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ON AN ANTIPERIODIC TYPE BOUNDARY VALUE PROBLEM FOR FIRST ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Nonimprovable, in a certain sense, sufficient conditions for the unique solvability of the boundary value problem

$$u'(t) = \ell(u)(t) + q(t), \qquad u(a) + \lambda u(b) = c$$

are established, where $\ell : C([a,b];R) \to L([a,b];R)$ is a linear bounded operator, $q \in L([a,b];R)$, $\lambda \in R_+$, and $c \in R$. The question on the dimension of the solution space of the homogeneous problem

$$u'(t) = \ell(u)(t), \qquad u(a) + \lambda u(b) = 0$$

is discussed as well.

INTRODUCTION

The following notation is used throughout.

R is the set of all real numbers, $R_+ = [0, +\infty[$.

$C([a,b];R)$ is the Banach space of continuous functions $u : [a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : a \le t \le b\}$.

$C([a,b];R_+) = \{u \in C([a,b];R) : u(t) \ge 0 \text{ for } t \in [a,b]\}$.

$\widetilde{C}([a,b];R)$ is the set of absolutely continuous functions $u : [a,b] \to R$.

$L([a,b];R)$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\,ds$.

$L([a,b];R_+) = \{p \in L([a,b];R) : p(t) \ge 0 \text{ for almost all } t \in [a,b]\}$.

$\mathcal{M}_{ab}$ is the set of measurable functions $\tau : [a,b] \to [a,b]$.

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];R) \to L([a,b];R)$.

$\mathcal{P}_{ab}$ is the set of linear operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C([a,b];R_+)$ into the set $L([a,b];R_+)$.

$[x]_+ = \frac{1}{2}(|x| + x)$, $[x]_- = \frac{1}{2}(|x| - x)$.

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By a solution of the equation

$$u'(t) = \ell(u)(t) + q(t)\,, \tag{1}$$

where $\ell \in \mathcal{L}_{ab}$, $q \in L([a,b];R)$, we understand a function $u \in \widetilde{C}([a,b];R)$ satisfying the equation (1) almost everywhere in $[a,b]$.

Consider the problem on the existence and uniqueness of a solution of (1) satisfying the boundary condition

$$u(a) + \lambda u(b) = c\,, \tag{2}$$

where $\lambda \in R_+$, $c \in R$.

The general boundary value problems for functional differential equations have been studied very intensively. There are a lot of general results (see, e.g., [1–27]), but still only a few effective criteria for the solvability of special boundary value problems for functional differential equations are known even in the linear case. In the present paper, we try to fill to some extent the existing gap. More precisely, in Section 1 we give nonimprovable effective sufficient conditions for the unique solvability of the problem (1), (2), and the condition for the homogeneous problem (1_0), (2_0) to have at most one–dimensional space of solutions. Sections 2 and 3 are devoted respectively to the proofs of the main results and the examples verifying their optimality.

All results will be concretized for the differential equation with deviating arguments of the form

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + q(t)\,, \tag{3}$$

where $p, g \in L([a,b];R_+)$, $q \in L([a,b];R)$, $\tau, \mu \in \mathcal{M}_{ab}$.

The special case of the discussed boundary value problem is the Cauchy problem (for $\lambda = 0$). In this case, the below Theorem 1.1 coincides with Theorem 1.3 obtained in [4]. The periodic type boundary value problem (for $\lambda < 0$) for the linear equation is studied in [14].

Along with the problem (1), (2) we consider the corresponding homogeneous problem

$$u'(t) = \ell(u)(t), \tag{1_0}$$

$$u(a) + \lambda u(b) = 0\,. \tag{2_0}$$

From the general theory of linear boundary value problems for functional differential equations the following result is well–known (see, e.g., [1, 2, 3, 19, 27]).

