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COMMON FIXED POINTS OF GREGUŠ TYPE MULTI–VALUED MAPPINGS

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Abstract. This work is considered as a continuation of [19,20,24]. The concepts of δ-compatibility and sub-compatibility of Li-Shan [19, 20] between a set-valued mapping and a single-valued mapping are used to establish some common fixed point theorems of Greguš type under a ϕ-type contraction on convex metric spaces. Extensions of known results, especially theorems by Fisher and Sessa [11] (Theorem B below) and Jungck [16] are thereby obtained. An example is given to support our extension.

1. Introduction

Fixed point theory of single-valued and multi-valued maps has been investigated extensively and applied to diverse problems during the last few decades. This theory provides techniques for solving a variety of applied problems in mathematical science and engineering (see e.g., [1, 2, 3, 23]).

In 1970, Takahashi [28] introduced a notion of convexity in metric spaces (see Definition 2.7) and generalized some fixed point theorems in Banach spaces. Subsequently, Ciric [6, 7], Gauy, Singh and Whitfield [14] and others have studied convex metric spaces and fixed point theorems.

In [13], Greguš proved the following theorem:

Theorem A. *Let C be a nonempty closed convex subset of a Banach space X and T be a mapping of C into itself satisfying the inequality*

$$||Tx - Ty|| \leq a||x - y|| + b||Tx - x|| + c||Ty - y|| \,,$$

for all x, y in C, where $a > 0$, $b \geq 0$, $c \geq 0$ and $a + b + c = 1$. Then T has a unique fixed point.

Fisher and Sessa [11] established a generalization of Theorem A as follows:

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Theorem B. *Let C be a nonempty closed convex subset of a Banach space X and T, f be two weakly commuting mappings of C into itself satisfying the inequality*

$$||Tx - Ty|| \leq a||fx - fy|| + (1-a)\max\{||Tx - fx||, ||Ty - fy||\},$$

for all x, y in C, where $0 < a < 1$. If f is linear and nonexpansive in C such that fC contains TC, then T and f have a unique common fixed point in C.

In recent years, common fixed points of Greguš type have been obtained by Ciric [4, 5], Davies and Sessa [8], Diviccaro, Fisher and Sessa [9], Jungck [16], Khan and Imdad [18], Murthy, Cho and Fisher [22] and Sessa and Fisher [26] in Banach spaces. On the other hand, Jungck [16] and Mukherjee and Verma [21] replaced linearity and nonexpansiveness by affine and continuity mappings, respectively. In [8, 22], the authors replaced nonexpansiveness, linearity and weak commutativity by continuity and compatibility. Also, Many theorems which are closely related to Greguš Theorem extended to multivalued mappings such as Li-Shan [19, 20] and Rashwan and Ahmed [24].

The aim of this paper is to prove some common fixed point theorems of Greguš type under a ϕ-contraction. Our results extend Theorems A, B and Jungck [16] to multi-valued mappings.

2. Basic Preliminaries

In the sequel, (X, d) denotes a metric space and $B(X)$ is the set of all nonempty bounded subsets of X. As in [10, 12], we define

$$\delta(A, B) = sup\{d(a, b) : a \in A, b \in B\},$$

for all A, B in $B(X)$. If A consists of a single point a, we write $\delta(A, B) = \delta(a, B)$. Also, if B contains a single point b, it yields that $\delta(A, B) = d(a, b)$.

It follows immediately from the definition of $\delta(A, B)$ that

$$\begin{aligned}
&\delta(A, B) = \delta(B, A) \geq 0, \\
&\delta(A, B) \leq \delta(A, C) + \delta(C, B), \\
&\delta(A, B) = 0 \quad \text{iff} \quad A = B = \{a\}, \\
&\delta(A, A) = \operatorname{diam} A,
\end{aligned}$$

for all $A, B, C \in B(X)$.

Definition 1.1 [10]**.** A sequence $\{A_n\}$ of nonempty subsets of X is said to be *convergent* to a subset A of X if:

(i) each point a in A is the limit of a convergent sequence $\{a_n\}$, where a_n is in A_n for $n \in N$ (N : the set of all positive integers),

(ii) for arbitrary $\epsilon > 0$, there exists an integer m such that $A_n \subseteq A_\epsilon$ for $n > m$, where A_ϵ denotes the set of all points x in X for which there exists a point a in A, depending on x, such that $d(x, a) < \epsilon$.

