'), (2), (3) *is uniquely solvable.*

Proof. The result is obtained from Theorem 3 by directly way. In this special case, Theorem 3 gives using (4) and (5) that

$$\tau_{ij}^0(t) = \begin{cases} 0 & t < \Delta \\ t-\Delta & t \ge \Delta \end{cases}; \qquad t \in I,\ i,j \in \{1,2\}\,,$$

$$\chi_I(\tau_{ij}(t)) = \begin{cases} 1 & t \ge \Delta \\ 0 & t < \Delta \end{cases}; \qquad t \in I,\ i,j \in \{1,2\}\,.$$

From (9) we obtain

$$A^1(t) = \int_0^t [\chi_I(s-\Delta)\mathbf{A}]\,ds = \mathbf{A}\int_0^t \chi_I(s-\Delta)\,ds = \mathbf{A}\max\{0, t-\Delta\}$$

and for $B^1(t),\ C^1(t),\ D^1(t)$ analogously. From (32) we obtain

$$||A||^1(t) = \int_0^t ||\chi_I(s-\Delta)\mathbf{A}||\ ds = ||\mathbf{A}|| \int_0^t ||\chi_I(s-\Delta)||\ ds = ||\mathbf{A}|| \max\{0, t-\Delta\}$$

and for $||B||^1(t),\ ||C||^1(t),\ ||D||^1(t)$ analogously. Clearly

$$\begin{aligned} ||\chi_I(s-\Delta)\mathbf{A}||_{L^\alpha} &= \left(\int_0^T ||\chi_I(s-\Delta)\mathbf{A}||^\alpha\ ds\right)^{\frac{1}{\alpha}} \\ &= ||\mathbf{A}|| \left(\int_0^T ||\chi_I(s-\Delta)||\ ds\right)^{\frac{1}{\alpha}} = ||\mathbf{A}|| \left(\max\{0; T-\Delta\}\right)^{\frac{1}{\alpha}} \end{aligned}$$

and for $||\chi_I(s-\Delta)\mathbf{B}||_{L^\alpha},\ ||\chi_I(s-\Delta)\mathbf{C}||_{L^\alpha},\ ||\chi_I(s-\Delta)\mathbf{D}||_{L^\alpha}$ analogously. The results for α^* we obtain by the same way. So

$$\frac{1}{\alpha} = 1 - \frac{2}{\beta} = \frac{\beta-2}{\beta} \quad \text{and} \quad \frac{1}{\alpha^*} = 1 - \frac{1}{\beta} = \frac{\beta-1}{\beta}\,,$$

if $\beta = +\infty$ we understand, for example, by $\frac{\beta-2}{\beta}$

$$\lim_{\beta\to+\infty} \frac{\beta-2}{\beta}\,.$$

Evidently, the condition (24) is not valid on segment $\langle 0, \Delta)$. Hence

$$\begin{aligned} &\left(\int_0^T \left(\int_0^t \left\|\chi_I(s-\Delta)\mathbf{A}\mathbf{x}(\tau_{11}^0(s))\right\|\ ds\right)^\beta dt\right)^{\frac{1}{\beta}} \\ &\qquad = \left(\int_\Delta^T \left(\int_\Delta^t ||\chi_I(s-\Delta)\mathbf{A}\mathbf{x}(s-\Delta)||\ ds\right)^\beta dt\right)^{\frac{1}{\beta}} = (*) \end{aligned}$$

and by Lemma 1, where $f = \chi_I(s-\Delta)\mathbf{A},\ g = \mathbf{x}(s-\Delta)$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ we obtain

$$(*) \le ||\chi_I(s-\Delta)\mathbf{A}||_{L^\alpha} \left[\int_\Delta^T \left(\int_\Delta^t ||\mathbf{x}(s-\Delta)||^{\frac{\beta}{2}}\ ds\right)^2 dt\right]^{\frac{1}{\beta}} = (**)$$

if $\xi = s - \Delta$, then $d\xi = ds$ and

$$\begin{aligned} (**) &= ||\chi_I(s-\Delta)\mathbf{A}||_{L^\alpha} \left[\int_\Delta^T \left(\int_0^{t-\Delta} ||\mathbf{x}(\xi)||^{\frac{\beta}{2}}\ d\xi\right)^2 dt\right]^{\frac{1}{\beta}} \\ &\le ||\chi_I(s-\Delta)\mathbf{A}||_{L^\alpha} \left[\int_0^T \left(\int_0^t ||\mathbf{x}(\xi)||^{\frac{\beta}{2}}\ d\xi\right)^2 dt\right]^{\frac{1}{\beta}}. \qquad \square \end{aligned}$$

Note 3. The system (1) can be rewrited as follows

$$\mathbf{y}'(t) = \sum_{i=1}^{4} \mathbf{P}_i(t)\mathbf{y}(\tau_i(t)) + \mathbf{q}(t)\,,$$

where

$$\mathbf{y} = \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix}, \quad \mathbf{P}_1(t) = \begin{pmatrix} \mathbf{A}(t) & \mathbf{\Theta}_n \\ \mathbf{\Theta}_n & \mathbf{\Theta}_n \end{pmatrix}, \quad \mathbf{P}_2(t) = \begin{pmatrix} \mathbf{\Theta}_n & \mathbf{B}(t) \\ \mathbf{\Theta}_n & \mathbf{\Theta}_n \end{pmatrix},$$
$$\mathbf{P}_3(t) = \begin{pmatrix} \mathbf{\Theta}_n & \mathbf{\Theta}_n \\ \mathbf{C}(t) & \mathbf{\Theta}_n \end{pmatrix}, \quad \mathbf{P}_4(t) = \begin{pmatrix} \mathbf{\Theta}_n & \mathbf{\Theta}_n \\ \mathbf{\Theta}_n & \mathbf{D}(t) \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{pmatrix},$$
$$\tau_1(t) = \tau_{11}(t)\,, \quad \tau_2(t) = \tau_{12}(t)\,, \quad \tau_3(t) = \tau_{21}(t)\,, \quad \tau_4(t) = \tau_{22}(t)\,.$$

So, the problem for arbitrary finite integer $N \in \Re_+$, $i = 1, \dots, N$ can be considered.

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