Theorem 0.1. *The problem* (1), (2) *is uniquely solvable if and only if the corresponding homogeneous problem* (1_0), (2_0) *has only the trivial solution.*

1. Main Results

1.1. Existence and Uniqueness Theorems.

Theorem 1.1. *Let* $\lambda \in \,]0,1]$, *the operator* ℓ *admit the representation* $\ell = \ell_0 - \ell_1$, *where*

$$\ell_0, \ell_1 \in \mathcal{P}_{ab}\,, \tag{4}$$

and let either

$$\|\ell_0(1)\|_L < 1 - \lambda^2\,, \tag{5}$$

$$\|\ell_1(1)\|_L < 1 - \lambda + 2\sqrt{1 - \|\ell_0(1)\|_L}\,, \tag{6}$$

or

$$1 - \lambda^2 \le \|\ell_0(1)\|_L\,, \tag{7}$$

$$\|\ell_0(1)\|_L + \lambda\|\ell_1(1)\|_L < 1 + \lambda\,. \tag{8}$$

Then the problem (1), (2) *has a unique solution.*

Remark 1.1. Let $\lambda \in [1, +\infty[$ and $\ell = \ell_0 - \ell_1$, where ℓ_0, ℓ_1 satisfy the condition (4). Define operator $\psi : L([a,b];R) \to L([a,b];R)$ by

$$\psi(w)(t) \stackrel{\text{def}}{=} w(a+b-t) \quad \text{for} \quad t \in [a,b]\,.$$

Let φ be a restriction of ψ to the space $C([a,b];R)$. Put $\mu = \frac{1}{\lambda}$, and

$$\widehat{\ell}_0(w)(t) \stackrel{\text{def}}{=} \psi(\ell_0(\varphi(w)))(t)\,, \qquad \widehat{\ell}_1(w)(t) \stackrel{\text{def}}{=} \psi(\ell_1(\varphi(w)))(t) \quad \text{for} \quad t \in [a,b]\,.$$

It is clear that if u is a solution of the problem (1_0), (2_0), then the function $v \stackrel{\text{def}}{=} \varphi(u)$ is a solution of the problem

$$v'(t) = \widehat{\ell}_1(v)(t) - \widehat{\ell}_0(v)(t)\,, \qquad v(a) + \mu v(b) = 0\,, \tag{9}$$

and vice versa, if v is a solution of the problem (9), then the function $u \stackrel{\text{def}}{=} \varphi(v)$ is a solution of the problem (1_0), (2_0).

It is also evident that

$$\|\widehat{\ell}_0(1)\|_L = \|\ell_0(1)\|_L\,, \qquad \|\widehat{\ell}_1(1)\|_L = \|\ell_1(1)\|_L\,.$$

Therefore, from Theorem 1.1 it immediately follows

Theorem 1.2. *Let $\lambda \in [1, +\infty[$, the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0$and ℓ_1 satisfy the condition* (4)*, and let either*

$$\|\ell_1(1)\|_L < 1 - \frac{1}{\lambda^2}\,,$$

$$\|\ell_0(1)\|_L < 1 - \frac{1}{\lambda} + 2\sqrt{1 - \|\ell_1(1)\|_L}\,,$$

or

$$1 - \frac{1}{\lambda^2} \le \|\ell_1(1)\|_L\,, \qquad \|\ell_0(1)\|_L + \lambda\|\ell_1(1)\|_L < 1 + \lambda\,.$$

Then the problem (1), (2) *has a unique solution.*

Remark 1.2. Below we give examples (see Examples 3.1–3.4) showing that neither of the strict inequalities (6) and (8) can be replaced by the nonstrict ones. According to Remark 1.1 and the above–said, neither of the strict inequalities in Theorem 1.2 can be replaced by the nonstrict ones.

For the equation of the type (3), from Theorems 1.1 and 1.2 we get the following assertions.

Corollary 1.1. *Let $\lambda \in]0,1]$, $p, g \in L([a,b];R_+)$, and let either*

$$\int_a^b p(s)\,ds < 1-\lambda^2, \qquad \int_a^b g(s)\,ds < 1-\lambda+2\sqrt{1-\int_a^b p(s)\,ds}\,,$$

or

$$1-\lambda^2 \le \int_a^b p(s)\,ds\,, \qquad \int_a^b p(s)\,ds+\lambda\int_a^b g(s)\,ds < 1+\lambda\,.$$