A is then said to be the *limit* of the sequence $\{A_n\}$.

Lemma 2.1 [10]. *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Lemma 2.2 [12]. *Let $\{A_n\}$ be a sequence in $B(X)$ and y be a point in X such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.*

Definition 2.2 [12]. A set-valued mapping F of X into $B(X)$ is said to be continuous at $x \in X$ if the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx whenever $\{x_n\}$ is a sequence in X converging to x in X. F is said to be continuous on X if it is continuous at every point in X.

Lemma 2.3 [12]. *Let $\{A_n\}$ be a sequence of nonempty subsets of X and z be in X such that $\lim_{n\to\infty} a_n = z$, z being independent of the particular choice of each $a_n \in A_n$. If a selfmap f of X is continuous, then $\{fz\}$ is the limit of the sequence $\{fA_n\}$.*

Definition 2.3 [27]. The mappings $F : X \to B(X)$ and $f : X \to X$ are said to be *weakly commuting* if $fFx \in B(X)$ and

$$\delta(Ffx, fFx) \leq \max\{\delta(fx, Fx), \operatorname{diam} fFx\}\,, \tag{2.1}$$

for all x in X.

Note that if F is a single-valued mapping, then the set $\{fFx\}$ consists of a single point. Therefore, $\operatorname{diam} fFx = 0$ for all $x \in X$ and condition (2.1) reduces to the condition given by Sessa [25], that is

$$d(Ffx, fFx) \leq d(fx, Fx)\,, \tag{2.2}$$

for all x in X.

Two *commuting mappings F and f* clearly *weakly commute* but *two weakly commuting F and f* do not necessarily *commute* as shown in [27].

In [15], Jungck generalized the concept of weakly commuting for single-valued mappings in the following way:

Definition 2.4. Two single-valued mappings f and g of a metric space (X, d) into itself are *compatible* if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

It can be seen that two *weakly commuting mappings* are *compatible* but the converse is false. Examples supporting this fact can be found in [15].

In [19], Li-Shan extended the definition 2.4 of compatibility to set-valued mappings as follows:

Definition 2.5. The mappings $f : X \to X$ and $F : X \to B(X)$ are *δ-compatible* if $\lim_{n\to\infty} \delta(Ffx_n, fFx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $fFx_n \in B(X)$, $Fx_n \to \{t\}$ and $fx_n \to t$ for some t in X.

Definition 2.6. The mappings $f : X \to X$ and $F : X \to B(X)$ are *subcompatible* if $\{t \in X : Ft = \{ft\}\} \subseteq \{t \in X : Fft = fFt\}$.

Remark 2.1. In [19], Li-Shan pointed out that the pair $\{F, f\}$ is δ-compatible $\Longrightarrow (F, f)$ is subcompatible but the converse is not true.

The following proposition of Jungck and Rhoades [17] is useful in the sequel:

Proposition 2.1. *Let (X, d) be a complete metric space. Suppose that $f : X \to X$ and $F : X \to B(X)$ and the pair $\{F, f\}$ is δ-compatible.*

(P_1) Suppose that the sequences $\{fx_n\}$ and $\{Fx_n\}$ converge to $t \in X$ and $\{t\}$, respectively. If f is continuous, then $Ffx_n \to \{ft\}$.

(P_2) If $\{ft\} = Ft$ for some $t \in X$, then $Fft = fFt$.

Now, we need some definitions due to Takahashi [28]:

Definition 2.7. Let X be a metric space and $I = [0, 1]$ be the closed unit interval. A continuous mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if there is $\lambda \in I$ such that for all $x, y, u \in X$

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y)\,.$$

X together with a convex structure is called *a convex metric space.*

Clearly, a Banach space or any convex subset of it is a convex metric space with $W(x, y, \lambda) = \lambda x + (1 - \lambda) y$. More generally, if X is a linear space with a translation invariant metric satisfying

$$d(\lambda x + (1 - \lambda) y, 0) \leq \lambda d(x, 0) + (1 - \lambda) d(y, 0)\,,$$

then X is a convex metric space.

Definition 2.8. Let X be a convex metric space. A nonempty subset K of X is convex if $W(x, y, \lambda) \in K$ whenever $x, y \in K$ and $\lambda \in I$.