Then the problem (3), (2) *has a unique solution.*

Corollary 1.2. *Let $\lambda \in [1,+\infty[$, $p, g \in L([a,b];R_+)$, and let either*

$$\int_a^b g(s)\,ds < 1-\frac{1}{\lambda^2}\,, \qquad \int_a^b p(s)\,ds < 1-\frac{1}{\lambda}+2\sqrt{1-\int_a^b g(s)\,ds}\,,$$

or

$$1-\frac{1}{\lambda^2} \le \int_a^b g(s)\,ds\,, \qquad \int_a^b p(s)\,ds+\lambda\int_a^b g(s)ds < 1+\lambda\,.$$

Then the problem (3), (2) *has a unique solution.*

1.2. On the Dimension of Solution Space of the Problem (1), (2).

Theorem 1.3. *Let the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where ℓ_0 and ℓ_1 satisfy the condition* (4), *and let either*

$$\|\ell_0(1)\|_L < 1\,, \qquad \|\ell_1(1)\|_L < 2+2\sqrt{1-\|\ell_0(1)\|_L}\,, \tag{10}$$

or

$$\|\ell_1(1)\|_L < 1\,, \qquad \|\ell_0(1)\|_L < 2+2\sqrt{1-\|\ell_1(1)\|_L}\,. \tag{11}$$

Then the space of solutions of the problem (1_0), (2_0) *is at most one–dimensional.*

Corollary 1.3. *Let either*

$$\int_a^b p(s)\,ds < 1\,, \qquad \int_a^b g(s)\,ds < 2+2\sqrt{1-\int_a^b p(s)\,ds}\,,$$

or

$$\int_a^b g(s)\,ds < 1\,, \qquad \int_a^b p(s)\,ds < 2+2\sqrt{1-\int_a^b g(s)\,ds}\,.$$

Then the space of solutions of the problem

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t))\,, \qquad u(a)+\lambda u(b) = 0$$

is at most one–dimensional.

Theorem 1.4. *Let the dimension of the solution space of the equation* (1_0) *is* $n < +\infty$. *Then the dimension of the solution space of the problem* (1_0), (2_0) *is either* n *or* $n-1$.

2. Proofs of the Main Results

Proof of Theorem 1.1. According to Theorem 0.1 it is sufficient to show that the homogeneous problem (1_0), (2_0) has no nontrivial solution.

Assume the contrary that the problem (1_0), (2_0) has a nontrivial solution u. From (2_0) it follows that u has a zero. Put

$$M = \max\{u(t) : t \in [a,b]\}\,, \qquad m = -\min\{u(t) : t \in [a,b]\}\,, \tag{12}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M\,, \qquad u(t_m) = -m\,. \tag{13}$$

Obviously, $M \geq 0$, $m \geq 0$, and at least one of these inequalities is strict. Without loss of generality we can assume that $t_M < t_m$.

The integration of (1_0) from a to t_M, and from t_m to b, in view of (4), (12) and (13), results in

$$M - u(a) = \int\limits_a^{t_M} [\ell_0(u)(s) - \ell_1(u)(s)]\, ds \leq M \int\limits_a^{t_M} \ell_0(1)(s) ds + m \int\limits_a^{t_M} \ell_1(1)(s)\, ds\,,$$

$$u(b) + m = \int\limits_{t_m}^{b} [\ell_0(u)(s) - \ell_1(u)(s)]\, ds \leq M \int\limits_{t_m}^{b} \ell_0(1)(s)\, ds + m \int\limits_{t_m}^{b} \ell_1(1)(s)\, ds\,.$$

Summing the last two inequalities and taking into account (2_0), (12) and the assumption $\lambda \in\,]0,1]$, we obtain

$$M + m - m(1+\lambda) \leq M + m + u(b)(1+\lambda) \leq M \int\limits_J \ell_0(1)(s)\, ds + m \int\limits_J \ell_1(1)(s)\, ds$$

and

$$M + m - M\left(1 + \frac{1}{\lambda}\right) \leq M + m - u(a)\left(1 + \frac{1}{\lambda}\right) \leq M \int\limits_J \ell_0(1)(s)\, ds + m \int\limits_J \ell_1(1)(s)\, ds\,,$$

where $J = [a, t_M] \cup [t_m, b]$. Thus

$$M - \lambda m \leq MC + mA, \tag{14}$$

and

$$m - \frac{1}{\lambda} M \leq MC + mA\,, \tag{15}$$

where

$$A = \int\limits_J \ell_1(1)(s)\, ds\,, \qquad C = \int\limits_J \ell_0(1)(s)\, ds\,. \tag{16}$$

On the other hand, the integration of (1_0) from t_M to t_m, on account of (4), (12) and (13), implies

$$M + m = \int_{t_M}^{t_m} [\ell_1(u)(s) - \ell_0(u)(s)]\, ds \le M \int_{t_M}^{t_m} \ell_1(1)(s)\, ds + m \int_{t_M}^{t_m} \ell_0(1)(s)\, ds\,.$$

Hence

$$M + m \le MB + mD\,, \tag{17}$$

where

$$B = \int_{t_M}^{t_m} \ell_1(1)(s)\, ds\,, \qquad D = \int_{t_M}^{t_m} \ell_0(1)(s)\, ds\,. \tag{18}$$

First suppose that $\|\ell_0(1)\|_L \ge 1$ holds. It is clear that it may happen only in the case where the conditions (7) and (8) are fulfilled. According to (8), $\|\ell_1(1)\|_L < 1$, hence $A < 1$ and $B < 1$. Therefore, from (15) and (17) it follows that

$$0 \le m(1-A) \le M\left(C + \frac{1}{\lambda}\right), \qquad 0 \le M(1-B) \le m(D-1)\,. \tag{19}$$

Consequently, (19) implies $M > 0$, $m > 0$, $D > 1$, and

$$0 < (1-A)(1-B) \le \left(C + \frac{1}{\lambda}\right)(D-1)\,. \tag{20}$$

Obviously,

$$(1-A)(1-B) \ge 1 - (A+B) = 1 - \|\ell_1(1)\|_L\,. \tag{21}$$

According to (8) and the assumption $\lambda \in\,]0,1]$, $\|\ell_0(1)\|_L < 1 + \frac{1}{\lambda}$ holds. Hence, $D - 1 < \frac{1}{\lambda}$, and so

$$\left(C + \frac{1}{\lambda}\right)(D-1) = \frac{1}{\lambda}D - \frac{1}{\lambda} + C(D-1) \le \frac{1}{\lambda}(C+D) - \frac{1}{\lambda} = \frac{1}{\lambda}\|\ell_0(1)\|_L - \frac{1}{\lambda}\,.$$

By the last inequality and (21), it follows from (20) that

$$1 - \|\ell_1(1)\|_L \le \frac{1}{\lambda}\|\ell_0(1)\|_L - \frac{1}{\lambda}\,,$$

which contradicts the inequality (8).

Now suppose that $\|\ell_0(1)\|_L < 1$. Obviously, $C < 1$, $D < 1$, and by (14) and (17) we get

$$0 \le M(1-C) \le m(A+\lambda)\,, \qquad 0 \le m(1-D) \le M(B-1)\,. \tag{22}$$

Consequently, (22) implies $M > 0$, $m > 0$, $B > 1$, and

$$0 < (1-C)(1-D) \le (A+\lambda)(B-1)\,. \tag{23}$$

It is clear that

$$(1-C)(1-D) \ge 1 - (C+D) = 1 - \|\ell_0(1)\|_L\,. \tag{24}$$

If (7) and (8) hold, then we obtain $\|\ell_1(1)\|_L < 1+\lambda$. Hence, $B-1<\lambda$, and

$$(A+\lambda)(B-1) = \lambda B - \lambda + A(B-1) \le \lambda(A+B) - \lambda = \lambda\|\ell_1(1)\|_L - \lambda\,.$$

By the last inequality and (24), it follows from (23) that

$$1 - \|\ell_0(1)\|_L \le \lambda\|\ell_1(1)\|_L - \lambda\,,$$

which contradicts the inequality (8).