Throughout this paper, a convex metric space will be denoted by (X, d, W). Let Φ be the set of all functions $\phi : [0, \infty) \longrightarrow [0, \infty)$ which satisfies the following conditions:

(i) ϕ is non-decreasing and continuous from the right,

(ii) $\phi(t) < t$ for each $t > 0$.

Let $F : X \to B(X), f : X \to X$ be mappings on a metric space X satisfying the following inequality:

$$\delta(Fx, Fy) \leq \phi(ad(fx, fy) + (1 - a) \max\{\delta(Fx, fx), \delta(Fy, fy)\})\,, \tag{2.3}$$

for all $x, y \in X$, where $0 < a < 1$ and $\phi \in \Phi$.

For our main theorems we need the following lemma, its proof is similar to that of Lemma 2.3 in [20]:

Lemma 2.4. *Let K be a nonempty closed subset of a complete metric space (X, d). If the mappings $f : K \to K$ and $F : K \to B(K)$ satisfy the condition* (2.3), *then*

(I) *F and f have at most one common fixed point u in K and further $Fu = \{u\}$;*

(II) *if $\{x_n\}$ is a sequence in K such that $\delta(Fx_n, fx_n) \to 0$, then there exists a $u \in K$ such that $Fx_n \to \{u\}$ and $fx_n \to u$.*

3. Main Results

The following theorem is useful in proving Theorem 3.2:

Theorem 3.1. *Let K be a nonempty closed subset of a complete metric space (X,d). Furthermore, let $F : K \to B(K)$ and $f : K \to K$ be a multivalued mapping and a single-valued mapping, respectively satisfying the inequality* (2.3).

(1) *If F and f have a unique common fixed point u in K and $Fu = \{u\}$, then* $\inf\{\delta(Fx, fx) : x \in K\} = 0$.

(2) *If* $\inf\{\delta(Fx, fx) : x \in K\} = 0$ *and F, f satisfy one of the following conditions:*

(U) *the pair $\{F, f\}$ is δ-compatible and f is continuous;*

(V) *the pair $\{F, f\}$ is δ-compatible, $FK \subseteq fK$ and F is continuous;*

(Z) *the pair $\{F, f\}$ is subcompatible and f is surjective,*

then F and f have a unique common fixed point u in K and $Fu = \{u\}$.

Proof. (1) Suppose that u is a unique common fixed point of F and f in K. Using the inequality (2.3), we obtain that

$$\delta(Fu, u) \leq \delta(Fu, Fu) \leq \phi((1-a)\delta(Fu, u)) < \delta(Fu, u)\,.$$

This contradiction implies that $Fu = \{u\}$. So, $\inf\{\delta(Fx, fx) : x \in K\} = 0$. To prove (2) let $\{x_n\}$ be a sequence such that

$$\delta(Fx_n, fx_n) \to \inf\{\delta(Fx, fx) : x \in K\} = 0\,.$$

By Lemma 2.4 (II), there exists a point $u \in K$ such that the sequences $\{fx_n\}$ and $\{Fx_n\}$ converge to u and $\{u\}$, respectively.

Now suppose that (U) holds. Since f is continuous, then Lemma 2.3 shows that the sequences $\{f^2x_n\}$ and $\{fFx_n\}$ converge to fu and $\{fu\}$, respectively. Proposition 2.1 (P_1) implies that the sequence $\{Ffx_n\}$ converges to $\{fu\}$. Applying the inequality (2.3), we get that

$$\delta(Ffx_n, Fx_n) \leq \phi(ad(f^2x_n, fx_n) + (1-a)\max\{\delta(f^2x_n, Ffx_n), \delta(Fx_n, fx_n)\})\,.$$

Letting $n \to \infty$, it implies from Lemma 2.1 that

$$d(fu, u) \leq \phi(ad(fu, u)) < ad(fu, u)) < d(fu, u)\,.$$

This contradiction demands that $fu = u$. From the inequality (2.3), it yields that

$$\delta(Fx_n, Fu) \leq \phi(ad(fx_n, fu) + (1-a)\max\{\delta(Fx_n, fx_n), \delta(Fu, fu)\})\,.$$

Letting $n \longrightarrow \infty$, it follows from Lemma 2.1 that

$$\delta(Fu, u) \leq \phi((1-a)\delta(Fu, u)) < \delta(u, Fu)\,.$$

This contradiction follows that $Fu = \{u\}$. Therefore, we know from Lemma 2.4 (I) that u is the unique common fixed point of F and f and $Fu = \{u\}$.