If (5) and (6) are satisfied, then according to

$$4(A+\lambda)(B-1) \le \big(A+B-(1-\lambda)\big)^2 = \Big(\|\ell_1(1)\|_L - (1-\lambda)\Big)^2$$

and (24), the inequality (23) implies

$$0 < 4(1 - \|\ell_0(1)\|_L) \le \Big(\|\ell_1(1)\|_L - (1-\lambda)\Big)^2\,,$$

which contradicts the inequality (6). □

To prove Theorem 1.3, we need the following

Lemma 2.1. *Let the condition* (10) *or* (11) *be fulfilled. Then there is no nontrivial solution* u *of the equation* (1_0) *satisfying*

$$u(a) = 0\,, \qquad u(b) = 0\,. \tag{25}$$

Proof. Suppose that (10) is fulfilled. The case where (11) holds can be proved analogously.

Assume the contrary that u is a nontrivial solution of (1_0) satisfying (25). First suppose that u does not change its sign. Without loss of generality we can assume that

$$u(t) \ge 0 \quad \text{for} \quad t \in [a,b]\,. \tag{26}$$

Put

$$\overline{M} = \max\{u(t) : t \in [a,b]\}\,, \tag{27}$$

and choose $t_0 \in\,]a,b[$ such that

$$u(t_0) = \overline{M}\,. \tag{28}$$

Obviously, $\overline{M} > 0$. The integration of (1_0) from a to t_0, in view of (4), (25)–(28), results in

$$\overline{M} = \int_a^{t_0} \ell_0(u)(s)\,ds - \int_a^{t_0} \ell_1(u)(s)\,ds \le \overline{M}\int_a^{t_0} \ell_0(1)(s)\,ds \le \overline{M}\|\ell_0(1)\|_L\,,$$

which contradicts the first inequality in (10).

Now suppose that u changes its sign. Define the numbers M and m by (12), and choose $t_M, t_m \in\,]a,b[$ such that (13) is fulfilled. Obviously, $M>0$, $m>0$. Without loss of generality we can assume that $t_M < t_m$. The integration of (1_0)

from a to t_M, from t_M to t_m, and from t_m to b, on account of (4), (12), (13), and (25), yields

$$M = \int_a^{t_M} \ell_0(u)(s)\,ds - \int_a^{t_M} \ell_1(u)(s)ds \le M\int_a^{t_M} \ell_0(1)(s)\,ds + m\int_a^{t_M} \ell_1(1)(s)\,ds, \tag{29}$$

$$M + m = \int_{t_M}^{t_m} \ell_1(u)(s)\,ds - \int_{t_M}^{t_m} \ell_0(u)(s)\,ds \le M\int_{t_M}^{t_m} \ell_1(1)(s)ds + m\int_{t_M}^{t_m} \ell_0(1)(s)\,ds\,, \tag{30}$$

$$m = \int_{t_m}^{b} \ell_0(u)(s)\,ds - \int_{t_m}^{b} \ell_1(u)(s)\,ds \le M\int_{t_m}^{b} \ell_0(1)(s)\,ds + m\int_{t_m}^{b} \ell_1(1)(s)\,ds\,. \tag{31}$$

Summing (29) and (31) we obtain

$$M + m \le M\int_J \ell_0(1)(s)\,ds + m\int_J \ell_1(1)(s)\,ds\,,$$

where $J = [a, t_M] \cup [t_m, b]$. From the last inequality and (30) it follows that

$$M(1 - C) \le m(A - 1)\,, \qquad m(1 - D) \le M(B - 1)\,, \tag{32}$$

where A, C and B, D are defined by (16) and (18). According to the first inequality in (10), $C < 1$, $D < 1$. Therefore $A > 1$, $B > 1$, and (32) implies

$$0 < (1 - C)(1 - D) \le (A - 1)(B - 1)\,. \tag{33}$$

Obviously,

$$(1 - C)(1 - D) \ge 1 - (C + D) = 1 - \|\ell_0(1)\|_L > 0\,,$$

$$4(A - 1)(B - 1) \le (A + B - 2)^2 = \Big(\|\ell_1(1)\|_L - 2\Big)^2\,.$$

By the last inequalities, from (33) we find

$$0 < 4(1 - \|\ell_0(1)\|_L) \le \Big(\|\ell_1(1)\|_L - 2\Big)^2\,,$$

which contradicts the second inequality in (10). $\square$

Proof of Theorem 1.3. Assume the contrary that the dimension of the space of solutions of the problem (1_0), (2_0) is greater than one. Then there exist at least two linearly independent solutions v and w of (1_0), (2_0). According to Lemma 2.1, $v(a) \ne 0$, $w(a) \ne 0$. Put

$$u(t) = v(a)w(t) - w(a)v(t) \quad \text{for} \quad t \in [a, b]\,.$$

Since v and w are linearly independent, then

$$u \not\equiv 0\,. \tag{34}$$

On the other hand, it is clear that u is a solution of (1_0) satisfying (25). By Lemma 2.1, $u \equiv 0$, which contradicts (34). $\square$

Proof of Theorem 1.4. Denote by k the dimension of the solution space of the problem (1_0), (2_0). Obviously, $n \geq 1$ and $k \leq n$. If $n = 1$, then the theorem is valid. Therefore, suppose that $n \geq 2$.

Let $k < n-1$. Then it is clear that there exist linearly independent solutions $u_1, \dots, u_n$ of equation (1_0) such that $u_1, \cdots, u_k$ are solutions of the problem (1_0), (2_0). Put

$$v_0(t) = (u_{n-1}(a) + \lambda u_{n-1}(b))u_n(t) - (u_n(a) + \lambda u_n(b))u_{n-1}(t)\,.$$

Since $u_{n-1}(a) + \lambda u_{n-1}(b) \neq 0$, $u_n(a) + \lambda u_n(b) \neq 0$, and u_{n-1}, u_n are linearly independent, then $v_0 \not\equiv 0$. On the other hand, v_0 is a solution of (1_0), (2_0) and $u_1, \dots u_k, v_0$ are linearly independent. Consequently, the number of linearly independent solutions of (1_0), (2_0) is $k+1$, which contradicts our assumption. $\square$

3. On Remark 1.2

Let $\lambda \in \,]0,1]$ (for the case $\lambda = 0$, see [4]). Denote by H the set of pairs $(x,y) \in R_+ \times R_+$ satisfying either

$$x \leq 1 - \lambda^2\,, \qquad y < 1 - \lambda + 2\sqrt{1-x}\,,$$

or

$$1 - \lambda^2 < x < 1 + \lambda\,, \qquad y < 1 + \frac{1}{\lambda} - \frac{x}{\lambda}\,.$$

According to Theorem 1.1, if $(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L) \in H$, then the problem (1), (2) has a unique solution.

Below we give the examples which show that for any pair $(x_0, y_0) \notin H$, $x_0 \geq 0$, $y_0 \geq 0$ there exist functions $h \in L([a,b];R)$ and $\tau \in \mathcal{M}_{ab}$ such that

$$\int_a^b [h(s)]_+ ds = x_0\,, \qquad \int_a^b [h(s)]_- ds = y_0\,, \tag{35}$$

and the problem

$$u'(t) = h(t)u(\tau(t))\,, \qquad u(a) + \lambda u(b) = 0 \tag{36}$$

has a nontrivial solution. Then by Theorem 0.1 there exist $q \in L([a,b];R)$ and $c \in R$ such that the problem (1), (2), where $\ell = \ell_0 - \ell_1$,

$$\ell_0(w)(t) \stackrel{\text{def}}{=} [h(t)]_+ w(\tau(t))\,, \qquad \ell_1(w)(t) \stackrel{\text{def}}{=} [h(t)]_- w(\tau(t))\,, \tag{37}$$

either has no solution or has infinitely many solutions.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin H$, then (x_0, y_0) belongs to at least one of the following sets

$$\begin{aligned}
&H_1 = \{(x,y) \in R \times R \;:\; 0 \leq y < 1,\ -\lambda y + 1 + \lambda \leq x\}\,,\\
&H_2 = \{(x,y) \in R \times R \;:\; 1 \leq x,\ 1 \leq y\}\,,\\
&H_3 = \left\{(x,y) \in R \times R \;:\; 1 - \lambda^2 \leq x < 1,\ -\frac{x}{\lambda} + 1 + \frac{1}{\lambda} \leq y\right\},\\
&H_4 = \left\{(x,y) \in R \times R \;:\; 0 \leq x \leq 1 - \lambda^2,\ 1 - \lambda + 2\sqrt{1-x} \leq y\right\}.
\end{aligned}$$