Now suppose that (V) holds. Then the sequence $\{Ffx_n\}$ converges to Fu. Let u_n be an arbitrary point in Fx_n for $n = 1, 2, \ldots$. Since $d(u_n, u) \leq \delta(Fx_n, u)$ and F is continuous, then we get that the sequence $\{Fu_n\}$ converges to Fu. By the inequality (2.3), we deduce that

$$\begin{aligned}\delta(Fu_n, Fu_n) &\leq \phi((1-a)\delta(Fu_n, fu_n)) \\ &\leq \phi((1-a)[\delta(Fu_n, Ffx_n) + \delta(Ffx_n, fFx_n)]) .\end{aligned}$$

As $n \to \infty$, the δ-compatibility of $\{F, f\}$ and Lemma 2.1 lead to

$$\delta(Fu, Fu) \leq \phi((1-a)\delta(Fu, Fu)) < \delta(Fu, Fu) .$$

This contradiction gives that $\delta(Fu, Fu) = 0$. From the inequality (2.3), we obtain that

$$\begin{aligned}\delta(Fu_n, Fx_n) &\leq \phi(ad(fu_n, fx_n) + (1-a)\max\{\delta(Fu_n, fu_n), \delta(Fx_n, fx_n)\}) \\ &\leq \phi(a[\delta(fFx_n, Ffx_n) + \delta(Ffx_n, fx_n)] \\ &\quad + (1-a)\max\{\delta(Fu_n, Ffx_n) + \delta(Ffx_n, fFx_n), \delta(Fx_n, fx_n)\}) .\end{aligned}$$

Since ϕ is continuous from the right and the pair $\{F, f\}$ is δ-compatible, as $n \to \infty$, using Lemma 2.1, we have that

$$\delta(Fu, u) \leq \phi(a\delta(Fu, u) + (1-a)\delta(Fu, Fu)) < a\delta(Fu, u) < \delta(Fu, u) .$$

This implies that $Fu = \{u\}$. Since $FK \subseteq fK$, then there exists a point w in K such that $fw = u$, it yields from inequality (2.3) that

$$\delta(Fx_n, Fw) \leq \phi(ad(fx_n, fw) + (1-a)\max\{\delta(Fx_n, fx_n), \delta(Fw, fw)\}) .$$

Letting $n \to \infty$, the last inequality becomes

$$\delta(u, Fw) \leq \phi((1-a)\delta(Fw, u)) < \delta(Fw, u) .$$

This contradiction implies that $Fw = \{u\}$. Since $\{F, f\}$ is δ-compatible and $\{fw\} = Fw$ for some $w \in K$, then Proposition 2.1 (P_2) leads to

$$\{u\} = Fu = Ffw = fFw = \{fu\} .$$

It follows from Lemma 2.4 (I) that u is the unique common fixed point of F and f and $Fu = \{u\}$.

Now suppose that (Z) holds. Then there exists a point v in K such that $fv = u$. From the inequality (2.3), we obtain that

$$\delta(Fv, Fx_n) \leq \phi(ad(fv, fx_n) + (1-a)\max\{\delta(Fv, fv), \delta(Fx_n, fx_n)\}) .$$

Letting $n \to \infty$, we get from Lemma 2.1 that

$$\delta(Fv, u) \le \phi((1-a)\delta(Fv, u)) < \delta(Fv, u)\,.$$

This contradiction implies that $Fv = \{u\}$. Since $\{F, f\}$ is subcompatible, we have that $Fu = Ffv = fFv = \{fu\}$. Using again the inequality (2.3), we deduce that

$$\delta(Fu, Fx_n) \le \phi(ad(fu, fx_n) + (1-a)\max\{\delta(Fu, fu), \delta(Fx_n, fx_n)\})\,.$$

As $n \to \infty$, Lemma 2.1 gives that

$$d(fu, u) \le \phi(ad(fu, u)) < ad(fu, u) < d(fu, u)\,.$$

It follows that $fu = u$. From Lemma 2.4 (I), u is the unique common fixed point of F and $Fu = \{u\}$. □