Example 3.1. Let $(x_0, y_0) \in H_1$. Put $a = 0$, $b = 2$, $\alpha = \frac{\lambda - \lambda y_0 + 1}{1 - y_0}$, $\beta = \frac{y_0}{1 - y_0}$, $t_0 = \frac{\lambda}{\alpha} + \frac{1}{x_0}$,

$$h(t) = \begin{cases} x_0 & \text{for} \quad t \in [0,1[\\ -y_0 & \text{for} \quad t \in [1,2] \end{cases}, \quad \tau(t) = \begin{cases} t_0 & \text{for} \quad t \in [0,1[\\ 1 & \text{for} \quad t \in [1,2] \end{cases}.$$

Then (35) holds, and the problem (36) has the nontrivial solution

$$u(t) = \begin{cases} -\alpha t + \lambda & \text{for} \quad t \in [0,1[\\ \beta(t-2) - 1 & \text{for} \quad t \in [1,2] \end{cases}.$$

Example 3.2. Let $(x_0, y_0) \in H_2$. Put $a = 0$, $b = 4$,

$$h(t) = \begin{cases} x_0 - 1 & \text{for} \quad t \in [0,1[\\ 1 - y_0 & \text{for} \quad t \in [1,2[\\ 1 & \text{for} \quad t \in [2,3[\\ -1 & \text{for} \quad t \in [3,4] \end{cases}, \quad \tau(t) = \begin{cases} 0 & \text{for} \quad t \in [0,2[\\ 3 & \text{for} \quad t \in [2,4] \end{cases}.$$

Then (35) holds, and the problem (36) has the nontrivial solution

$$u(t) = \begin{cases} 0 & \text{for} \quad t \in [0,2[\\ t - 2 & \text{for} \quad t \in [2,3[\\ 4 - t & \text{for} \quad t \in [3,4] \end{cases}.$$

Example 3.3. Let $(x_0, y_0) \in H_3$. Put $a = 0$, $b = 2$, $\alpha = \frac{\lambda x_0}{1 - x_0}$, $\beta = \frac{1 - x_0 + \lambda}{1 - x_0}$, $t_0 = 2 - \frac{1}{y_0} - \frac{1}{\beta}$,

$$h(t) = \begin{cases} x_0 & \text{for} \quad t \in [0,1[\\ -y_0 & \text{for} \quad t \in [1,2] \end{cases}, \quad \tau(t) = \begin{cases} 1 & \text{for} \quad t \in [0,1[\\ t_0 & \text{for} \quad t \in [1,2] \end{cases}.$$

Then (35) holds, and the problem (36) has the nontrivial solution

$$u(t) = \begin{cases} \alpha t + \lambda & \text{for} \quad t \in [0,1[\\ \beta(2-t) - 1 & \text{for} \quad t \in [1,2] \end{cases}.$$

Example 3.4. Let $(x_0, y_0) \in H_4$. Put $a = 0$, $b = 5$, $\alpha = \sqrt{1 - x_0}$, $\beta = 1 - y_0 + 2\alpha - \lambda$, $t_0 = 3 - \alpha$,

$$h(t) = \begin{cases} \lambda - \alpha & \text{for} \quad t \in [0,1[\\ -\alpha & \text{for} \quad t \in [1,2[\\ -1 & \text{for} \quad t \in [2,3[\\ \beta & \text{for} \quad t \in [3,4[\\ x_0 & \text{for} \quad t \in [4,5] \end{cases}, \quad \tau(t) = \begin{cases} 5 & \text{for} \quad t \in [0,1[\\ 1 & \text{for} \quad t \in [1,3[\\ t_0 & \text{for} \quad t \in [3,4[\\ 5 & \text{for} \quad t \in [4,5] \end{cases}.$$

Then (35) holds, and the problem (36) has the nontrivial solution

$$u(t) = \begin{cases} (\alpha-\lambda)t+\lambda & \text{for} \quad t\in[0,1[\\ \alpha^2(1-t)+\alpha & \text{for} \quad t\in[1,2[\\ \alpha(3-t)-\alpha^2 & \text{for} \quad t\in[2,3[\\ -\alpha^2 & \text{for} \quad t\in[3,4[\\ x_0(5-t)-1 & \text{for} \quad t\in[4,5] \end{cases}.$$

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