Now, we are ready to prove the following theorem:

Theorem 3.2. *Let K be a nonempty closed subset of a complete convex metric space (X, d, W) and $F : K \to B(K)$, $f : K \to K$ be mappings satisfying the inequality* (2.3) *. If fK is a convex subset of X such that $FK \subseteq fK$ and F, f satisfy one of the three conditions in Theorem 3.1, then F and f have a unique common fixed point u in K and $Fu = \{u\}$.*

Proof. Let x_0 be an arbitrary point in K. Since $FK \subseteq fK$, we choose points x_1, x_2, x_3 in K such that $fx_1 \in Fx$, $fx_2 \in Fx_1$, $fx_3 \in Fx_2$. For $i = 1, 2, 3$, we obtain from the inequality (2.3) that

$$\begin{aligned}\delta(Fx_i, fx_i) &\le \delta(Fx_i, Fx_{i-1}) \\ &\le \phi(ad(fx_i, fx_{i-1}) + (1-a)\max\{\delta(Fx_i, fx_i), \delta(Fx_{i-1}, fx_{i-1})\}) \\ &\le \phi(a\delta(Fx_{i-1}, fx_{i-1}) + (1-a)\max\{\delta(Fx_i, fx_i), \delta(Fx_{i-1}, fx_{i-1})\})\,.\end{aligned}$$

If $\delta(Fx_i, fx_i) \ge \delta(Fx_{i-1}, fx_{i-1})$, then

$$\delta(Fx_i, fx_i) \le \phi(\delta(Fx_i, Fx_i)) < \delta(Fx_i, fx_i)\,.$$

This contradiction implies that

$$\delta(Fx_i, fx_i) < \delta(Fx_{i-1}, fx_{i-1})\,,$$

for $i = 1, 2, 3$. It follows that

$$\delta(Fx_i, fx_i) < \delta(Fx_0, fx_0)\,, \tag{3.1}$$

for $i = 1, 2, 3$. Since fK is convex, then there exists w in K such that

$$fw = W(fx_2, fx_3, \frac{1}{2}) \in W(Fx_1, Fx_2, \frac{1}{2})\,,$$

where $W(Fx_1, Fx_2, \frac{1}{2}) = \cup\{W(e, m, \frac{1}{2}) : e \in Fx_1, m \in Fx_2\}$. $\square$

Using the inequalities (2.3) and (3.1), we have from the definition of convex structure that

$$\begin{aligned} d(fx_1, fw) &\leq \delta(fx_1, W(Fx_1, Fx_2, \frac{1}{2})) \\ &\leq \frac{1}{2}[\delta(fx_1, Fx_1) + \delta(fx_1, Fx_2)] \\ &\leq \frac{1}{2}[\delta(fx_1, Fx_1) + \delta(Fx_0, Fx_2)] \\ &< \frac{1}{2}[\delta(fx_0, Fx_0) + \phi(ad(fx_0, fx_2) \\ &\quad + (1-a)\max\{\delta(fx_0, Fx_0), \delta(Fx_2, fx_2)\})] \\ &< \frac{a+2}{2}\delta(Fx_0, fx_0)\,. \end{aligned} \tag{3.2}$$

Also, we have from the inequality (3.1) and the definition of convex structure that

$$\begin{aligned} d(fx_2, fw) &= \delta(fx_2, W(fx_2, fx_3, \frac{1}{2})) \\ &\leq \frac{1}{2}[d(fx_2, fx_2) + d(fx_2, fx_3)] \\ &\leq \frac{1}{2}\delta(Fx_2, fx_2) < \frac{1}{2}\delta(fx_0, Fx_0)\,. \end{aligned} \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned} \delta(Fw, fw) &\leq \delta(Fw, W(Fx_1, Fx_2, \frac{1}{2})) \\ &\leq \frac{1}{2}[\delta(Fw, Fx_1) + \delta(Fw, Fx_2)] \\ &\leq \frac{1}{2}[\phi(ad(fw, fx_1) + (1-a)\max\{\delta(Fw, fw), \delta(Fx_1, fx_1)\}) \\ &\quad + \phi(ad(fw, fx_2) + (1-a)\max\{\delta(Fw, fw), \delta(Fx_2, fx_2)\})] \\ &< \frac{a}{2}[d(fw, fx_1) + d(fw, fx_2)] \\ &\quad + (1-a)\max\{\delta(Fx_0, fx_0), \delta(Fw, fw)\} \\ &< \frac{a(3+a)}{4}\delta(Fx_0, fx_0) + (1-a)\max\{\delta(Fx_0, fx_0), \delta(Fw, fw)\}\,. \end{aligned}$$

If $\delta(Fx_0, fx_0) \geq \delta(Fw, fw)$, then

$$\delta(Fw, fw) < \frac{4 + a^2 - a}{4}\delta(Fx_0, fx_0)\,.$$

If $\delta(Fx_0, fx_0) \leq \delta(Fw, fw)$, then

$$\delta(Fw, fw) < \frac{3+a}{4}\delta(Fx_0, fx_0)\,.$$

Take $\alpha = \max\{\frac{4+a^2-a}{4}, \frac{3+a}{4}\}$. It is clear that $0 \le \alpha < 1$. we obtain that

$$\delta(Fw, fw) < \alpha\delta(Fx_0, fx_0)\,.$$

Therefore

$$\begin{aligned}\inf\{\delta(Fx_0, fx_0) : x_0 \in K\} &\le \inf\{\delta(Fw, fw) : fw = W(fx_2, fx_3, \frac{1}{2})\}\\ &< \alpha \inf\{\delta(Fx_0, fx_0) : x_0 \in K\}\,.\end{aligned}$$

So, $\inf\{\delta(Fx_0, fx_0) : x_0 \in K\} = 0$. Hence, we have from Theorem 3.1 (2) that F and f have a unique common fixed point u in K and $Fu = \{u\}$.

Remark 3.1. In Theorem 3.2, if F is a single-valued mapping of K into itself and $\phi(t) = kt$, for all $t > 0$, where $k \in (0, 1)$, we obtain a generalization of Theorem B for weakly commuting mappings.

Remark 3.2. In Theorem 3.2, if F is a single-valued mapping of K into itself and $\phi(t) = kt$, for all $t > 0$, where $k \in (0, 1)$, we obtain a generalization of Theorem 2.1 for compatible mappings of Jungck [16].

Now, we give an example to show the greater generality of Theorem 3.2 over Theorem B.

Example. Let $X = [0, \infty)$ with the Euclidean metric d and define

$$fx = x^3 + 3x^2 + 3x, \quad Fx = [0, \frac{x^3}{6}],$$

for all x in X. Suppose that $K = [0, 10]$ and $\phi(t) = \frac{1}{3}t$. For all $x, y \in X$,

$$\begin{aligned}\delta(Fx, Fy) &= \max\{\frac{x^3}{6}, \frac{y^3}{6}\}\\ &= \frac{1}{3}\frac{1}{2}\max\{x^3, y^3\}\\ &\le \frac{1}{3}\frac{1}{2}\max\{(x^3 + 3x^2 + 3x), (y^3 + 3y^2 + 3y)\}\\ &= \frac{1}{3}\frac{1}{2}\max\{\delta(fx, Fx), \delta(fy, Fy)\}\\ &\le \frac{1}{3}[\frac{1}{2}d(fx, fy) + (1 - \frac{1}{2})\max\{\delta(fx, Fx), \delta(fy, Fy)\}]\\ &= \phi(\frac{1}{2}d(fx, fy) + (1 - \frac{1}{2})\max\{\delta(fx, Fx), \delta(fy, Fy)\})\,,\end{aligned}$$

i.e., condition (2.3) is satisfied. Also we fined that

$$fx_n \to 0\,,\ Fx_n \to \{0\} \quad \text{if} \quad x_n \to 0 \quad \text{and} \quad \delta(Ffx_n, fFx_n) \to 0 \quad \text{as} \quad x_n \to 0\,.$$

Also, we get $fFx_n \in B(X)$, i.e., f and F are δ-compatible and hence they are subcompatible. It is obvious that f and F are continuous, $FK \subseteq fK$ and f is surjective. So, all assumptions of Theorem 3.2 satisfy and 0 is the unique common fixed point. Note that the extension of Theorem B to multi-valued mappings is not applicable because F and f are not weakly commuting mappings at $x = 1$ and hence Theorem B is not applicable.

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